The index of isolated umbilics on surfaces of non-positive curvature

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The index of isolated umbilics on surfaces of non-positive curvature

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Abstract. It is shown that if a C^2 surface $M \subset \mathbb{R}^3$ has negative curvature on the complement of a point $q \in M$, then the $\mathbb{Z}/2$-valued Poincaré-Hopf index at q of either distribution of principal directions on $M - \{q\}$ is non-positive. Conversely, any non-positive half-integer arises in this fashion. The proof of the index estimate is based on geometric-topological arguments, an index theorem for symmetric tensors on Riemannian surfaces, and some aspects of the classical Poincaré-Bendixson theory.

To Professor Jorge Sotomayor,
on the occasion of his 70th birthday

1. Introduction

The distributions of principal directions on a surface in $\mathbb{R}^3$, defined on the complement of the umbilical set (i.e., the locus where the principal curvatures coincide), have been the object of intense scrutiny since the early days of differential geometry. For both technical and geometric reasons, most of these investigations were conducted under the hypothesis that the surface is real analytic, or at least of class C^3, although one needs only C^2 regularity in order for the fields of principal directions to be continuous.

The aim of this work is to establish an estimate for the local index of these fields, under a natural curvature restriction, but with optimal regularity:

Theorem 1.1. *Let $M \subset \mathbb{R}^3$ be a C^2 surface, $q \in M$. Assume that the Gaussian curvature of M is negative at every point of M other than q. Then, the $\mathbb{Z}/2$-valued Poincaré-Hopf index at q of either distribution of principal directions on $M - \{q\}$ is non-positive.*

We observe that the theorem is sharp. Indeed, if the Gaussian curvature remains negative at q, then the distributions of principal directions extend continuously to M, and so the index is zero. On the other hand, given any negative number $j \in \mathbb{Z}/2$ one can construct a surface as in the statement of the theorem, even a minimal one (i.e., with vanishing mean curvature), that has an isolated umbilic of index j ([SX3]).

When the surface in question is minimal, the conclusion in Theorem 1.1 can be verified using the holomorphic data in the Weierstrass representation of the surface.

The main point of the present work is that, surprisingly, Theorem 1.1 applies to surfaces that are merely C^2, and not only to those C^ω surfaces that are minimal. Since complex analysis is no longer available in this more general setting, new tools have to be introduced in order to estimate the index. Loosely speaking, the replacement for complex analysis is, when properly augmented, the classical qualitative theory of planar dynamical systems.

Umbilic points are notoriously elusive geometric objects. For instance, on a surface of positive curvature the index of an isolated umbilic need not be positive. In other words, the "dual" statement of Theorem 1.1 does not hold. Indeed, inverting surfaces satisfying the hypotheses of Theorem 1.1 on suitable spheres – a process that does not change the index of the umbilic –, one can produce surfaces of *positive* Gaussian curvature exhibiting an umbilic whose index is any prescribed *negative* half-integer.

By analogy with the above mentioned sharpness of Theorem 1.1, one might naively expect that every *positive* half-integer could be realized as the index of an isolated umbilic on a surface of *positive* curvature. In stark opposition to this expectation, it is actually predicted that on any sufficiently regular surface, without any curvature restrictions whatsoever, the index of an isolated umbilic should be at most *one*. This is the well-known local Carathéodory conjecture, also known as the Loewner conjecture. We refer the reader to [Bol], [Bro], [FNB]–[GH], [GS]–[Ham2], [Iva]–[Xav] for a sample of the many works, old and new, on this very challenging problem, as well as on various aspects of the global study of principal foliations.

It is worth to point out that, if true, the local Carathéodory conjecture would imply, by the Poincaré-Hopf theorem, the Carathéodory conjecture (the statement that any C^2 immersion of the 2-sphere into $\mathbb{R}^3$ has at least two umbilical points;

see [GK] for a solution of this conjecture that has been proposed recently).

To put our results in perspective, we re-iterate that Theorem 1.1 is sharp and verifies the C^2 version of the local Carathéodory conjecture in a geometrically important special case, *but with a stronger conclusion*. Thus, Theorem 1.1 represents a contribution to the interface between classical differential geometry and classical dynamical systems that stands on its own, since it cannot be subsumed by the resolution of the Carathéodory conjecture. On the other hand, we hasten to add that there is no expectation that the present method can be used to tackle the said conjecture, given our strong reliance on negative curvature.

Over the years, the task of estimating the index of isolated umbilics has proven to be an arduous one, often involving lengthy and intricate arguments (e.g., [Iva]). Against this backdrop, it is pleasing that the proof of Theorem 1.1, albeit delicate in its own right, is rather conceptual. The arguments are based on an index theorem for abstract symmetric tensors on Riemannian surfaces, elements of the classical qualitative theory of two-dimensional dynamical systems, and a modicum of topology and classical differential geometry. Although the main result is new, the subject matter lends itself to a more expository style and, accordingly, full details are provided.

2. An index theorem for abstract symmetric tensors

A continuous symmetric tensor field A of type (1,1), defined on a Riemannian surface M, is said to have an isolated A-singularity at $q \in M$ if there exists a neighborhood V of q such that the eigenvalues of $A(p)$ are unequal for any $p \in V - \{q\}$. This condition is equivalent to the requirement that the traceless part $A(p) - \frac{1}{2}(\mathrm{tr} A(p))I$ of $A(p)$ be non-zero for all $p \neq q$. If the eigenvalues of $A(q)$ are distinct, then q is automatically an isolated A-singularity (one then thinks of q as being a "removable" A-singularity).

Under the above conditions, there are two continuous line fields on $M - \{q\}$, not necessarily orientable, which correspond to the diagonalizing directions of A. Given a continuous field of directions ξ on $V - \{q\}$, denote by $A\xi$ the field of directions obtained by applying $A(p)$ to any vector generating the one-dimensional subspace $\xi(p)$ of $T_p M$. We write $j(A)$ for the index at q of either of the two fields of diagonalizing directions of A. Similarly, $j(\eta)$ stands for the index at q of a line field η with an isolated singularity at q.

For the more general definition of local index of a totally real n-form (the case $n = 1$ corresponds to a direction field), as well as the proof of the two-dimensional Poincare-Hopf theorem in this context, see [FNB].

Theorem 2.1. *Let A be a continuous symmetric tensor field of type $(1,1)$ on a Riemannian 2-manifold M, and $q \in M$ an isolated A-singularity. Then, for every continuous field of directions ξ on a punctured neighborhood of q, one has*

$$(2.1) \qquad 2\,j(A) = j\big((A - \tfrac{1}{2}(\operatorname{tr}A)I)\,\xi\big) + j(\xi).$$

Remarks. One should think of ξ as being a "test" line field. For instance, if ξ is chosen to be one of the continuous fields of eigendirections of A, each term on the right hand side of (2.1) equals $j(A)$. As another illustration to see that 2 is the correct factor in the left hand side, let A be a continuous symmetric tensor field on a compact orientable surface M of non-zero genus, with the property that the set F_1 where A is a multiple of the identity is finite. Let ξ be a continuous line field on M with a finite set F_2 of singularities. Applying (2.1) around each point in $F_1 \cup F_2$, and summing, one sees from the Poincaré-Hopf theorem that both sides of (2.1) equal $2\chi(M)$.

Proof. Formula (2.1) was first established in [SX3], under the more restrictive assumptions that A is a smooth tensor and ξ is a smooth vector field; here, A and ξ are only assumed to be continuous, and ξ is allowed to be an unorientable line field. This extra generality requires a new line of argument.

Let $\lambda \geq \mu$ be the eigenvalues of A and U a neighborhood of q such that $\lambda(p) > \mu(p)$ whenever $p \in U - \{q\}$. The continuous distributions D_λ and D_μ of eigenspaces determined by A on $U - \{q\}$ have the same index at q, and this common value is, by definition, the index $j(A)$ of A at q. Let $B = A - \tfrac{1}{2}(\operatorname{tr}A)I$, so that $\operatorname{tr}B = 0$ everywhere, and let

$$\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$$

be the matrix representation of B with respect to an orthonormal frame $\{e_1, e_2\}$ in (a possibly smaller neighborhod) U. Observe that the continuous vector field $X := ae_1 + be_2$ has no zeros on $U - \{q\}$.

Let $C \subset U$ be a Jordan curve around q and $\gamma : [0,1] \to U$ a positive parametrization of C (relative to the orientation determined by $\{e_1, e_2\}$ on U). Let $V(t) = \cos\theta(t)\,e_1 + \sin\theta(t)\,e_2$ be a continuous unit vector field along γ such that $V(t)$ generates $D_\lambda(\gamma(t))$, for every $t \in [0,1]$. By definition,

$$(2.2) \qquad j(D_\lambda) = \frac{\theta(1) - \theta(0)}{2\pi}.$$

(Notice that since $V(1) = \pm V(0)$, $\theta(1)$ differs from $\theta(0)$ by a multiple of π, and thus $j(D_\lambda) \in \tfrac{1}{2}\mathbb{Z}$.) Consider the continuous vector field W along γ defined by

(2.3) $$W(t) = \cos(2\,\theta(t))\,e_1 + \sin(2\,\theta(t))\,e_2.$$

We claim that $W(t)$ is orthogonal to $(-be_1 + ae_2)(\gamma(t))$, for all $t \in [0,1]$. In fact, since $V(t)$ is an eigenvector of $A(\gamma(t))$ (and hence of $B(\gamma(t))$) and $\sin\theta(t)\,e_1 - \cos\theta(t)\,e_2$ is orthogonal to $V(t)$, one has

$$
\begin{aligned}
0 &= \big\langle B\big(V(t)\big), \sin\,\theta(t)\,e_1 - \cos\,\theta(t)\,e_2\big\rangle \\
&= \big\langle \cos\,\theta(t)(ae_1 + be_2) + \sin\,\theta(t)(be_1 - ae_2), \sin\,\theta(t)\,e_1 - \cos\,\theta(t)\,e_2\big\rangle \\
&= -b\big(\cos^2\theta(t) - \sin^2\theta(t)\big) + 2a\,\sin\,\theta(t)\,\cos\,\theta(t) \\
&= -b\,\cos(2\,\theta(t)) + a\,\sin(2\,\theta(t)) \\
&= \big\langle \cos\big(2\,\theta(t)\big)\,e_1 + \sin(2\,\theta(t))\,e_2, -be_1 + ae_2\big\rangle
\end{aligned}
$$

(2.4) $$= \big\langle W(t), -be_1 + ae_2\big\rangle,$$

which proves the claim. $\square$

   Since X is orthogonal to $-be_1 + ae_2$, it follows from the claim above that

(2.5) $$\frac{X}{|X|} \circ \gamma = \pm W.$$

Let ξ be a continuous field of directions on $U - \{q\}$. If $Z(t) = \cos\varphi(t)\,e_1 + \sin\varphi(t)\,e_2$ is a continuous vector field along γ such that, for all $t \in [0,1]$, $Z(t)$ generates $\xi(\gamma(t))$, then

(2.6) $$j(\xi) = \frac{\varphi(1) - \varphi(0)}{2\pi}.$$

From (2.3) and (2.5), we obtain

$$
\begin{aligned}
B\big(Z(t))\big) &= [a\cos\varphi + b\sin\varphi]e_1 + [b\cos\varphi - a\sin\varphi]e_2 \\
&= \pm|X|\big\{[\cos(2\theta)\cos\varphi + \sin(2\theta)\sin\varphi]e_1 \\
&\qquad\qquad + [\sin(2\theta)\cos\varphi - \cos(2\theta)\sin\varphi]e_2\big\}
\end{aligned}
$$

(2.7) $$= \pm|X|\big\{\cos\big(2\,\theta - \varphi\big)e_1 + \sin\big(2\,\theta - \varphi\big)e_2\big\}.$$

Since $Z(t)$ generates $\xi(\gamma(t))$, the equality above shows that, for all $t \in [0,1]$,

(2.8) $$\cos\big(2\,\theta(t) - \varphi(t)\big)e_1 + \sin\big(2\,\theta(t) - \varphi(t)\big)e_2$$

generates $(B\xi)(\gamma(t))$. The lemma now follows from (2.2) and (2.6):

$$
\begin{aligned}
j(B\xi) &= \frac{[2\,\theta(1) - \varphi(1)] - [2\,\theta(0) - \varphi(0)]}{2\pi} \\
&= \frac{2\,\theta(1) - 2\,\theta(0)}{2\pi} - \frac{\varphi(1) - \varphi(0)}{2\pi} \\
&= 2j(D_\lambda) - j(\xi) \\
&= 2j(A) - j(\xi).
\end{aligned}
$$

3. Gradients and degenerate local homeomorphisms

The lemma below is well known for the usual gradient of a planar function. Here, we work in the context of arbitrary Riemannian surfaces.

Lemma 3.1. *Let f be a C^1 function defined on an open set U of a C^2 Riemannian surface M, and $q \in U$ an isolated critical point of f. Then, the Poincaré-Hopf index of ∇f at q is at most one. Furthermore, the index of ∇f at q is one if and only if f has a strict local maximum, or minimum, at q.*

Proof. Taking U to be a coordinate neighborhood, $U = \varphi(W)$, $W \subset \mathbb{R}^2$, we may consider a continuous tensor $\tilde{J}$ on U corresponding to rotation by $\pi/2$ in the tangent spaces of M. Since $\tilde{J}$ can be continuously deformed into the identity through pointwise invertible tensors, the index at q of ∇f satisfies $j(\nabla f) = j(\tilde{J}\nabla f)$. Notice that $\tilde{J}\nabla f$ is tangent to the level curves of f. Likewise, $\varphi^*(\tilde{J}\nabla f)$ is tangent to the level curves of $f \circ \varphi$ on W. Hence

$$j(\nabla f) = j(\tilde{J}\nabla f) = j\big(\varphi^*(\tilde{J}\nabla f)\big) = j\big(J\nabla_0(f \circ \varphi)\big),$$

where the last two indices are computed at $\varphi^{-1}(q)$, J stands for the usual complex structure in $\mathbb{R}^2$, and ∇_0 is the Euclidean gradient. It follows from the Poincaré-Bendixson theory (see, e.g., the exercise on p. 173 of [Har]) that

$$j(J\nabla_0(f \circ \varphi)) \leq 1,$$

with equality holding if and only if $\varphi^{-1}(q)$ is a point of local maximum, or minimum, of $f \circ \varphi$. In particular, $j(\nabla f) \leq 1$, and equality holds if and only if q is an extremum of f.   □

Under the hypotheses of Theorem 1.1, if q is an umbilic point then the Gaussian curvature necessarily has to vanish at q, and so the Gauss map is not a local diffeomorphism. However, one can still prove that the Gauss map is open. More generally, using arguments from algebraic topology, it is possible to argue that a continuous map must be open if it is a local homeomorphism on the complement of a sufficiently "thin" subset of its domain ([Chu, p. 354]). Fortunately, in the special case that concerns us, an elementary proof is available:

Lemma 3.2. *Let $U \subset \mathbb{R}^n$ be open, $n \geq 2$, $q \in U$, $F : U \to \mathbb{R}^n$ continuous. If the restriction of F to $U - \{q\}$ is a local homeomorphism, then F is an open map.*

Proof. (The simple example $f(x) = x^2$ shows the need to have $n \geq 2$.) Assume, by contradiction, that F is not an open map. Since the restriction of F to $U - \{q\}$

is a local homeomorphism, there exists an open set $V \subset U$, with $q \in V$, such that $F(q) \in \partial F(V)$. Let B be an open ball centered at q such that $\overline{B} \subset V$. We are going to need claims (i) and (ii) below:

(i) For every $y \in \partial F(B)$, $y \neq F(q)$, there exists $x \in \partial B$ satisfying $F(x) = y$. Indeed, let (x_k) be a sequence in B with $F(x_k) \to y$. Passing to a subsequence, we can suppose $x_k \to x \in \overline{B}$. Hence $F(x_k) \to F(x)$, and so $F(x) = y$. Since $y \neq F(q)$, and the image of every point in $B - \{q\}$ belongs to the interior of $F(B)$, one has $x \in \partial B$.

(ii) Every ball D centered at $F(q)$ contains a point $y \in \partial F(B)$ distinct from $F(q)$.

Observe that $F(q) \in \partial F(B)$, otherwise $F(q) \in \text{int} F(B) \subset \text{int} F(V)$. Since $F(q) \in D \cap \partial F(B)$, one cannot have $D \subset F(B)$. Let then $z \in D - F(B)$. Using the continuity of F and the fact that the restriction of F to $B - \{q\}$ is a local homeomorphism, we see that $D \cap \text{int} F(B) \neq \varnothing$. Since $n \geq 2$, one can choose $w \in D \cap \text{int} F(B)$ such that $F(q)$ lies outside the segment $\overline{wz}$ joining w to z. Since $\overline{wz}$ joins a point in the interior of $F(B)$ to a point in the complement of $F(B)$, it must contain a point $y \in \partial F(B)$, which is necessarily distinct from $F(q)$. Since $y \in \overline{wz} \subset D$, (ii) follows.

Applying (ii) to a sequence of balls $D = D_k$ centered at $F(q)$, with radii tending to zero, one sees that there exists a sequence (y_k) in $\partial F(B)$, with $y_k \neq F(q)$ for all k, such that $y_k \to F(q)$. By (i), $y_k = F(x_k)$ for some sequence (x_k) in ∂B. Passing to a subsequence, we can assume that $x_k \to x \in \partial B$. By continuity, $F(x_k) \to F(x)$, and so $F(x) = F(q)$. We have then found a point $x \in \partial B \subset V - \{q\}$ whose image by F belongs to $\partial F(V)$, contradicting the fact that the restriction of F to $V - \{q\}$ is a local homeomorphism. $\qquad\square$

4. A special homotopy and the proof of Theorem 1.1

Let $U \subset M$ be a neighborhood of q on which a C^1 normal (Gauss) map $\xi : U \to S^2$ is defined. Choose $a \in S^2$ such that $0 < \langle a, \xi(q) \rangle < 1$. Shrinking U, one may assume that $0 < \langle a, \xi(p) \rangle < 1$ if $p \in U$.

Denote by α the second fundamental form of M, and consider the height function $f : M \to \mathbb{R}$, $f(x) = \langle x, a \rangle$. In particular, the (intrinsic) gradient and Hessian of f satisfy
(4.1)
$$\nabla f(p) = a - \langle a, \xi(p) \rangle \xi(p), \quad \text{Hess} f(p)(v, v) = \langle \xi(p), a \rangle \alpha(v, v), \quad v \in T_p M.$$

The first equation expresses the gradient of the restriction as the orthogonal projection of the space gradient into the tangent space. The formula for the

Hessian of the restriction of a function is standard in submanifold geometry, and can be found, say, in ([Daj, p. 46]).

Writing $H_f(p)$ and $A(p)$ for the linear endomorphisms associated to the quadratic forms $\operatorname{Hess} f(p)$ and $\alpha(p)$ on T_pM, respectively, it is clear from (4.1) that the indices of the continuous symmetric tensors A and H_f satisfy $j(H_f) = j(A)$.

Using the fact that $\det A$, being the Gaussian curvature, is negative away from q by hypothesis, it is easy to see that

$$A(p) - \frac{t}{2}\big(\operatorname{tr} A(p)\big)I, \ \ 0 \le t \le 1,$$

is a homotopy through invertible maps whenever $p \ne q$. Indeed, in a diagonalizing basis at p, the operator above has the matrix representation

$$\begin{bmatrix} \lambda(p)(1 - \frac{t}{2}) - \mu(p)\frac{t}{2} & 0 \\ 0 & \mu(p)(1 - \frac{t}{2}) - \lambda(p)\frac{t}{2} \end{bmatrix}.$$

Since $t \in [0,1]$ and the principal curvatures satisfy $\lambda(p) > 0 > \mu(p)$ if $p \ne q$, it is clear that the diagonal elements are non-zero.

It follows from the invariance of the degree under homotopies which do not introduce further zeros that, for every continuous non-vanishing vector field η on $U - \{q\}$,

$$j(A\eta) = j\Big(\big(A - \frac{1}{2}(\operatorname{tr} A)I\big)\eta\Big).$$

Hence, by (4.1), Theorem 2.1 and the fact that $\langle \xi(p), a \rangle \ne 0$ for all $p \in U$,

$$(4.2) \qquad\qquad 2j(A) = j(A\eta) + j(\eta) = j(H_f\eta) + j(\eta).$$

Applying (4.2) with $\eta = \nabla f$ (which is a permissible choice, since $0 < \langle a, \xi(p) \rangle < 1$ implies $\nabla f(p) \ne 0$), and using the general formula

$$(4.3) \qquad\qquad H_\psi \nabla \psi = \nabla\Big(\frac{1}{2}|\nabla \psi|^2\Big),$$

which is valid on any Riemannian manifold, with $\psi = f$, one has

$$(4.4) \qquad 2j(A) = j(H_f \nabla f) + j(\nabla f) = j\Big(\nabla\big(\frac{1}{2}|\nabla f|^2\big)\Big) + j(\nabla f).$$

Manifestly, (4.4) provides a formula for the index of the umbilic q (i.e., the index of the tensor field A) in terms of the indices of two gradient fields.

To conclude the proof of the theorem, we must show that $j(A)$ is non-positive. If q is not an umbilical (planar) point, the distributions of principal directions extend continuously across q, in which case $j(A)$ vanishes. Hence, we may assume that q is an umbilic, that is, $\lambda(q) = \mu(q) = 0$. In particular, $A(q) = 0$.

Since $\nabla f(p) \neq 0$ for $p \in U$, the term $j(\nabla f)$ in (4.4) vanishes. Hence, it remains to argue that $j(\nabla(\frac{1}{2}|\nabla f|^2)) \leq 0$. According to Lemma 3.1, one must show that the function $h := \frac{1}{2}|\nabla f|^2$ has an isolated critical point at q, which is neither a local maximum nor a local minimum. From (4.1) and (4.3), one has $\nabla h(p) = \nabla(\frac{1}{2}|\nabla f|^2)(p) = \langle a, \xi(p)\rangle A(p)\nabla f(p)$. Since $A(q) = 0$, the point q is critical for h. In order to see that no point p in $U - \{q\}$ is critical for h, observe that $\langle a, \xi(p)\rangle > 0$, $\nabla f(p) \neq 0$ and $A(p)$ is invertible (since the Gaussian curvature $\det A(p)$ is negative for $p \neq q$, by hypothesis). Hence q is an isolated critical point of h.

We now proceed to show that h has neither a local maximum nor a minimum at q. A direct calculation gives $\sqrt{2h(p)} = |\nabla f(p)| = \sin\theta(p)$, where $\theta(p) \in [0, \pi]$ is the angle between the vectors a and $\xi(p)$. Since $0 < \langle a, \xi(q)\rangle < 1$, it follows that $\theta(q) \in (0, \frac{\pi}{2})$. Therefore, in order to show that h does not have an extremum at q, it suffices to argue that the Gauss map $\xi : U \to S^2$ is open. But this is a consequence of Lemma 3.2 and the inverse function theorem, since the Jacobian determinant of the normal map is the Gaussian curvature which, by hypothesis, is negative away from q.

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