Almost-periodic actions on the real line

L'Enseignement Mathématique (2) 59 (2013), 183–194

ALMOST-PERIODIC ACTIONS ON THE REAL LINE

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ABSTRACT. A homeomorphism of the real line is *almost-periodic* if the set of its conjugates by the translations is relatively compact in the compact open topology. Our main result states that an action of a finitely generated group on the real line without global fixed points is conjugated to an action by almost-periodic homeomorphisms without almost fixed points. This is equivalent to saying that the real line together with the translation flow can be compactified as an orbit of a free action of $\mathbf{R}$ on a compact space, together with an action of the group by homeomorphisms *without* global fixed points. As an application we give an alternative proof of Witte's theorem: an amenable left orderable group is locally indicable.

1. INTRODUCTION

A group G is *left-orderable* if there exists a total order on G which is invariant by left translations. Many groups are left-orderable: free groups, surface groups, braid groups, conjecturally every lattice in $\mathrm{SO}(1,3)$ is virtually left-orderable, etc. See e.g. [8] for interesting references. On the other hand, no group with Kazhdan property (T) is known to be left-orderable, and lattices in semi-simple Lie groups of rank ≥ 2 are believed not to be, see Witte's work [10] where this is proved if the $\mathbf{Q}$-rank of the lattice is ≥ 2, and [5, Problem 7.3, p. 389], where this problem is explicitly stated.

The reason for such a belief, at least in the case of lattices, comes from a conjecture by Zimmer, stating that a lattice of a semi-simple Lie group of rank ≥ 2 does not act faithfully on a 1-dimensional manifold by homeomorphisms. Since a countable group is left-orderable if and only if it has a faithful action by orientation preserving homeomorphisms on the real line, see [5, Theorem 6.8, p. 374], Zimmer's conjecture predicts that a lattice in a semi-simple Lie group of rank ≥ 2 is not left-orderable.

Ghys proved in [4] the following very interesting statement: if a lattice of a semi-simple Lie group of rank ≥ 2 acts on the circle by homeomorphisms, then it has a finite orbit. From this he was able to deduce Zimmer's conjecture in the case where the action is of class C^1. However, if the action is only continuous, the conjecture is still open.

Hence, it is tempting to ask the following question: *is it true that a finitely generated left-orderable group acts on the circle by homeomorphisms without a finite orbit?* A positive answer would imply that a lattice in a semi-simple Lie group of rank ≥ 2 is not left-orderable. This would also solve the following conjecture by Linnell: a finitely generated left-orderable group either contains a free group on two generators, or it has a non trivial morphism to the integers. Indeed, if a group acts on the circle without a finite orbit, then either it is semi-conjugated to an action by rotations (and in this case we easily find a non trivial morphism to the integers), or it has no invariant measure, and a result of Margulis ensures that it contains a free group on two generators, see [7].

The goal of this note is to give an "almost" positive answer to the question asked above. Namely, we prove the following:

THEOREM 1.1. *Let G be a finitely generated left-orderable group. Then there is a compact space X, a free action of $\mathbf{R}$ on X, and an action of G on X without global fixed points, which preserves the $\mathbf{R}$-orbits and acts on them by orientation preserving homeomorphisms.*

We think of this action of G on X as being an almost action of the group G on the circle. Indeed X can be approximated by circles in the following way: consider a long segment contained in an $\mathbf{R}$-orbit, with close extremities in X, and glue them together to obtain a circle (such a segment exists by compactness). The action of G on X does not provide an action of G on this circle, but almost! Observe moreover that G does not have any finite orbit on X.

We believe that this construction can serve for the resolution of the above mentioned problems. To motivate its interest, we give an alternative proof of a theorem by Witte, which is a partial version of Linnell's conjecture, see [11]: if a finitely generated left-orderable group is amenable then it has a non trivial morphism to the integers. The interested reader can go directly to Section 6 to find the proof of Witte's theorem assuming Theorem 1.1.

For the construction of the space X, we introduce the following notion of almost-periodicity for homeomorphisms of the real line: a homeomorphism is called *almost-periodic* if the set of its conjugates by the translations $\tau_s(t) = t+s$ is relatively compact in $\mathrm{Homeo}^+(\mathbf{R})$ for the compact open topology, see Section 2 for more details. We prove that every action of a finitely generated group G on the real line without fixed points is conjugated to an action by almost-periodic homeomorphisms, without *almost fixed points*, that is

$$\inf_{t\in\mathbf{R}} \max_{g\in S} |g(t) - t| > 0 ,$$

where S is a finite generating set of G, see Theorem 4.1. This easily permits us to construct our space X, together with the flow and the G-action, see Lemma 2.2.

An important device used to construct almost-periodic actions, is to conjugate a given action by homeomorphisms on the real line to an action by bi-Lipschitz homeomorphisms. This is interesting on its own, and is the content of Section 3.

ACKNOWLEDGEMENTS. I warmly thank the anonymous referee for his careful reading and his numerous comments and suggestions. It is a pleasure to thank Victor Kleptsyn and Andrès Navas for stimulating conversations on group actions on the real line, and Marie-Claude Arnaud, who invented a nice argument that simplifies the proof of Theorem 3.1.

2. ALMOST-PERIODIC REPRESENTATIONS

The group of homeomorphisms of the real line preserving the orientation, $\mathrm{Homeo}^+(\mathbf{R})$, is equipped with the compact open topology, which turns it into a topological group. A homeomorphism $h\colon \mathbf{R} \to \mathbf{R}$ is called *almost-periodic* if the set

$$\{\tau_s^{-1} \circ h \circ \tau_s \mid s \in \mathbf{R}\}$$

is relatively compact in $\mathrm{Homeo}^+(\mathbf{R})$, where $\tau_s(t) = s + t$. The set of almost-periodic and orientation preserving homeomorphisms is denoted $APH^+(\mathbf{R})$.

PROPOSITION 2.1. *$APH^+(\mathbf{R})$ is a subgroup of* $\mathrm{Homeo}^+(\mathbf{R})$.

Proof. This is a consequence of the continuity of composition and inverse operations on $\mathrm{Homeo}^+(\mathbf{R})$ considered with the compact open topology. □

An action of a group on the line whose image is contained in $APH^+(\mathbf{R})$ will be called *almost-periodic*. There are various ways to construct a faithful almost-periodic action of a left-orderable and countable group G on the line. The simplest is to begin with a faithful action by homeomorphisms on the interval, and to extend it to the line by conjugating by the powers of the translation $t \mapsto t+1$. Hence, $APH^+(\mathbf{R})$ contains a copy of any left-orderable and countable group.

We will now provide the construction of a compact foliated space together with a G-action on it preserving the leaves, beginning with an almost-periodic action of G on the real line.

LEMMA 2.2. *Let G be a finitely generated group and let $\rho_0 \colon G \to \mathrm{Homeo}^+(\mathbf{R})$ be an action of G on the real line by orientation preserving homeomorphisms. Then ρ_0 is almost-periodic if and only if there exists a flow $\Phi = \{\Phi_s\}_{s\in\mathbf{R}}$ acting freely on a compact space X, an action of G on X by homeomorphisms preserving every Φ-orbit together with their orientation, and a point $x_0 \in X$, such that for every $g \in G$ and every $t \in \mathbf{R}$ we have*

$$g \cdot \Phi_t(x_0) = \Phi_{\rho_0(g)(t)}(x_0)\,. \tag{2.1}$$

Moreover, we can suppose that the Φ-orbit of ρ_0 is dense in X.

Proof. First, let us prove that if there exists a compact space X together with a flow Φ and a G-action verifying (2.1), then the representation ρ_0 is almost-periodic. For every $x \in X$, we can lift the action of G on X to an action $\rho_x \colon G \to \mathrm{Homeo}^+(\mathbf{R})$ verifying

$$g \cdot \Phi_t(x) = \Phi_{\rho_x(g)(t)}(x)\,,$$

which is well-defined since the $\mathbf{R}$-action Φ is free. Moreover, since the G-action on X is continuous, for every $g \in G$, the map $x \in X \mapsto \rho_x(g) \in \mathrm{Homeo}^+(\mathbf{R})$ is continuous. Hence, the set of elements $\rho_x(g)$, for $x \in X$, is compact. Now, for every $s, t \in \mathbf{R}$ and every $x \in X$, we have

$$g \cdot \Phi_t(\Phi_s(x)) = g \cdot \Phi_{t+s}(x) = \Phi_{\rho_x(g)(t+s)}(x) = \Phi_{\rho_x(g)(t+s)-s}(\Phi_s(x))\,,$$

so that we get the formula

$$\rho_{\Phi_s(x)}(g) = \tau_s^{-1} \circ \rho_x(g) \circ \tau_s\,.$$

Hence, for every $g \in G$, the conjugates of $\rho_x(g)$ by the translations τ_s stay in a compact set, which proves that ρ_x is almost-periodic for every $x \in X$, and in particular for $x = x_0$ we deduce that $\rho_0 = \rho_{x_0}$ is almost-periodic.

Before proving the reciprocal statement, let us make explicit an almost-periodic homeomorphism: the homeomorphism σ defined by

$$\sigma(t) = t + \frac{1}{3}\big(\sin(t) + \sin(\sqrt{2}t)\big). \tag{2.2}$$

Indeed, the diffeomorphism of the 2-torus $\mathbf{R}^2/2\pi\mathbf{Z}^2$ defined by the formula

$$\overline{\sigma}(u,v) = \Big(u + \frac{1}{3}\big(\sin(u) + \sin(v)\big),\ v + \frac{\sqrt{2}}{3}\big(\sin(u) + \sin(v)\big)\Big)$$

preserves the orbits of the irrational linear flow defined by

$$\Psi_t = \exp\big(t(\tfrac{\partial}{\partial u} + \sqrt{2}\tfrac{\partial}{\partial v})\big)\,,$$

and we have $\overline{\sigma}(\Psi_t(x_0)) = \Psi_{\sigma(t)}(x_0)$ for every $t \in \mathbf{R}$, with $x_0 = (0,0)$. This gives an example of an almost-periodic homeomorphism. It is interesting to note that σ does not commute with any translation, as the reader can easily check, hence giving a non trivial example of an almost-periodic homeomorphism.

Let us now provide the existence of the compact space X, together with the actions of $\mathbf{R}$ and G on X verifying (2.1), starting with an almost-periodic action ρ_0. Introduce the set $APA^+(G)$ of almost-periodic actions of G on the real line. It can be seen as a closed subset of $APH^+(\mathbf{R})^S$, where S is a finite generating set of G. Define the *translation flow* $\{\Phi_s\}_{s\in\mathbf{R}}$ acting on $APA^+(G)$ by conjugation by the translations, namely

$$\Phi_s(\rho)(g) := \tau_{-s} \circ \rho(g) \circ \tau_s \tag{2.3}$$

for every $\rho \in APA^+(G)$ and $g \in G$. This is a topological flow acting on $APA^+(G)$. Denote by X the closure of the Φ-orbit of ρ_0. This is a compact Φ-invariant subset of $APA^+(G)$, since ρ_0 is almost-periodic.

A priori, this action of the flow Φ on X has no reason to be free. However, it is possible to make it free by the following procedure. Observe that if ρ_σ is the action of $\mathbf{Z}$ on $\mathbf{R}$ mapping 1 to the homeomorphism σ defined in (2.2), then the closure of the Φ-orbit of ρ_σ in $APA^+(\mathbf{Z})$ is homeomorphic to the 2-torus $\mathbf{R}^2/2\pi\mathbf{Z}^2$, and the action of Φ on this torus is the linear irrational flow described above. Hence, upon replacing G by the free product $G*\mathbf{Z}$ and extending ρ_0 so that the free generator is mapped to the homeomorphism σ defined in (2.2), we may assume that X admits a Φ-equivariant continuous map to the 2-torus, and in particular acts freely on X.

We claim that the formula

$$g \cdot \rho := \tau_{-\rho(g)(0)} \circ \rho \circ \tau_{\rho(g)(0)} \tag{2.4}$$

defines an action of G on $APA^+(G)$. One can verify this by a tedious computation, but here is an elegant argument due to the referee. Consider the actions of $\mathbf{R}$ and G on the product $APA^+(G) \times \mathbf{R}$ given by

$$s \cdot (\rho, t) := (\Phi_s(\rho), t - s) \quad \text{and} \quad g \cdot (\rho, t) = \big(\rho, \rho(g)(t)\big).$$

An element of $APA^+(G) \times \mathbf{R}$ can be thought of as an action of G on the real line together with a marker. The action by the reals is given by translating the marker, while conjugating the representation by the same translation, and the G-action is given by acting on the marker using the action given by the first coordinate, while letting the representation unchanged. These two actions commute, by an easy computation. Hence there is a natural action of G on the quotient of $APA^+(G) \times \mathbf{R}$ by $\mathbf{R}$, which naturally identifies with $APA^+(G)$ via the embedding $\rho \in APA^+(G) \mapsto (\rho, 0) \in \mathbf{R} \times APA^+(G)$. The action of G on $APA^+(G)$ induced by this identification is given by the formula (2.4).

By construction, the action of G on $APA^+(G)$ preserves each Φ-orbit, and is conjugated to ρ on it. More precisely, we have:

$$g \cdot \Phi_s(\rho) = \Phi_{\rho(g)(s)}(\rho),$$

for every $s \in \mathbf{R}$, $\rho \in APA^+(G)$, and $g \in G$. Hence, G preserves the set X, and the lemma is proved. □

3. A bi-Lipschitz conjugation theorem

We denote by $\mathrm{Bilip}^+(\mathbf{R})$ the group of orientation preserving bi-Lipschitz homeomorphisms of the real line. For every $h \in \mathrm{Bilip}^+(\mathbf{R})$, we denote by $K(h)$ the minimum of the numbers $K \geq 1$ such that

$$\forall x, y \in \mathbf{R} \qquad K^{-1} \cdot |y - x| \leq |h(y) - h(x)| \leq K \cdot |y - x|. \tag{3.1}$$

We equip $\mathrm{Bilip}^+(\mathbf{R})$ with the topology of uniform convergence on compact subsets of $\mathbf{R}$.

THEOREM 3.1. *A finitely generated group of homeomorphisms of the real line is conjugated to a group acting by Lipschitz homeomorphisms.*

Proof. Our proof is inspired by a discussion with Marie-Claude Arnaud. In [3], we give a more conceptual (but more elaborate) proof based on probabilistic arguments. Equivalent results were proved for transverse pseudo-group of foliations, or groups acting on the circle, see [1, Proposition 2.5] and [2, Théorème D].

Let $\lambda = f(t)dt$ be a probability measure on $\mathbf{R}$ with a smooth and positive density f such that for $|t|$ large enough, we have $f(t) = 1/t^2$. The following observation will be central in what follows: if, for some constant $L \geq 1$, a homeomorphism h from the real line to itself satisfies

$$(3.2) \qquad h_*\lambda \leq L\lambda \quad \text{and} \quad (h^{-1})_*\lambda \leq L\lambda\,,$$

then h is Lipschitz. To prove this fact, first observe that $\lambda([t, +\infty)) = \frac{1}{t}$, for t a large positive number (and similarly $\lambda((-\infty, t]) = \frac{1}{|t|}$ if t is a large negative number). Thus, the left part of (3.2) shows that for $|t|$ large enough, $|h(t)| \leq L|t|$. The density of $(h^{-1})_*\lambda$ being given by $h'(t)f(h(t))$, the right part of (3.2) gives the bound $h'(t) \leq \frac{Lf(t)}{f(h(t))}$ for almost every t. Thus, up to sets of Lebesgue measure 0, h' is bounded on every compact interval, and for $|t|$ large enough we have $h'(t) \leq L^3$; this proves that h' is bounded, and hence h is Lipschitz.

Denote by G a finitely generated subgroup of $\mathrm{Homeo}^+(\mathbf{R})$, and let S be a finite system of generators for G. Let $\varphi \in L^1(G)$ be a function with positive values such that, for every element $h \in G$, there is a constant L_h such that $\varphi(hg) \leq L_h\varphi(g)$; for instance one can take $\varphi(g) = \alpha^{\|g\|}$ with α a small enough positive number, where $\|g\|$ is the minimum length of a word in the elements of S which equals g. Normalize the function φ so that $\sum_{g\in G}\varphi(g) = 1$, and introduce the probability measure on $\mathbf{R}$ defined by

$$\nu := \sum_{g\in G} \varphi(g)\cdot g_*\lambda\,.$$

Observe that for every $h \in G$, we have

$$h_*\nu = \sum_{g\in G} \varphi(g)\cdot (hg)_*\lambda \leq L\nu\,,$$

where $L = L_{h^{-1}}$.

The measure ν has full support and no atoms. Thus, there exists a homeomorphism ϕ from the real line to itself which maps ν to λ. Denote $h^\Phi = \Phi \circ h \circ \Phi^{-1}$. We have

$$h^\Phi_*\lambda = \Phi_*h_*\nu \leq L\Phi_*\nu = L\lambda\,.$$

From the discussion above, we deduce that G^Φ is contained in $\mathrm{Bilip}^+(\mathbf{R})$. $\square$

4. Actions without almost-fixed points

Let G be a finitely generated group with finite generating set S and $\rho \in APA^+(G)$ an almost-periodic action of G on $\mathbf{R}$. We say that ρ has an *almost fixed point* if

$$\inf_{t\in\mathbf{R}} \sup_{g\in S} |\rho(g)(t) - t| = 0\,.$$

This property is equivalent to the following: the action of G on the compact space constructed in Lemma 2.2 has a global fixed point in the closure of the orbit $\Phi_{\mathbf{R}}(\rho_0)$. It is not immediate to construct almost-periodic actions without almost fixed points. The main new device of this note is to provide such a construction, if the group is finitely generated and left-orderable:

THEOREM 4.1. *An orientation preserving action of a finitely generated group on the real line is topologically conjugated to an almost-periodic action. Moreover, if the original action has no fixed point, then it is possible to find a conjugacy to an almost-periodic action without almost fixed points.*

This section is devoted to the proof of Theorem 4.1. Let S be a finite symmetric system of generators of G and let $K > 1$ and $0 < C < D$ some constants. We denote by $R = R(G, S, K, C, D)$ the set of representations $\rho\colon G \to \mathrm{Bilip}^+(\mathbf{R})$ such that for every $g \in S$, $K(\rho(g)) \leq K$ and for every $t \in \mathbf{R}$:

$$t - D \leq \min_{g\in S} \rho(g)(t) \leq t - C \leq t + C \leq \max_{g\in S} \rho(g)(t) \leq t + D\,. \tag{4.1}$$

LEMMA 4.2. *Let $\rho_0\colon G \to \mathrm{Bilip}^+(\mathbf{R})$ be an action of a finitely generated group G without global fixed points. There are constants $K > 1$ and $C, D > 0$ and a finite symmetric generating set S of G such that the set R contains a representation conjugated to ρ_0. Moreover, R is a compact set.*

Proof. It is sufficient to prove the statement in the case where G is a finitely generated subgroup of $\mathrm{Bilip}^+(\mathbf{R})$ and $\rho_0 = id$. Let S be a finite symmetric generating set of G and K be a constant such that for every $g \in S$, $K(g) \leq K$. The condition (4.1) might not be satisfied, e.g. when the action is affine. So we will need to modify our action and build a new one. To do so, we define a sequence of points $t_n \in \mathbf{R}$ for every $n \in \mathbf{Z}$ by $t_0 = 0$ and $t_{n+1} = \max_{g\in S} g(t_n)$, or equivalently $t_{n-1} = \min_{g\in S} g(t_n)$ since S is symmetric. Since G has no fixed point on the real line, we have

$$\lim_{n\to\pm\infty} t_n = \pm\infty\,.$$

We let φ be the homeomorphism from the real line to itself which sends t_n to n, and is affine on the intervals $[t_n, t_{n+1}]$. We claim that the action of G

on the real line defined by $\rho(g) = \varphi \circ g \circ \varphi^{-1}$ belongs to $R(G, \bar{S}, K^6, 1, 4)$ for the generating set $\bar{S} = S \cup S^2$.

To prove this, we remark that the distortion of the sequence t_n is uniformly bounded; more precisely for every integer $n \in \mathbf{Z}$, setting $\delta_n = t_{n+1} - t_n$, we have

$$K^{-1} \cdot \delta_{n+1} \leq \delta_n \leq K \cdot \delta_{n+1}\,. \tag{4.2}$$

To see this, write $t_{n+1} = g_n(t_n)$, where $g_n \in S$. By definition $g_n(t_{n+1}) \leq t_{n+2}$, and because g_n is a K-bi-Lipschitz map, we get

$$t_{n+2} - t_{n+1} \geq g_n(t_{n+1}) - g_n(t_n) \geq K^{-1} \cdot (t_{n+1} - t_n)\,,$$

hence the second inequality in (4.2). The first one is obtained by analogous considerations. This implies that φ is close to being affine on $[t_{n-1}, t_{n+2}]$; more precisely, for every pair of points $w, z \in [t_{n-1}, t_{n+2}]$, we have

$$\frac{|z-w|}{K \cdot \delta_n} \leq |\varphi(z) - \varphi(w)| \leq \frac{K \cdot |z-w|}{\delta_n}\,.$$

We are now able to prove that for every $g \in S$, the map $\rho(g)$ is Lipschitz and $K(\rho(g)) \leq K^3$. It suffices to prove that $\rho(g)$ is Lipschitz on every interval of the form $[n, n+1]$ with Lipschitz constant K^3. Consider two points $x, y \in [n, n+1]$ and define $w = \varphi^{-1}(x)$, $z = \varphi^{-1}(y)$: we have

$$\begin{aligned} |\rho(g)(y) - \rho(g)(x)| &\leq |\varphi(g(z)) - \varphi(g(w))| \\ &\leq \frac{K \cdot |g(z) - g(w)|}{\delta_n} \leq \frac{K^2 \cdot |z-w|}{\delta_n} \leq K^3 |y-x|\,. \end{aligned}$$

By construction, for every element $g \in S$,

$$x - 2 \leq \rho(g)(x) - x \leq x + 2\,,$$

because the nearest integer points after and before x are moved a distance less than 1 by $\rho(g)$. Moreover, for every $n \in \mathbf{Z}$, we have $\rho(g_{n+1}g_n)(n) = n+2$. Hence, for every $x \in \mathbf{R}$ we have $\rho(g_{n+1}g_n)(x) \geq x+1$, n being the integer part of x. Hence, we have proved that ρ belongs to $R(G, \bar{S}, K^6, 1, 4)$. □

Let us finish this section by giving the proof of Theorem 4.1. Let ρ_0 denote an orientation preserving action of a finitely generated group G on the real line. Upon replacing G by $G * \mathbf{Z}$ and extending ρ_0 so that the free generator is mapped to a non trivial translation, we can assume that the group has no fixed point. Hence it is only necessary to prove the second part of Theorem 4.1. By Theorem 3.1, this action is conjugated to an action by bi-Lipschitz homeomorphisms, and by Lemma 4.2, there exist constants $C, D, K > 0$ and a finite symmetric set S of G such that the set R contains an element ρ_1 which is conjugated to ρ_0. This set can be seen as a closed subset of $\mathrm{Bilip}^+(\mathbf{R})^S$, and as such is equipped with the product topology; the relations (3.1) and (4.1) imply that R is a compact set, by the Arzelà-Ascoli theorem. Moreover, the same relations show that the translation flow Φ defined by (2.3) preserves R. Hence, every element of R is an almost-periodic action of G, and condition (4.1) shows that this action has no almost fixed points. Hence, ρ_1 is an almost-periodic action of G on $\mathbf{R}$ without almost fixed points which is conjugated to ρ_0, and Theorem 4.1 is proved.

5. Proof of Theorem 1.1

Let us provide the proof of Theorem 1.1. Let G be a finitely generated left-orderable group. It is a classical fact that G has a faithful action on the real line by orientation preserving homeomorphisms, and without global fixed points. We recall the idea of the construction, and refer to [5, Theorem 6.8, p. 374] for details. Let $n \in \mathbf{N} \mapsto g_n \in G$ be a bijection. We consider a sequence of distinct real numbers a_n defined inductively as follows. Set $a_0 = 0$, and suppose $a_1, \ldots, a_n$ have been defined. If g_{n+1} is bigger (resp. smaller) than g_k for all $k \leq n$, set $a_{n+1} = \sup_{k \leq n} a_k + 1$ (resp. $a_{n+1} = \inf_{k \leq n} a_k - 1$). If not, find an ordering of the integers from 0 to n such that $g_{i_0} < g_{i_1} < \ldots < g_{i_n}$ and let $k \leq n$ be such that $g_{i_k} < g_{n+1} < g_{i_{k+1}}$. In this case, define $a_{n+1} = \frac{a_{i_k} + a_{i_{k+1}}}{2}$. The sequence (a_n) constructed in this way has the same order as the sequence (g_n). The action of G on itself by left translation preserves the order, hence the action of G on the subset $\{a_n\} \subset \mathbf{R}$ via the isomorphism $G \simeq \mathbf{N} \simeq \{a_n\}$ also preserves the natural order of $\mathbf{R}$. We leave to the reader to check that this action can be extended to a faithful action of G by orientation preserving homeomorphisms on the real line without any global fixed point.

By Theorem 4.1, this action is conjugated to an almost-periodic action ρ_0 on the real line, without almost fixed points. Consider the space X constructed in Lemma 2.2, together with the free action of $\mathbf{R}$ and the G-action. Since ρ_0 has no almost fixed point and since $\Phi_{\mathbf{R}}(\rho_0)$ is dense in X, G has no fixed point in X. Moreover, G preserves every Φ-orbit. Thus, Theorem 1.1 is proved. $\square$

6. An alternative proof of Witte's theorem

Let G be a finitely generated left-orderable group, and X a compact space equipped with a free action of $\mathbf{R}$ and an action of G on it, as described by Theorem 1.1. Suppose that G is amenable. Then there exists a probability measure m on X which is invariant by G. Consider the conditional measures of m along the orbits of the translation flow. These are Radon measures on m-almost every Φ-orbit, well-defined up to multiplication by a positive constant. We denote by m_l this Radon measure on a Φ-orbit l. More precisely, in a flow box $[0,1]\times\Lambda$ where the flow Φ is given by the formula $\Phi_s(t,l)=(t+s,l)$, we desintegrate the measure m as

$$m(dt,dl)=m_l(dt)\ \overline{m}(dl)$$

where $\overline{m}$ is the image of m under the projection $[0,1]\times\Lambda\to\Lambda$ and the measures m_l are measures on the unit interval, see [9, Section 3]. The measures m_l depend on the flow box; however, they are well defined up to a positive constant on almost every orbit.

Because G preserves m, and is countable, for m-almost every Φ-orbit l of X and every element g of G, the measure g_*m_l is a positive constant times m_l:

$$g_*m_l=c_l(g)m_l\,,\quad \text{where}\quad c_l(g)>0\,.$$

If m_l is not preserved [1]) by G, the map $g\mapsto\log c_l$ is a non trivial morphism from G to $\mathbf{R}$, and hence we deduce the existence of a non trivial morphism to $\mathbf{Z}$. If not, m_l is preserved by G, and either m_l is atomic or not. In the first case, m_l has an atom whose orbit is discrete, and the group acts as a translation on it, giving rise to a non trivial morphism to the integers. In the second case, the action is semi-conjugated to an action by translations, which defines a non trivial morphism to the real numbers, and hence to the integers. In all cases, there is a non trivial morphism from G to $\mathbf{Z}$.

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(Reçu le 23 février 2011 ; version révisée reçue le 16 mai 2012)

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