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ORBIFOLDS AS STACKS?

by Eugene LERMAN

ABSTRACT. The first goal of this survey paper is to argue that if orbifolds are groupoids, then the collection of orbifolds and their maps has to be thought of as a 2-category. Compare this with the classical definition of Satake and Thurston of orbifolds as a 1-category of sets with extra structure and/or with the "modern" definition of orbifolds as proper étale Lie groupoids up to Morita equivalence.

The second goal is to describe two complementary ways of thinking of orbifolds as a 2-category: (1) the weak 2-category of foliation Lie groupoids, bibundles and equivariant maps between bibundles and (2) the strict 2-category of Deligne-Mumford stacks over the category of smooth manifolds.

1. INTRODUCTION

Orbifolds are supposed to be generalizations of manifolds. While manifolds are modeled by open balls in the Euclidean spaces, orbifolds are supposed to be modeled by quotients of open balls by linear actions of finite groups. Orbifolds were first defined in the 1950's by Satake [25, 26]. The original definition had a number of problems. The chief problem was the notion of maps of orbifolds: different papers of Satake had different definitions of maps and it was never clear if maps could be composed. Additionally:

1. The group actions were required to be effective (and there was a spurious condition on the codimension of the set of singular points). The requirement of effectiveness created a host of problems: there were problems in the definition of suborbifolds and of vector (orbi-)bundles over the orbifolds. A quotient of a manifold by a proper locally free action of a Lie group was not necessarily an orbifold by this definition.

2. There were problems with pullbacks of vector (orbi-)bundles — they were not defined for all maps.

Over the years various patches to the definition have been proposed. See, for example, Chen and Ruan [6], Haefliger [8, 9], Moerdijk [20], Moerdijk and Pronk [22]. In particular Moerdijk's paper on orbifolds as groupoids has been quite influential among symplectic topologists. At about the same time the notion that orbifolds are Deligne-Mumford/geometric stacks over the category of manifolds started to be mooted.

There are two points to this paper.

1. If one thinks of orbifolds as groupoids then orbifolds have to be treated as a 2-*category*: it is not enough to have maps between groupoids, one also has to have maps between maps. This point is not new; I have learned it from [13]. Unfortunately it has not been widely accepted, and it bears repetition.

2. There are two complementary ways of thinking of orbifolds as a 2-category. One way uses bibundles as maps. The other requires embedding Lie groupoids into the 2-category of stacks. Since stacks and the related mental habits are not familiar to many differential geometers I thought it would be useful to explain what stacks are. While there are several such introductions already available [19, 3, 12], I feel there is room for one more, especially for the one with the emphasis on "why".

I will now outline the argument for thinking of orbifolds as a 2-category (the possibly unfamiliar terms are defined in subsequent sections). The simplest solution to all of the original problems with Satake's definition is to start afresh. We cannot glue together group actions, but we can glue together action groupoids. Given an action of a finite group, the corresponding groupoid is étale and proper. This leads one to think of a C^∞ orbifold (or, at least, of an orbifold atlas) as a proper étale Lie groupoid. The orbit spaces of such groupoids are Hausdorff, and locally these groupoids look like action groupoids for linear actions of finite groups. Since a locally free proper action of a Lie group on a manifold should give rise to an orbifold, limiting oneself to étale groupoids is too restrictive. A better class of groupoids consists of Lie groupoid equivalent to proper étale groupoids. These are known as *foliation groupoids*.

If orbifolds are Lie groupoids, what are maps? Since many geometric structures (metrics, forms, vector fields, etc.) are sections of vector bundles, hence maps, one cannot honestly do differential geometry on orbifolds without addressing this question first.

Since groupoids are categories, their morphisms are functors. But our groupoids are smooth, so we should require that the functors are smooth too (as maps on objects and arrows). One quickly discovers that these morphisms are not enough. The problem is that there are many smooth functors that are equivalences of categories and that have no smooth inverses. So, at the very least, we need to formally invert these smooth equivalences. But groupoids and functors are not just a category; there are also natural transformations between functors. This feature is dangerous to ignore for two reasons. First of all, it is "widely known" that the space of maps between two orbifolds is some sort of an infinite dimensional orbifold. So if one takes the point of view that orbifolds are groupoids, then the space of maps between two orbifolds should be a groupoid and not just a set. The most natural groupoid structure comes from natural transformations between functors. There are other ways to give the space of maps between two orbifolds the structure of a groupoid, but I do not find these approaches convincing.

The second reason has to do with gluing maps. Differential geometers glue maps all the time. For example, when we integrate a vector field on a manifold, we know that a flow exists locally by an existence theorem for a system of ODEs. We then glue together these local flows to get a global flow. However, if we take the category of Lie groupoids, identify isomorphic functors and then invert the equivalences (technically speaking we *localize at the equivalences*), the morphisms in the resulting category will not be uniquely determined by their restrictions to elements of an open cover. We will show that *any* localization of the category of groupoid will have this feature, regardless of how it is constructed! See Lemma 3.41 below.

Having criticized the classical and "modern" approaches to orbifolds, I feel compelled to be constructive. I will describe two *geometrically* compelling and complementary ways to localize Lie groupoids at equivalences as a 2-category. These are:

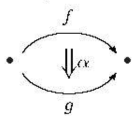
1. replace functors by bibundles and natural transformations by equivariant maps of bibundles or
2. embed groupoids into the 2-category of stacks.

ACKNOWLEDGEMENTS. I have benefited from a number of papers on stacks in algebraic and differential geometry: Metzler [19], Behrend-Xu [3], Vistoli [29], Behrend *et al.* [2] and Heinloth [12] to name a few. Many definitions and arguments are borrowed from these papers. There are now several books on Lie groupoids. I have mostly cribbed from Moerdijk-Mrčun [21]. The paper by Laurent-Gengoux *et al.* [15] has also been very useful. I have benefited from conversations with my colleagues. In particular I would like to thank Matthew Ando, Anton Malkin, Tom Nevins and Charles Rezk.

1.1 Conventions and notation

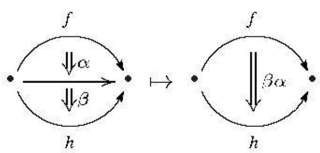
We assume that the reader is familiar with the notions of categories, functors and natural transformations. Given a category C we denote its collection of objects by C_0; C_0 is not necessarily a set. The reader may pretend that we are working in the framework of Von Neumann–Bernays–Gödel (NBG) axioms for set theory. But for all practical purposes set theoretic questions, such as questions of *size* will be swept under the rug, i.e., ignored. We denote the class of morphisms of a category C by C_1. Given two objects $x, y \in C_0$ we denote the collection of all morphisms from x to y by $Hom_C(x, y)$ or by $C(x, y)$, depending on what is less cumbersome.

1.2 A note on 2-categories

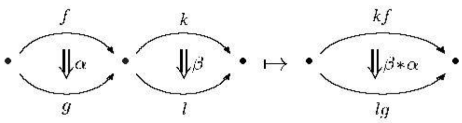
We will informally use the notions of *strict* and *weak* 2*-categories*. For formal definitions the reader may wish to consult Borceux [4]. Roughly speaking a *strict* 2*-category* A is an ordinary category A that in addition to objects and morphisms has morphisms between morphisms, which are usually called 2*-morphisms* (to distinguish them from ordinary morphisms which are called 1*-morphisms*). We will also refer to 1-morphisms as (1-)*arrows*. The prototypical example is Cat, the category of categories. The objects of Cat are categories, 1-morphisms (1-arrows) are functors and 2-morphisms (2-arrows) are natural transformations between functors. We write $\alpha\colon f \Rightarrow g$ and 

, when there is a 2-morphism α from a 1-morphism f to a 1-morphism g.

Natural transformations can be composed in two different ways:

vertically: 

and

horizontally: 

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The two compositions are related by a 4-*interchange law* that we will not describe. Axiomatizing this structure gives rise to the notion of a *strict* 2-*category*.

Note that for every 1-arrow f in a 2-category we have a 2-arrow $id_f\colon f \Rightarrow f$. A 2-arrow is *invertible* if it is invertible with respect to the vertical composition. So it makes sense to talk about two 1-arrows in a 2-category being isomorphic.

Weak 2-*categories* (also known as *bicategories*) also have objects, 1-arrows and 2-arrows, but the composition of 1-arrows is no longer required to be strictly associative. Rather, given a triple of composable 1-arrows f, g, h one requires that $(fg)h$ is isomorphic to $f(gh)$. That is, one requires that there is an invertible 2-arrow $\alpha\colon (fg)h \Rightarrow f(gh)$. As in a strict 2-category it makes sense to talk about two 1-arrows in a weak 2-category being isomorphic (the vertical composition of 2-arrows is still strictly associative). If $f\colon x \to y$ is an arrow in a weak 2-category for which there is an arrow $g\colon y \to x$ with fg isomorphic to 1_y and gf isomorphic to 1_x we say that f is *weakly invertible* and that g is a weak inverse of f.

2. Orbifolds as groupoids

In this section we define *proper étale Lie groupoids*. A comprehensive reference on Lie groupoids is [18]. Proposition 2.23 below is the main justification for thinking of proper étale groupoids as orbifolds (or orbifold atlases): locally they look like finite groups acting linearly on a disk in some Euclidean space. Proper étale Lie groupoids are not the only groupoids we may think of as orbifolds. For example, a locally free proper action of a Lie group on a manifold defines a groupoid that is also, in some sense, an orbifold. We will explain in what sense such an action groupoid is equivalent to an étale groupoid. This requires the notion of a pullback of a groupoid along a map. We start by recalling the definition of a fiber product of sets.

Definition 2.1. Let $f\colon X \to Z$ and $g\colon Y \to Z$ be two maps of sets. The *fiber product* of f and g, or more sloppily the *fiber product of* X *and* Y *over* Z is the set

$$X \times_{f,Z,g} Y \equiv X \times_Z Y := \{(x, y) \mid f(x) = g(y)\}.$$

Remark 2.2. If $f\colon X \to Z$ and $g\colon Y \to Z$ are continuous maps between topological spaces then the fiber product $X \times_Z Y$ is a subset of $X \times Y$ and hence is naturally a topological space (it is closed if Z is Hausdorff). If $f\colon X \to Z$

and $g\colon Y \to Z$ are smooth maps between manifolds, then the fiber product is *not* in general a manifold. It *is* a manifold if the map $(f, g)\colon X \times Y \to Z \times Z$ is transverse to the diagonal Δ_Z.

DEFINITION 2.3. A *groupoid* is a small category (objects form a set) where all morphisms are invertible.

Thus a groupoid G consists of the set of objects (0-morphisms) G_0 , the set of arrows (1-morphisms) G_1 together with five structure maps: $s\colon G_1 \to G_0$ (source), $t\colon G_1 \to G_0$ (target), $u\colon G_0 \to G_1$ (unit), m (multiplication) and $inv\colon G_1 \to G_1$ (inverse) satisfying the appropriate identities. We think of an element $\gamma \in G_1$ as an arrow from its source x to its target y: $x \xrightarrow{\gamma} y$. Thus $s(\gamma) = x$ and $t(\gamma) = y$. For each object $x \in G_0$ we have the identity arrow $x \xrightarrow{1_x} x$, and $u(x) = 1_x$. Note that $s(u(x)) = t(u(x)) = x$. Arrows with the matching source and target can be composed: $x \bullet \xrightarrow{\gamma} \bullet y \xrightarrow{\sigma} \bullet z$, with $x \bullet \xrightarrow{\sigma\circ\gamma} \bullet z$.

Therefore the multiplication map m is defined on the fiber product

$$G_1 \times_{G_0} G_1 \equiv G_1 \times_{s, G_0, t} G_1 := \{(\sigma, \gamma) \in G_1 \times G_1 \mid s(\sigma) = t(\gamma)\};$$

$$m\colon G_1 \times_{G_0} G_1 \to G_1, \quad m(x \bullet \xrightarrow{\gamma} \bullet y \xrightarrow{\sigma} \bullet z) = x \bullet \xrightarrow{\sigma\circ\gamma} \bullet z.$$

Since all 1-arrows are invertible by assumption (G is a groupoid) there is the inversion map

$$inv\colon G_1 \to G_1, \quad inv(x \bullet \xrightarrow{\gamma} \bullet y) = x \bullet \xleftarrow{\gamma^{-1}} \bullet y.$$

The five maps are subject to identities, some of which we already mentioned.

NOTATION 2.4. We will write $G = \{G_1 \rightrightarrows G_0\}$ when we want to emphasize that a groupoid G has the source and target maps.

EXAMPLE 2.5. A group is a groupoid with one object.

EXAMPLE 2.6 (Sets are groupoids). Let M be a set, $G_0 = G_1 = M$, $s, t = id\colon M \to M$, $inv = id$ etc. Then $\{M \rightrightarrows M\}$ is a groupoid with all the arrows being the identity arrows.

EXAMPLE 2.7 (Action groupoid). A left action of a group Γ on a set X defines an *action groupoid* as follows: we think of a pair $(g, x) \in \Gamma \times X$ as an arrow from x to $g \cdot x$, where $\Gamma \times X \ni (g, x) \mapsto g \cdot x \in X$ denotes the action.

Formally $G_1 = \Gamma \times X$, $G_0 = X$, $s(g,x) = x$, $t(g,x) = g \cdot x$, $u(x) = (1,x)$, where $1 \in \Gamma$ is the identity element, $inv(g,x) = (g^{-1}, g \cdot x)$ and the multiplication is given by

$$(h, g \cdot x)(g,x) = (hg,x)\,.$$

DEFINITION 2.8 (Orbit space/Coarse moduli space). Let G be a groupoid. Then

$$\sim := \{(x,y) \in G_0 \times G_0 \mid \text{there is } \gamma \in G_1 \text{ with } x \xrightarrow{\gamma} y\}$$

is an equivalence relation on G_0. We denote the quotient $G_0/{\sim}$ by G_0/G_1 and think of the projection $G_0 \to G_0/G_1$ as the orbit map. We will refer to the set G_0/G_1 as the *orbit space* of the groupoid G. Note that if $G = \{\Gamma \times X \rightrightarrows X\}$ is an action groupoid, then $G_0/G_1 = X/\Gamma$. The orbit space G_0/G_1 is also referred to as the *coarse moduli space* of the groupoid G.

DEFINITION 2.9 (Maps/Morphisms of groupoids). A *map/morphism* ϕ from a groupoid G to a groupoid H is a *functor*. That is, there is a map $\phi_0\colon G_0 \to H_0$ on objects, a map $\phi_1\colon G_1 \to H_1$ on arrows that makes the diagram

$$\begin{array}{ccc} G_1 & \xrightarrow{\phi_1} & H_1 \\ \big\downarrow{\scriptstyle (s,t)} & & \big\downarrow{\scriptstyle (s,t)} \\ G_0 \times G_0 & \xrightarrow{(\phi_0,\phi_0)} & H_0 \times H_0 \end{array}$$

commute and preserves the (partial) multiplication and the inverse maps.

REMARK 2.10. Note that $\phi_0 = s \circ \phi_1 \circ u$, where $u\colon G_0 \to G_1$ is the unit map. For this reason we will not distinguish between a functor $\phi\colon G \to H$ and the corresponding map on the set of arrows $\phi_1\colon G_1 \to H_1$.

Next we define Lie groupoids. Roughly speaking a *Lie groupoid* is a groupoid in the category of manifolds. Thus the spaces of arrows and objects are manifolds and all the structure maps s, t, u, m, inv are smooth. Additionally one usually assumes that the spaces of objects and arrows are Hausdorff and paracompact (except in foliation theory where this assumption is usually dropped).

There is a small problem with the above definition: in general there is no reason for the fiber product $G_1 \times_{G_0} G_1$ of a Lie groupoid $\{G_1 \rightrightarrows G_0\}$ to be a manifold. Therefore one cannot talk about the multiplication being smooth. This problem is corrected by assuming that the source and target maps $s, t\colon G_1 \to G_0$ are submersions. We therefore have:

DEFINITION 2.11. A *Lie groupoid* is a groupoid G such that the set G_0 of objects and the set G_1 of arrows are (Hausdorff paracompact) manifolds, the source and target maps $s, t\colon G_1 \to G_0$ are submersions and all the rest of the structure maps are smooth as well.

REMARK 2.12. Since $inv^2 = id$, inv is a diffeomorphism. Since $s \circ inv = t$, the source map s is a submersion if and only if the target map t is a submersion.

REMARK 2.13. The coarse moduli space G_0/G_1 of a Lie groupoid G is naturally a topological space.

EXAMPLE 2.14 (Manifolds as Lie groupoids). Let M be a manifold, $G_0 = G_1 = M$, $s, t = id\colon M \to M$, $inv = id$ etc. Then $\{M \rightrightarrows M\}$ is a Lie groupoid with all the arrows being the identity arrows.

EXAMPLE 2.15 (Action Lie groupoid). Let Γ be a Lie group acting smoothly on a manifold M. Then the action groupoid $\Gamma \times M \rightrightarrows M$ is a Lie groupoid.

EXAMPLE 2.16 (Cover Lie groupoid). Let M be a manifold with an open cover $\{U_\alpha\}$. Let $\mathcal{U} = \bigsqcup U_\alpha$ be the disjoint union of the sets of the cover and $\bigsqcup_{\alpha,\beta} U_\alpha \cap U_\beta$ the disjoint union of double intersections. More formally

$$\bigsqcup_{\alpha,\beta} U_\alpha \cap U_\beta = \mathcal{U} \times_M \mathcal{U},$$

where $\mathcal{U} = \bigsqcup U_i \to M$ is the evident map. We define $s\colon U_\alpha \cap U_\beta \to U_\alpha$ and $t\colon U_\alpha \cap U_\beta \to U_\beta$ to be the inclusions. Or, more formally, we have two projection maps $s, t\colon \mathcal{U} \times_M \mathcal{U} \to \mathcal{U}$. We think of a point $x \in U_\alpha \cap U_\beta$ as an arrow from $x \in U_\alpha$ to $x \in U_\beta$. One can check that $\mathcal{U} \times_M \mathcal{U} \rightrightarrows \mathcal{U}$ is a Lie groupoid. Alternatively it is the pull-back of the groupoid $M \rightrightarrows M$ by the "inclusion" map $\mathcal{U} \to M$ (see Definition 2.25 below).

REMARK 2.17. Occasionally it will be convenient for us to think of *a cover of a manifold M as a surjective local diffeomorphism $\phi\colon \mathcal{U} \to M$*. Here is a justification: If $\{U_i\}$ is an open cover of M then $\mathcal{U} = \bigsqcup U_i$ and

$\phi\colon \mathcal{U} \to M$ is the "inclusion". Conversely, if $\phi\colon \mathcal{U} \to M$ is a surjective local diffeomorphism then there is an open cover $\{V_i\}$ of $\mathcal{U}$ so that $\phi|_{V_i}\colon V_i \to M$ is an open embedding. Moreover the "inclusion" $\bigsqcup \phi(V_i) \to M$ "factors" through $\phi\colon \mathcal{U} \to M$. So any cover in the traditional sense is a cover in the generalized sense. And any cover in the new sense gives rise to a cover in the traditional sense.

DEFINITION 2.18 (Proper groupoid). A Lie groupoid G is *proper* if the map $(s,t)\colon G_1 \to G_0 \times G_0$, which sends an arrow to the pair of points (source, target), is proper.

DEFINITION 2.19 (Étale groupoid). A Lie groupoid G is *étale* if the source and target maps $s,t\colon G_1 \to G_0$ are local diffeomorphisms.

EXAMPLE 2.20. An action groupoid for an action of a finite group is étale and proper. A cover groupoid $\mathcal{U} \times_M \mathcal{U} \rightrightarrows \mathcal{U}$ is étale and proper. An action groupoid $\Gamma \times M \rightrightarrows M$ is proper if and only if the action is proper (by definition of a proper action).

DEFINITION 2.21 (Restriction of a Lie groupoid). Let G be a Lie groupoid and $U \subset G_0$ an open set. Then $s^{-1}(U) \cap t^{-1}(U)$ is an open submanifold of G_1 closed under multiplication and taking inverses, hence forms the space of arrows of a Lie groupoid whose space of objects is U. We call this groupoid *the restriction of* G *to* U and denote it by $G|_U$.

REMARK 2.22. We will see that the restriction is a special case of a pull-back construction defined below (Definition 2.25).

We can now state the proposition that justifies thinking of proper étale Lie groupoids as orbifolds. It asserts that any such groupoid looks locally like a linear action of a finite group on an open ball in some $\mathbf{R}^n$. More precisely, we have:

PROPOSITION 2.23. *Let G be a proper étale Lie groupoid. Then for any point $x \in G_0$ there is an open neighborhood $U \subset G_0$ so that the restriction $G|_U$ is isomorphic to an action groupoid $\Lambda \times U \rightrightarrows U$ where Λ is a finite group. That is, there is an invertible functor $f\colon G|_U \to \{\Lambda \times U \rightrightarrows U\}$. Moreover, we may take U to be an open ball in some Euclidean space centered at the origin and the action of Λ to be linear.*

Proof. This is a special (easy) case of Theorem 2.3 in [30]. For proper étale *effective* groupoids the result was proved earlier in [22].

REMARK 2.24. One occasionally runs into an idea that a proper étale Lie groupoid G is an atlas on its coarse moduli space G_0/G_1. Indeed, there is an analogy with atlases of manifolds: if M is a manifold and $\{U_i\}$ is a cover by coordinate charts then M is the coarse moduli space of the cover groupoid $\{\mathcal{U} \times_M \mathcal{U} \rightrightarrows \mathcal{U}\}$, where $\mathcal{U} = \bigsqcup U_i$. This idea leads to endless trouble.

Next I would like to explain how to obtain a proper étale Lie groupoid from a proper and locally free action of a Lie group on a manifold.

DEFINITION 2.25. The *pull-back* of a groupoid G by a map $f: N \to G_0$ is the groupoid f^*G with the space of objects N, the space of arrows

$$\begin{aligned}(f^*G)_1 &:= (N \times N) \times_{G_0 \times G_0} G_1 \\ &= \{(x, y, g) \in N \times N \times G_1 \mid s(g) = f(x), t(g) = f(y)\} \\ &= \{(x, y, g) \in N \times N \times G_1 \mid f(x) \bullet \xrightarrow{\;g\;} \bullet f(y)\},\end{aligned}$$

the source and target maps $s(x, y, g) = x$, $t(x, y, g) = y$ and multiplication given by $(y, z, h)(x, y, g) = (x, z, hg)$. Note that the maps $f_0 = f: N \to G_0$ and $f_1: f^*G_1 \to G_1$, $f_1(x, y, g) = g$, form a functor $\widetilde{f}: f^*G \to G$.

It is not always true that the pull-back of a Lie groupoid by a smooth map is a Lie groupoid: we need the space of arrows $(f^*G)_1$ to be a manifold and the source and target maps to be submersions. The following condition turns out to be sufficient.

PROPOSITION 2.26. *Let G be a Lie groupoid and $f: N \to G_0$ a smooth map. Consider the fiber product*

$$N \times_{f, G_0, s} G_1 = \{(x, g) \in N \times G_1 \mid f(x) = s(g)\}.$$

If the map $N \times_{f, G_0, s} G_1 \to G_0$, $(x, g) \mapsto t(g)$ is a submersion, then the pullback groupoid f^*G is a Lie groupoid and the functor $\widetilde{f}: f^*G \to G$ defined above is a smooth functor.

Proof. See, for example, [21], pp. 121–122.

REMARK 2.27. If the map $N \times_{f, G_0, s} G_1 \to G_0$, $(x, g) \mapsto t(g)$ is a *surjective* submersion then the functor $\widetilde{f}: f^*G \to G$ is an equivalence of groupoids in the sense of Definition 3.5 below.

EXAMPLE 2.28. Let G be a Lie groupoid, U an open subset of the space of objects G_0. The inclusion map $\iota\colon U \hookrightarrow G_0$ satisfies the conditions of the proposition above and so the pull-back groupoid ι^*G is a Lie groupoid. It is not hard to see that ι^*G is the restriction $G|_U$ of G to U.

Next recall that an action of a Lie group Γ on a manifold M is *locally free* if for all points $x \in M$ the stabilizer group

$$\Gamma_x := \{g \in \Gamma \mid g \cdot x = x\}$$

is discrete. An action of Γ on M is *proper* if the map

$$\Gamma \times M \to M \times M\,, \qquad (g,x) \mapsto (x, g\cdot x)$$

is proper (this is exactly the condition for the action groupoid $\{\Gamma \times M \rightrightarrows M\}$ to be proper). A *slice* for an action of Γ on M at a point $x \in M$ is an embedded submanifold $\Sigma \subset M$ with $x \in \Sigma$ so that

1. Σ is preserved by the action of Γ_x: for all $s \in \Sigma$ and $g \in \Gamma_x$, we have $g \cdot s \in \Sigma$.
2. The set $\Gamma \cdot \Sigma := \{g \cdot s \mid g \in \Gamma, s \in \Sigma\}$ is open in M.
3. The map $\Gamma \times \Sigma \to \Gamma \cdot \Sigma \subset M$, $(g,s) \mapsto g \cdot s$ descends to a diffeomorphism $(\Gamma \times \Sigma)/\Gamma_x \to \Gamma \cdot \Sigma$ (here Γ_x acts on $\Gamma \times \Sigma$ by $a \cdot (g,s) = (ga^{-1}, a \cdot s)$).

Thus, for every point $s \in \Sigma$ the orbit $\Gamma \cdot s$ intersects the slice Σ in a unique Γ_x orbit. A classical theorem of Palais asserts that a proper action of a Lie group Γ on a manifold M has a slice at every point of M.

With these preliminaries out of the way, consider a proper locally free action of a Lie group Γ on a manifold M. Pick a collection of slices $\{\Sigma_\alpha\}_{\alpha \in A}$ so that every Γ orbit intersects a point in one of these slices: $\Gamma \cdot \bigcup \Sigma_\alpha = M$. Let $\mathcal{U} = \bigsqcup \Sigma_\alpha$ and $f\colon \mathcal{U} \to M$ be the "inclusion" map: for each $x \in \Sigma_\alpha$, $f(x) = x \in M$. The fact that Σ_α's are slices implies (perhaps after a moment of thought) that Proposition 2.26 applies with $G = \{\Gamma \times M \rightrightarrows M\}$ and $f\colon \mathcal{U} \to M$. We get a pullback Lie groupoid f^*G, which is, by construction, étale. By Remark 2.27 the functor $\widetilde{f} : f^*\{\Gamma \times M \rightrightarrows M\} \to \{\Gamma \times M \rightrightarrows M\}$ is an equivalence of groupoids. Note that $\widetilde{f}$ is not surjective and may not be injective either. In particular, it is not invertible. Reasons for thinking of it as some sort of an isomorphism are explained in the next section.

Note that if we pull G back further by the inclusion $\Sigma_\beta \hookrightarrow \sqcup \Sigma_\alpha$, we get an action groupoid of the form $\Lambda \times \Sigma_\beta \rightrightarrows \Sigma_\beta$ where Λ is a discrete compact group, that is, a finite group.

EXAMPLE 2.29. An industrious reader may wish to work out the example of the action of $\mathbf{C}^{\times} = \{z \in \mathbf{C} \mid z \neq 0\}$ on $\mathbf{C}^2 \smallsetminus \{0\}$ given by $\lambda \cdot (z_1, z_2) = (\lambda^p z_1, \lambda^q z_2)$ for a pair of positive integers (p, q). The reader will only need two slices: $\mathbf{C} \times \{1\}, \{1\} \times \mathbf{C} \subset \mathbf{C}^2 \smallsetminus \{0\}$.

3. LOCALIZATION AND ITS DISCONTENTS

At this point in our discussion of orbifolds we reviewed the reasons for thinking of smooth orbifolds as Lie groupoids. If orbifolds are Lie groupoids then their maps should be smooth functors. It will turn out that many such maps that should be invertible are not. We therefore need to enlarge our supply of available maps. We start by recalling various notions of two categories being "the same". More precisely recall that there are two equivalent notions of equivalence of categories.

Recall our notation: if A is a category, then A_0 denotes its collection of objects and $A(a, a')$ denotes the collection of arrows between two objects $a, a' \in A_0$.

DEFINITION 3.1. A functor $F\colon A \to B$ is *full* if for any $a, a' \in A_0$ the map $F\colon A(a, a') \to B(F(a), F(a'))$ is onto. It is *faithful* if $F\colon A(a, a') \to B(F(a), F(a'))$ is injective. A functor that is full and faithful is *fully faithful*.

A functor $F\colon A \to B$ is *essentially surjective* if for any $b \in B_0$ there is $a \in A_0$ and an invertible arrow $\gamma \in B_1$ from $F(a)$ to b.

EXAMPLE 3.2. Let Vect denote the category of finite-dimensional vector spaces over $\mathbf{R}$ and linear maps. Let Mat be the category of real matrices. That is, the objects of Mat are non-negative integers. A morphism from n to m in Mat is an $n \times m$ real matrix. The functor Mat $\to$ Vect which sends n to $\mathbf{R}^n$ and a matrix to the corresponding linear map is fully faithful and essentially surjective.

The following theorem is a basic result in category theory.

THEOREM 3.3. *A functor $F\colon A \to B$ is fully faithful and essentially surjective if and only if there is a functor $G\colon B \to A$ with two natural isomorphisms (invertible natural transformations) $\alpha\colon FG \Rightarrow id_A$ and $\beta\colon GF \Rightarrow id_B$.*

DEFINITION 3.4. A functor $F\colon A \to B$ satisfying one of the two equivalent conditions of the theorem above is called an *equivalence of categories*. We think of the functor $G\colon B \to A$ above as a (weak) inverse of F.

There is no analogous theorem for C^∞ functors between Lie groupoids: there are many fully faithful essentially surjective smooth functors between Lie groupoids with no continuous (weak) inverses. The simplest examples come from cover groupoids. If $\mathcal{U} \times_M \mathcal{U} \rightrightarrows \mathcal{U}$ is a cover groupoid associated to a cover $\mathcal{U} \to M$ of a manifold M then the natural functor $\{\mathcal{U} \times_M \mathcal{U} \rightrightarrows \mathcal{U}\} \to \{M \to M\}$ is fully faithful and essentially surjective and has no continuous weak inverse (unless one of the connected components of $\mathcal{U}$ is all of M).

Additionally, not every fully faithful and essentially surjective smooth functor between two Lie groupoids should be considered an equivalence of Lie groupoids (just like not every smooth bijection between manifolds is a diffeomorphism). The accepted definition is (see for example [21]):

DEFINITION 3.5. A smooth functor $F\colon G \to H$ from a Lie groupoid G to a Lie groupoid H is an *equivalence* of Lie groupoids if

1. the induced map
$$G_1 \to (G_0 \times G_0) \times_{(F,F),H_0\times H_0,(s,t)} H_1\,, \quad \gamma \mapsto (s(\gamma), t(\gamma), F(\gamma))$$
is a diffeomorphism;
2. the map $G_0 \times_{F,H_0,t} H_1 \to H_0$, $(x, h) \mapsto s(h)$ is a surjective submersion.

REMARK 3.6. The first condition implies that F is fully faithful and the second that it is essentially surjective.

REMARK 3.7. In literature this notion of equivalence variously goes by the names of "essential" and "weak" equivalences to distinguish it from "strict" equivalence: a smooth functor of Lie groupoids $F\colon G \to H$ is a *strict equivalence* if there is a smooth functor $L\colon H \to G$ with two smooth natural isomorphisms (invertible natural transformations) $\alpha\colon FL \Rightarrow id_G$ and $\beta\colon LF \Rightarrow id_H$. We will not use the notion of strict equivalence of Lie groupoids in this paper.

EXAMPLE 3.8. As we pointed out above, if $f\colon \mathcal{U} \to M$ is a surjective local diffeomorphism then the functor $\widetilde{f}\colon \{\mathcal{U} \times_M \mathcal{U} \rightrightarrows \mathcal{U}\} \to \{M \rightrightarrows M\}$ is an equivalence of Lie groupoids in the sense of Definition 3.5.

EXAMPLE 3.9. As we have seen in the previous section, if a Lie group Γ acts locally freely and properly on a manifold M, $\mathcal{U} = \bigsqcup \Sigma_\alpha$ is a collection of slices with $\Gamma \cdot \bigcup \Sigma_\alpha = M$ and $f^*\{\Gamma \times M \rightrightarrows M\}$ is the pullback of the action groupoid along $f\colon \mathcal{U} \to M$, then the functor $\tilde{f}\colon f^*\{\Gamma \times M \rightrightarrows M\} \to \{\Gamma \times M \rightrightarrows M\}$ is an equivalence of Lie groupoids. This is a reason for thinking of the action groupoid $\{\Gamma \times M \rightrightarrows M\}$ as an orbifold.

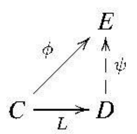
REMARK 3.10. We cannot fully justify the correctness of Definition 3.5. And indeed the reasons for it being "correct" are somewhat circular. If one embeds the category of Lie groupoids either into the Hilsum-Skandalis category of groupoids and generalized maps (see below) or into stacks (stacks are defined in the next section), the functors that become invertible are precisely the equivalences and nothing else! But why define the generalized maps or to embed groupoids into stacks? To make equivalences invertible, of course!

Let us recapitulate where we are. An orbifold, at this point, should be a Lie groupoid equivalent to a proper étale Lie groupoid. If this is the case, what should be the maps between orbifolds? Smooth functors have to be maps in our category of orbifolds, but we need a more general notion of a map to make equivalences invertible. There is a standard construction in category theory called *localization* that allows one to formally invert a class of morphisms. This is the subject of the next subsection.

3.1 LOCALIZATION OF A CATEGORY

Let C be a category and W a subclass of morphisms of C (i.e. $W \subset C_1$). A *localization of C with respect to W* is a category D and a functor $L\colon C \to D$ with the following properties:

1. For any $w \in W$, $L(w)$ is invertible in D.
2. If $\phi\colon C \to E$ is a functor with the property that $\phi(w)$ is invertible in E for all $w \in W$ then there exists a unique map $\psi\colon D \to E$ so that $\psi \circ L = \phi$, that is,



commutes.

REMARK 3.11. The second condition is there to make sure, among other things, that the localization D is not the trivial category with one object and one morphism.

The next two results are old and well known. The standard reference is Gabriel-Zisman [7]. We include them for completeness.

LEMMA 3.12. *If a localization $L\colon C \to D$ of C with respect to $W \subset C_1$ exists, then it is unique.*

Proof. This is a simple consequence of the universal property of the localization. If $L'\colon C \to D'$ is another functor satisfying the two conditions above then there are functors $\psi\colon D \to D'$ and $\tau\colon D' \to D$ so that $\psi \circ L = L'$ and $\tau \circ L' = L$. Hence $\tau \circ \psi \circ L = L$. Since $id_D \circ L = L$ as well, $\tau \circ \psi = id_D$ by uniqueness. Similarly $\psi \circ \tau = id_{D'}$.

NOTATION 3.13. We may and will talk about *the* localization of C with respect to W and denote it by $\pi_W\colon C \to C[W^{-1}]$.

LEMMA 3.14. *The localization $\pi_W\colon C \to C[W^{-1}]$ of a category C with respect to a subclass W of arrows always exists.*

REMARK 3.15. Some readers may be bothered by the issues of *size*: the construction we are about to describe may produce a category where the collections of arrows between pairs of objects may be too big to be mere sets. Later on we will apply Lemma 3.14 to the category of Lie groupoids. There is a standard solution to this "problem". One applies the argument below only to small categories, whose collection of objects are sets. What about the category Gpoid of Lie groupoids which is not small (the collection of all Lie groupoids is a proper class)? There is a standard solution to this problem as well. Fix the disjoint union $\mathbf{E}$ of Euclidean spaces of all possible finite dimensions; $\mathbf{E} := \mathbf{R}^0 \sqcup \mathbf{R}^1 \sqcup \ldots \sqcup \mathbf{R}^n \sqcup \ldots$. Given a Lie groupoid G, we consider its space of objects G_0 as being embedded in its space G_1 of arrows. By the Whitney embedding theorem the manifold G_1 may be embedded in some Euclidean space $\mathbf{R}^n \subset \mathbf{E}$. It follows that the category Gpoid of Lie groupoids is equivalent to the category of $\mathbf{E}$Gpoid of Lie groupoids embedded in $\mathbf{E}$. Clearly $\mathbf{E}$Gpoid is small.

Proof of Lemma 3.14. The idea of the construction of $C[W^{-1}]$ is to keep the objects of C the same, to add to the arrows of C the formal inverses of the arrows in W and to divide out by the appropriate relations. Here are the details.

Recall that a *directed graph* $\mathcal{G}$ consists of a class of objects $\mathcal{G}_0$, a class of arrows $\mathcal{G}_1$ and two maps $s, t\colon \mathcal{G}_1 \to \mathcal{G}_0$ (source and target). In other words, for us a directed graph is a "category without compositions".

Given a category C and a subclass $W \subset C_1$, let W^{-1} be the class consisting of formal inverses of elements of W: for each $w \in W$ we have exactly one $w^{-1} \in W^{-1}$ and conversely. We then have a directed graph $\widetilde{C}[W^{-1}]$ with objects C_0 and arrows $C_1 \sqcup W^{-1}$.

A directed graph $\mathcal{G}$ generates a free category $F(\mathcal{G})$ on $\mathcal{G}$: the objects of $F(\mathcal{G})$ are objects $\mathcal{G}_0$ of $\mathcal{G}$ and arrows are *paths*. That is, an arrow in $F(\mathcal{G})_1$ from $x \in \mathcal{G}_0$ to $y \in \mathcal{G}_0$ is a finite sequence $(\gamma_n, \gamma_{n-1}, \ldots, \gamma_1)$ of elements of $\mathcal{G}_1$ with $s(\gamma_1) = x$ and $t(\gamma_n) = y$ (think: $x \xrightarrow{\gamma_1} \bullet \xrightarrow{\gamma_2} \ldots \xrightarrow{\gamma_n} y$). In addition, for every $x \in \mathcal{G}_0$ there is an empty path $()_x$ from x to x. Paths are composed by concatenation:

$$(\sigma_m, \ldots, \sigma_1)(\gamma_n, \ldots, \gamma_1) = (\sigma_m, \ldots, \sigma_1, \gamma_n, \ldots, \gamma_1).$$

We now construct $C[W^{-1}]$ from the category $F(\widetilde{C}[W^{-1}])$ by dividing out the arrows of $F(\widetilde{C}[W^{-1}])$ by an equivalence relation. Namely let $\sim$ be the equivalence relation generated by the following equations:

1. $()_x \sim (1_x)$ for all $x \in G_0$ (1_x is the identity arrow in C_1 for an object $x \in C_0$).
2. $(\sigma)(\gamma) \sim (\sigma\gamma)$ for any pair of composable arrows in C.
3. For any $x \xrightarrow{w} y \in W$, $(w, w^{-1}) \sim (1_y)$ and $(w^{-1}, w) \sim (1_x)$.

Thus we set $C[W^{-1}]_0 = C_0$ and $C[W^{-1}]_1 = F(\widetilde{C}[W^{-1}])_1/\sim$. We have the evident functor $\pi_W\colon C \to C[W^{-1}]$ induced by the inclusion of C into the directed graph $\widetilde{C}[W^{-1}]$.

It remains to check that $\pi_W\colon C \to C[W^{-1}]$ is a localization. Note first that for any $w \in W$ the arrow $\pi_W(w)$ is invertible in $C[W^{-1}]$ by construction of $C[W^{-1}]$. If $\phi\colon C \to E$ is any functor such that $\phi(w)$ is invertible for any $w \in W$, then ϕ induces a map $\widetilde{\phi}\colon \widetilde{C}[W^{-1}] \to E$: $\widetilde{\phi}(w^{-1}) := \phi(w)^{-1}$. This map drops down to a functor $\psi\colon C[W^{-1}] \to E$ with $\psi([w^{-1}]) = \phi(w)^{-1}$ for all $w \in W$ (here $[w^{-1}]$ denotes the equivalence class of the path (w^{-1}) in $F(\widetilde{C}[W^{-1}])$).

We now come to a subtle point. It may be tempting to apply the localization construction to the category Gpoid whose objects are Lie groupoids, morphisms are functors and the class W consists of equivalences, and then take the category of orbifolds to be the subcategory whose objects are isomorphic to proper étale Lie groupoids. Let us not rush. First of all, it will not at all be clear what the morphisms in $\mathsf{Gpoid}[W^{-1}]$ *are*, since they are defined by generators and relations. A more explicit construction would be more useful. Secondly, Gpoid is really a 2-category: there are also natural transformations between functors. We are thus confronted with three choices:

(1) Forget about natural transformations and localize; we get a category.

(2) Identify isomorphic functors and then localize. [1]) We get, perhaps, a smaller category.

(3) Localize Gpoid as a 2-category.

It is not obvious what the correct choice is. Option (1) is never used; perhaps it is not clear how to do it geometrically. Option (2) is fairly popular [11, 20, 16]. There are several equivalent geometric ways of carrying it out. We will review the one that uses isomorphism classes of bibundles. It is essentially due to Hilsum and Skandalis [14]. We will prove that it is, indeed, a localization. We will show that it has the unfortunate feature that maps from one orbifold to another do not form a sheaf: we cannot reconstruct a map from its restrictions to elements of an open cover. We will argue that this feature of option (2) is unavoidable: it does not depend on the way the localization is constructed. For this reason I think that choosing option (2) is a mistake.

There is another reason to be worried about option (2). It is "widely known" that the loop space of an orbifold is an orbifold. So if we take the point of view that an orbifold is a groupoid, the loop space of an orbifold should be a groupoid as well. But if we think of the category of orbifolds as a 1-category the space of arrows between two orbifolds is just a set and not a category in any natural sense. There are, apparently, ways to get around this problem [5, 11, 17], but I do not understand them.

There are many ways of carrying out option (3), localizing Gpoid as a 2-category. Let me single out three:

- Pronk constructed a calculus of fractions and localized Gpoid as a weak 2-category [23]. She also proved that the resulting 2-category is equivalent to the strict 2-category of geometric stacks over manifolds.
- One can embed the strict 2-category Gpoid into a weak 2-category Bi whose objects are Lie groupoids, 1-arrows are bibundles and 2-arrows

[1]) Two smooth functors $f, g \colon G \to H$ between two Lie groupoids are *isomorphic* if there is a natural transformation $\alpha \colon G_0 \to H_1$ from f to g. Note that since all arrows in H_1 are invertible, α is automatically a natural isomorphism.

equivariant diffeomorphisms between bibundles. We will explain the construction of Bi in the next subsection.

- One can embed Gpoid into the strict 2-category of stacks over manifolds. We will explain this in Section 4.

In the rest of the section we discuss option (2) in details. We start by introducing *bibundles* and reviewing some of their properties. Thereby we will introduce the weak 2-category Bi. Next we will discuss a concrete localization of the category of Lie groupoids due to Hilsum and Skandalis; it amounts to identifying isomorphic 1-arrows in Bi. We will then demonstrate that localizing groupoids as 1-category is problematic no matter which particular localization is being used.

3.2 Bibundles

DEFINITION 3.16. A *right action* of a Lie groupoid H on a manifold P consists of the following data:

1. a map $a\colon P \to H_0$ (anchor) and
2. a map
$$P \times_{a,H_0,t} H_1 \to P\,, \qquad (p,h) \mapsto p\cdot h\,, \text{ (the action)}$$
(as usual $t\colon H_1 \to H_0$ denotes the target map) such that
   (a) $a(p\cdot h) = s(h)$ for all $(p,h) \in P \times_{a,H_0,t} H_1$;
   (b) $(p\cdot h_1)\cdot h_2 = p\cdot(h_1 h_2)$ for all appropriate $p \in P$ and $h_1, h_2 \in H_1$;
   (c) $p \cdot 1_{a(p)} = p$ for all $p \in P$.

DEFINITION 3.17. A manifold P with a right action of a Lie groupoid H is a *principal (right) H-bundle over B* if there is a surjective submersion $\pi\colon P \to B$ so that

1. $\pi(p\cdot h) = \pi(p)$ for all $(p,h) \in P \times_{a,H_0,t} H_1$, that is, π is H-invariant; and
2. the map $P \times_{a,H_0,t} H_1 \to P \times_B P$, $(p,h) \mapsto (p, p\cdot h)$ is a diffeomorphism, that is, H acts freely and transitively on the fibers of $\pi\colon P \to B$.

EXAMPLE 3.18. For a Lie groupoid H the target map $t\colon H_1 \to H_0$ makes H_1 into a principal H-bundle with the action of H being the multiplication on the right (the anchor map is $s\colon H_1 \to H_0$). This bundle is sometimes called the *unit* principal H-bundle for the reasons that may become clear later.

Principal H-bundles pull back: if $\pi\colon P \to B$ is a principal H-bundle and $f\colon N \to B$ is a map then the pullback

$$f^*P := N \times_B P \to N$$

is a principal H-bundle as well. The action of H on f^*P is the restriction of the action of H on the product $N \times P$ to $N \times_B P \subset N \times P$. It is not difficult to check that $f^*P \to N$ is indeed a principal H-bundle.

LEMMA 3.19. *A principal H-bundle $\pi\colon P \to B$ has a global section if and only if P is isomorphic to a pull-back of the principal H-bundle $H_1 \xrightarrow{t} H_0$.*

Proof. Since $P \to B$ is H-principal we have a diffeomorphism

$$P \times_{a,H_0,t} H_1 \to P \times_B P\,, \qquad (p,h) \mapsto (p, p\cdot h)\,.$$

Its inverse is of the form $(p_1,p_2) \mapsto (p_1, d(p_1,p_2)) \in P \times_{a,H_0,t} H_1$, where $d(p_1,p_2)$ is the unique element h in H_1 so that $p_2 = p_1 \cdot h$. The map

$$d\colon P \times_B P \to H_1 \quad \text{(“the division map”)}$$

is smooth. Note that $d(p,p) = 1_{a(p)}$. If $\sigma\colon B \to P$ is a section of $\pi\colon P \to B$, define $\widetilde{f}\colon P \to H_1$ by

$$\widetilde{f}(p) = d(\sigma(\pi(p)), p)\,.$$

Then

$$p = \sigma(\pi(p)) \cdot \widetilde{f}(p) \quad \text{for all } p \in P\,.$$

Note that $\widetilde{f}$ is H-equivariant: observe that for all $(p,h) \in P \times_{H_0} H_1$

$$\sigma(\pi(p\cdot h)) \cdot \widetilde{f}(p) \cdot h = p \cdot h = \sigma(\pi(p\cdot h))\widetilde{f}(p\cdot h)\,.$$

Hence, since P is H-principal, $\widetilde{f}(p) \cdot h = \widetilde{f}(p \cdot h)$.
Consequently we get a map

$$\varphi\colon P \to f^*H_1\,, \qquad \varphi(p) = (\pi(p), \widetilde{f}(p))\,,$$

where $f\colon B \to H_0$ is defined by $f(b) = a(\sigma(b))$. The map φ has a smooth inverse $\psi\colon f^*H_1 \to P$, $\psi(b,h) = \sigma(b)\cdot h$, hence φ is a diffeomorphism.

Conversely, since $H_1 \xrightarrow{t} H_0$ has a global section, namely $u(x) = 1_x$ for $x \in H_0$, any pullback of $H_1 \xrightarrow{t} H_0$ has a global section as well.

REMARK 3.20. It is useful to think of principal groupoid bundles with global sections as trivial principal bundles.

The next result is technical and will not be needed until we start discussing stacks in the next section. It should be skipped on the first reading.

COROLLARY 3.21. *Let G be a Lie groupoid, $\xi_1 \to N$, $\xi_2 \to N$ two principal G-bundles with anchor maps a_1, a_2 respectively. Any G-equivariant map $\psi \colon \xi_1 \to \xi_2$ inducing the identity on N is a diffeomorphism.*

Proof. Note that $a_2 \circ \psi = a_1$; this is necessary for ψ to intertwine the two G-actions.

Since ψ is G-equivariant and induces the identity map on the base N, for any open set $U \subset N$, $\psi(\xi_1|_U) \subset \xi_2|_U$. Therefore it is enough to show that for any sufficiently small subset U of N the map $\psi \colon \xi_1|_U \to \xi_2|_U$ is a diffeomorphism. Since $\xi_1 \to N$ is a submersion, it has local sections. The two observations above allow us to assume that $\xi_1 \to N$ has a global section $\sigma \colon N \to \xi_1$.

We have seen in the proof of Lemma 3.19 that the section σ together with the "division map" $d \colon \xi_1 \times_N \xi_1 \to G_1$ defines a G-equivariant diffeomorphism

$$\widetilde{f} \colon \xi_1 \to f^*(G_1 \to G_0),$$

where $f = a_1 \circ \sigma$. Similarly the section $\psi \circ \sigma \colon N \to \xi_2$ together with the division map for ξ_2 defines a G-equivariant diffeomorphism

$$\widetilde{h} \colon \xi_2 \to h^*(G_1 \to G_0),$$

where $h = a_2 \circ (\psi \circ \sigma)$. Since $(a_2 \circ \psi) \circ \sigma = a_1 \circ \sigma$, we have $h = f$. By tracing through the definitions one sees that

$$\psi = (\widetilde{h})^{-1} \circ \widetilde{f}.$$

Hence ψ is a diffeomorphism.

DEFINITION 3.22. A *left* action of a Lie groupoid G on a manifold M is

1. A map $a_L = a \colon M \to G_0$ (the (left) anchor) and
2. a map

$$G_1 \times_{s,G_0,a} M \to M, \qquad (\gamma, x) \mapsto \gamma \cdot x, \text{ (the action)},$$

such that

(a) $1_{a(x)} \cdot x = x$ for all $x \in M$,
(b) $a(\gamma \cdot x) = t(\gamma)$ for all $(\gamma, x) \in G_1 \times_{s,G_0,a} M$,
(c) $\gamma_2 \cdot (\gamma_1 \cdot x) = (\gamma_2 \gamma_1) \cdot x$ for all appropriate $\gamma_1, \gamma_2 \in G_1$ and $x \in M$.

REMARK 3.23. Given a right action $a_R\colon M \to G_0$, $M \times_{G_0} G_1 \to M$ of a Lie groupoid G on a manifold M, we get a left action of G on M by composing it with the inversion map $G_1 \to G_1$, $\gamma \mapsto \gamma^{-1}$.

REMARK 3.24. If $f\colon G \to H$ is a smooth functor between two Lie groupoids then the pullback

$$f_0^* H_1 = G_0 \times_{f_0, H_0, t} H_1 \xrightarrow{\pi} G_0$$

of the principal H-bundle $H_1 \xrightarrow{t} H_0$ by $f_0\colon G_0 \to H_0$ is a principal H-bundle. In addition it has a left G-action:

$$G_1 \times_{s, G_0, \pi} (G_0 \times_{f_0, H_0, t} H_1) \to (G_0 \times_{f_0, H_0, t} H_1), \qquad (g, (x, h)) \mapsto (t(g), f_1(g)h).$$

This left G-action commutes with the right H-action.

The manifold $f_0^* H_1$ with the commuting actions of G and H constructed above is an example of a *bibundle* from G to H, which we presently define.

DEFINITION 3.25. Let G and H be two Lie groupoids. A *bibundle* from G to H is a manifold P together with two maps $a_L\colon P \to G_0$, $a_R\colon P \to H_0$ such that

1. there is a left action of G on P with respect to an anchor a_L and a right action of H on P with respect to an anchor a_R;
2. $a_L\colon P \to G_0$ is a principal H-bundle;
3. a_R is G-invariant: $a_R(g \cdot p) = a_R(p)$ for all $(g, p) \in G_1 \times_{H_0} P$;
4. the actions of G and H commute.

If P is a bibundle from a Lie groupoid G to a Lie groupoid H we write $P\colon G \to H$.

DEFINITION 3.26. Two bibundles $P, Q\colon G \to H$ are *isomorphic* if there is a diffeomorphism $\alpha\colon P \to Q$ which is G–H equivariant: $\alpha(g \cdot p \cdot h) = g \cdot \alpha(p) \cdot h$ for all $(g, p, h) \in G_1 \times_{G_0} P \times_{H_0} H_1$.

REMARK 3.27 (Bibundles defined by functors). By Remark 3.24 any functor $f\colon G \to H$ defines a bibundle

$$\langle f \rangle := f_0^* H_1 = G_0 \times_{f, H_0, t} H_1\colon G \to H.$$

The bibundle $\langle id_G \rangle$ corresponding to the identity functor $id_G\colon G \to G$ is G_1 with G acting on G_1 by left and right multiplications.

Note that $\langle f \rangle \to G_0$ has a global section $\sigma(x) := (x, f(1_x))$.

EXAMPLE 3.28. A map $f\colon M \to N$ between two manifolds tautologically defines a functor $f\colon \{M \rightrightarrows M\} \to \{N \rightrightarrows N\}$. The corresponding bibundle $\langle f \rangle$ is simply the graph $\mathrm{graph}(f)$ of f. It is not hard to show that a converse is true as well: any bibundle $P\colon \{M \rightrightarrows M\} \to \{N \rightrightarrows N\}$ is a graph of a function $f_P\colon M \to N$.

Note also that given two maps $f\colon M \to N$, $g\colon M' \to N$, an equivariant map of bibundles $\phi\colon \mathrm{graph}(f) \to \mathrm{graph}(g)$ has to be of the form $\phi(x, f(x)) = (h(x), g(h(x)))$ for some map $h\colon M \to M'$. That is, $\phi\colon \mathrm{graph}(f) \to \mathrm{graph}(g)$ corresponds to $h\colon M \to M'$ with the diagram $\begin{array}{ccc} M & \xrightarrow{f} & \\ {\scriptstyle h}\downarrow & & N \\ M' & \xrightarrow{g} & \end{array}$ commuting. This example is also important for embedding the category of manifolds into the 2-category of stacks.

EXAMPLE 3.29. Let M be a manifold and Γ a Lie group. As we have seen a number of times the manifold M defines the groupoid $\{M \rightrightarrows M\}$. The group Γ defines the action groupoid $\{\Gamma \rightrightarrows *\}$ for the action of Γ on a point $*$. A bibundle $P\colon \{M \rightrightarrows M\} \to \{\Gamma \rightrightarrows *\}$ is a principal Γ-bundle over M. A bibundle P is isomorphic to a bibundle of the form $\langle f \rangle$ for some functor $f\colon \{M \rightrightarrows M\} \to \{\Gamma \rightrightarrows *\}$ only if it has a global section, that is, only if it is trivial. Thus *there are many more bibundles than functors*.

Note, however, that any principal Γ-bundle $P \to M$ is *locally* trivial. Hence, after passing to an appropriate cover $\phi\colon \mathcal{U} \to M$, the bibundle $\phi^* P\colon \{\mathcal{U} \times_M \mathcal{U} \rightrightarrows \mathcal{U}\} \to \{\Gamma \rightrightarrows *\}$ is isomorphic to $\langle f \rangle$ for some functor $f\colon \{\mathcal{U} \times_M \mathcal{U} \rightrightarrows \mathcal{U}\} \to \{\Gamma \rightrightarrows *\}$. This is a special case of Lemma 3.37 below.

Note also that the functor $f\colon \{\mathcal{U} \times_M \mathcal{U} \rightrightarrows \mathcal{U}\} \to \{\Gamma \rightrightarrows *\}$ is a Čech 1-cocycle on M with coefficients in Γ with respect to the cover $\mathcal{U}$.

REMARK 3.30. Bibundles can be composed: if $P\colon G \to H$ and $Q\colon H \to K$ are bibundles, we define their *composition* to be the quotient of the fiber product $P \times_{H_0} Q$ by the action of H:

$$Q \circ P := (P \times_{H_0} Q)/H.$$

This makes sense: Since $Q \to H_0$ is a principal K-bundle, the fiber product $P \times_{H_0} Q$ is a manifold. Since the action of H on P is principal, the action of H on $P \times_{H_0} Q$ given by $(p, q) \cdot h = (p \cdot h, h^{-1} \cdot q)$ is free and proper. Hence the quotient $(P \times_{H_0} Q)/H$ is a manifold. Since the action of H on $P \times_{H_0} Q$ commutes with the actions of G and K, the quotient $(P \times_{H_0} Q)/H$ inherits the actions of G and K. Finally, since $Q \to H_0$ is a principal K-bundle, $(P \times_{H_0} Q)/H \to G_0$ is a principal K-bundle.

REMARK 3.31. The composition of bibundles is not strictly associative: if P_1, P_2, P_3 are three bibundles then $P_1 \circ (P_2 \circ P_3)$ is not the same manifold as $(P_1 \circ P_2) \circ P_3$. On the other hand the two bibundles are *isomorphic* in the sense of Definition 3.26: there is an equivariant diffeomorphism $\alpha\colon P_1 \circ (P_2 \circ P_3) \to (P_1 \circ P_2) \circ P_3$. This is the reason why we end up with a weak 2-category when we replace functors by bibundles.

REMARK 3.32. A natural transformation $\alpha\colon f \Rightarrow g$ between two functors $f, g\colon K \to L$ gives rise to an isomorphism $\langle\alpha\rangle\colon \langle f\rangle \to \langle g\rangle$ of the corresponding bibundles.

REMARK 3.33. If a bibundle $P\colon G \to H$ is G-principal, then it defines a bibundle $P^{-1}\colon H \to G$: switch the anchor maps, turn the left G-action into the right G-action and the right H-action into a left H-action. Indeed, the compositions $P^{-1} \circ P$ and $P^{-1} \circ P$ are isomorphic to $\langle id_G\rangle$ and $\langle id_H\rangle$ respectively.

We summarize (without proof):

1. The collection (Lie groupoids, bibundles, isomorphisms of bibundles) is a weak 2-category. We denote it by Bi.
2. The strict 2-category of Lie groupoids, smooth functors and natural transformations embeds into Bi. For this reason bibundles are often referred to as "generalized morphisms."

The lemma below allows us to start justifying our notions of equivalence of Lie groupoids.

LEMMA 3.34. *A functor $f\colon G \to H$ is an equivalence of Lie groupoids if and only if the corresponding bibundle $\langle f\rangle\colon G \to H$ is G-principal, hence (weakly) invertible.*

Proof. Recall that a functor $f\colon G \to H$ is an equivalence of Lie groupoids if and only if two conditions hold (cf. Definition 3.5):

1. the map $\varphi\colon G_1 \to (G_0 \times G_0) \times_{(f,f),H_0 \times H_0,(s,t)} H_1$, $\varphi(\gamma) = (s(\gamma), t(\gamma), f(\gamma))$ is a diffeomorphism and
2. the map $b\colon G_0 \times_{F,H_0,t} H_1 \to H_0$, $b(x,h) = s(h)$ is a surjective submersion.

Recall also that $\langle f\rangle = G_0 \times_{f,H_0,t} H_1$ and that the right anchor $a_R\colon \langle f\rangle \to H_0$ is precisely the map b, while the left anchor is the projection on the first factor: $a_L(x,h) = x$. Tautologically a_R is a surjective submersion if and only if b is a surjective submersion.

Suppose that G acts freely and transitively on the fibers of $a_R\colon \langle f\rangle \to H_0$. That is, suppose $a_R\colon \langle f\rangle \to H_0$ is a principal G-bundle. Then the map

$$\psi\colon G_1 \times_{s,G_0,a_L} (G_0 \times_{f,H_0,t} H_1) \to \langle f\rangle \times_{H_0} \langle f\rangle, \quad \psi(g,x,h) = ((x,h),(t(g), f(g)h'))$$

is a diffeomorphism. Hence it has a smooth inverse. Thus for any $(x,h),(x',h') \in G_0 \times H_1$ with $f(x) = t(h)$, $f(x') = t(h')$ and $s(h) = s(h')$ there is a unique $g \in G_1$ depending smoothly on x, x', h and h' with $s(g) = x$, $t(g) = x'$ and $h' = f(g)h$. Therefore for any $x, y \in G_0$ and any $h' \in H_1$ with $s(h') = f(x)$ and $t(h') = f(y)$ there is a unique $g \in G_1$ depending smoothly on x, y and h' so that $h' = f(g)1_{f(x)}$. That is, the map

$$\varphi\colon G_1 \to (G_0 \times G_0) \times_{(f,f),H_0\times H_0,(s,t)} H_1$$

has a smooth inverse. Therefore if $\langle f\rangle \to H_0$ is left G-principal bundle then f is an equivalence of Lie groupoids.

Conversely suppose φ has a smooth inverse. Then for any $((x,h),(x',h')) \in \langle f\rangle \times_{H_0} \langle f\rangle$ there is a unique $g \in G_1$ with $s(g) = x'$, $t(g) = x$ and $f(g) = h(h')^{-1}$. Hence the map ψ has a smooth inverse. Therefore, if $f\colon G \to H$ is an equivalence of Lie groupoids, then $\langle f\rangle \to H_0$ is left G-principal bundle.

COROLLARY 3.35. *Let G be a Lie groupoid and $\phi\colon \mathcal{U} \to G_0$ a cover (a surjective local diffeomorphism). Then the bibundle $\langle\widetilde{\phi}\rangle$ defined by the induced functor $\widetilde{\phi}\colon \phi^*G \to G$ is invertible.*

Proof. We have seen that the functor $\widetilde{\phi}\colon \phi^*G \to G$ is an equivalence. The result follows from Lemma 3.34 above.

LEMMA 3.36. *Let $P\colon G \to H$ be a bibundle from a Lie groupoid G to a Lie groupoid H. Then P is isomorphic to $\langle f\rangle$ for some functor $f\colon G \to H$ if and only if $a_L\colon P \to G_0$ has a global section.*

Proof. We have seen that for a functor $f\colon G \to H$ the map

$$a_L\colon G_0 \times_{H_0} H_1 \to G_0$$

has a global section.

Conversely, suppose we have a bibundle $P\colon G \to H$ and the principal H-bundle $a_L\colon P \to G_0$ has a global section. Then by Lemma 3.19 the bundle $P \to G_0$ is isomorphic to $G_0 \times_{\phi, H_0, t} H_1$ for some map $\phi\colon G_0 \to H_0$. Therefore we may assume that $P = G_0 \times_{\phi, H_0, t} H_1$. Now the left action of G on P defines a map $f\colon G_1 \to H_1$ by

$$g \cdot (t(g)1_{\phi(t(g))}) = (s(g), 1_{\phi(s(g))}) \cdot f(g)\,.$$

The map f is well defined since the action of H is principal. Finally the map f preserves multiplication: if $z \overset{g_2}{\to} y \overset{g_1}{\to} x$ are two composable arrows in G_1 then, on one hand,

$$g_2\ (g_1 \cdot (x, 1_{\phi(x)}) = g_2 \cdot (y, 1_{\phi(y)}) \cdot f(g_1) = ((z, 1_{\phi(z)}) \cdot f(g_2)) \cdot f(g_1)$$

and on the other,

$$(g_2 g_1) \cdot (x, 1_{\phi(x)}) = (z, 1_{\phi(z)}) \cdot f(g_2 g_1)\,.$$

Hence $f(g_2)f(g_1) = f(g_2 g_1)$, that is, f is a functor.

LEMMA 3.37. *Let $P\colon G \to H$ be a bibundle from a groupoid G to a groupoid H. There is a cover $\phi\colon \mathcal{U} \to G_0$ and a functor $f\colon \phi^*G \to H$ so that*

$$P \circ \langle \widetilde{\phi} \rangle \overset{\simeq}{\longrightarrow} \langle f \rangle\,,$$

where $\widetilde{\phi}\colon \phi^*G \to G$ is the induced functor and $\overset{\simeq}{\longrightarrow}$ an isomorphism of bibundles.

Proof. Since $a_L\colon P \to G_0$ is an H-principal bundle, it has local sections $\sigma_i\colon U_i \to P$ with $\bigcup U_i = G$. Let $\mathcal{U} = \bigsqcup U_i$ and $\phi\colon \mathcal{U} \to G_0$ be the inclusion. Then $\phi^*P \to \mathcal{U}$ has a global section. Hence, by Lemma 3.36 there is a functor $f\colon \phi^*G \to H$ with $\langle f \rangle = \phi^*P$.

3.3 HILSUM-SKANDALIS CATEGORY OF LIE GROUPOIDS

Recall that Bi denotes the weak 2-category with objects Lie groupoids, 1-arrows bibundles and 2-arrows equivariant maps between bibundles. The 2-arrows are always invertible. Recall that Gpoid denotes the (2-)category of Lie groupoids, functors and natural transformations.

DEFINITION 3.38. Define the 1-category Gp to be the category with objects Lie groupoids and arrows the *isomorphism classes* $[f]$ of smooth functors.

Define the 1-category HS (for Hilsum and Skandalis [14], who invented it) to be the category constructed out of Bi by identifying isomorphic bibundles.

There is an evident functor $\tilde{z}\colon \mathsf{Gpoid} \to \mathsf{HS}$ which is the identity on objects and takes a functor f to the equivalence class of the bibundle $\langle f \rangle$ defined by f : $\tilde{z}(f) = [\langle f \rangle]$. Clearly it drops down to a faithful functor

$$z\colon \mathsf{Gp} \to \mathsf{HS}\,, \qquad z(G \overset{[f]}{\to} H) = (G \overset{[\langle f\rangle]}{\to} H)\,.$$

By abuse of notation let W denote the collection of isomorphism classes of equivalences in Gp :

$$W = \{[w] \mid w \in \mathsf{Gpoid}_1 \text{ is an equivalence}\}\,.$$

PROPOSITION 3.39. *The functor* $z\colon \mathsf{Gp} \to \mathsf{HS}$ *defined above localizes* Gp *at the class of equivalences* W. *That is,* z *induces an equivalence of categories* $\mathsf{Gp}[W^{-1}] \to \mathsf{HS}$.

Proof. By Lemma 3.34, $z([w])$ is invertible in HS for any equivalence w. Thus the content of the proposition is the universal property of the functor $z\colon \mathsf{Gp} \to \mathsf{HS}$. Suppose $\Phi\colon \mathsf{Gp} \to \mathsf{E}$ is a functor that sends isomorphism classes of equivalences to invertible arrows. We want to construct a functor $\Psi\colon HS \to \mathsf{E}$ so that

$$\Psi \circ z = \Phi\,.$$

As the first step, for an object $G \in \mathsf{HS}_0$ define $\Psi(G) = \Phi(G)$. Next let $P\colon G \to H$ be a bibundle. We want to define $\Psi([P])$. By Lemma 3.37 we can factor P as

$$P \simeq \langle f' \rangle \circ \langle w' \rangle^{-1}$$

for some equivalence $w'\colon G' \to G$ and a functor $f'\colon G' \to G$. Define

$$\Psi([P]) = \Phi([f'])\,\Phi([w'])^{-1}.$$

We need to check that this is well defined and that Ψ preserves compositions. Suppose $w''\colon G'' \to G$ and $f''\colon G'' \to G$ is another choice of factorization. Let

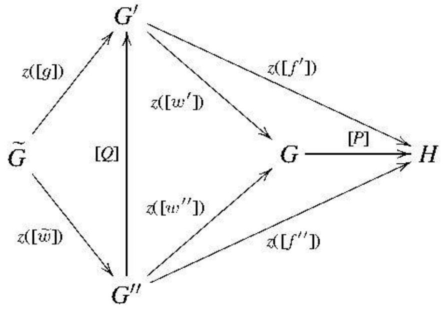
$$[Q] = z[w']^{-1} z[w']\colon G'' \to G'.$$

Then $[Q]$ can be factored as well:

$$[Q] = z([g])\,z([\tilde{w}])^{-1}$$

for some equivalence $\tilde{w}\colon \tilde{G} \to G''$ and some functor $g\colon \tilde{G} \to G'$.

The diagram



commutes in HS. Hence

$$z([f''])\,z([\widetilde{w}]) = z([f'])\,z([g])\,. \tag{3.1}$$

Since z is faithful,

$$[f''][\widetilde{w}] = [f'][g]$$

in Gp. Hence, in E,

$$\begin{aligned}\Phi([f''])\,\Phi([\widetilde{w}]) &= \Phi([f'])\,\Phi([g]) = \Phi([f'])\,\Phi([w'])^{-1}\Phi([w'])\,\Phi([g]) \\ &= \Phi([f'])\,\Phi([w'])^{-1}\Phi([w''])\,\Phi([\widetilde{w}])\,,\end{aligned}$$

where we used the fact that z is faithful and (3.1). Since $\Phi([\widetilde{w}])$ is invertible,

$$\Phi([f'']) = \Phi([f'])\,\Phi([w'])^{-1}\Phi([w''])\,.$$

Therefore

$$\Phi([f''])\,\Phi([w''])^{-1} = \Phi([f'])\,\Phi([w'])^{-1},$$

and Ψ is well-defined.

A similar argument shows that Ψ preserves multiplication.

DEFINITION 3.40 (Morita equivalent groupoids). Two Lie groupoids are *Morita equivalent* if there they are isomorphic in the localization $\mathsf{Gp}[W^{-1}]$ of the category of groupoids at equivalences. In particular, G and H are Morita equivalent if there is a bibundle $P\colon G \to H$ with the action of G being principal.

We finally come to the punchline of the section: the localization of the category of Lie groupoids at equivalences as a 1*-category* has problems.

LEMMA 3.41. *There are a cover* $\{U_1, U_2\}$ *of* S^1 *and two morphisms* $f, g\colon S^1 \to \{\mathbf{Z}/2 \rightrightarrows *\}$ *in* $\mathsf{Gp}[W^{-1}]$ *so that* $f|_{U_i} = g|_{U_i}$ *(*$i = 1, 2$*) but* $f \neq g$.

Proof. In the category HS a morphism from a manifold M (that we think of as the groupoid $\{M \rightrightarrows M\}$) to a groupoid G is the equivalence class of a bibundle P from $\{M \rightrightarrows M\}$ to G. An action of $\{M \rightrightarrows M\}$ on P is simply a map $a_L \colon P \to M$. So a bibundle from M to G is a principal G-bundle and an HS morphism from M to G is the equivalence class of some principal G-bundle over M. Hence an HS morphism from S^1 to $\{\mathbf{Z}/2 \rightrightarrows *\}$ is the class of a principal $\mathbf{Z}/2$ bundle over S^1 (cf. Example 3.29). There are two such classes: the class of the trivial bundle a and the class of the nontrivial bundle b. Now cover S^1 by two contractible open sets U_1 and U_2. Any principal S^1 bundle over a contractible open set is trivial. Therefore $a|_{U_i} = b|_{U_i}$, $i = 1, 2$. This gives us the two morphisms in HS from S^1 to $\{\mathbf{Z}/2 \rightrightarrows *\}$ with the desired properties. Let $F\colon \mathsf{HS} \to \mathsf{Gp}[W^{-1}]$ denote an equivalence of categories, which exists by Proposition 3.39. Then $f = F(a)$ and $g = F(b)$ are the desired morphisms in $\mathsf{Gp}[W^{-1}]$.

It may be instructive to note how this problem does not arise in the weak 2-category Bi. In Bi the 1-arrows are not isomorphism classes of bibundles but actual bibundles. Let $P_1 \to S^1$ denote a trivial $\mathbf{Z}/2$ principal bundle and $P_2 \to S^1$ a nontrivial one. Over the open sets U_1, U_2 we have isomorphisms $\varphi_i\colon P_1|_{U_i} \xrightarrow{\simeq} P_2|_{U_i}$, rather than equalities, as we had with their isomorphism classes. These local isomorphisms obviously do not glue together to form a global isomorphism from P_1 to P_2. They cannot, because P_1 and P_2 are not isomorphic. And they do not because they do not agree on double intersections: $\varphi_1|_{P_1|_{U_1 \cap U_2}} \neq \varphi_2|_{P_1|_{U_1 \cap U_2}}$.

At this point we can agree that the right setting for orbifolds is the weak 2-category Bi and declare our mission accomplished. That is, a *smooth orbifold* would be a Lie groupoid weakly isomorphic in Bi (i.e., Morita equivalent) to a proper étale Lie groupoid. We would call such groupoids *orbifold groupoids*. A *map* between two orbifolds would be a smooth bibundle.

The geometry of orbifolds would proceed along the lines of Moerdijk's paper [20]. For example, let us define *vector orbi-bundles*. The definition is modeled on the case where the orbifold is a manifold with an action of a finite group. That is, suppose a finite group Γ acts on a manifold M. A vector bundle over the orbifold "M/Γ" is a Γ-equivariant vector bundle $E \to M$. Hence, in general, a vector bundle over an orbifold groupoid G is a vector bundle $E \to G_0$ over the space of objects together with a linear left action of G on E (*linear* means that the map $G_1 \times_{G_0} E \to E$ is a vector bundle map). A bit of work shows that one can pull back a vector bundle by a bibundle.

On the other hand, there is still something awkward in this set-up, since the composition of bibundles is not strictly associative. This gets particularly strange when we start thinking about *flows* of vector fields, or, more generally, group actions. For example, let the circle S^1 act on itself by translations. Now take an open cover $\mathcal{U} \to S^1$ and form the cover groupoid $G = \{\mathcal{U} \times_{S^1} \mathcal{U} \rightrightarrows \mathcal{U}\}$. The induced functor $G \to \{S^1 \rightrightarrows S^1\}$ is weakly invertible, so we get an "action" of S^1 on G. The word "action" is in quotation marks because for any two elements of the group $\lambda, \lambda' \in S^1$ and the corresponding isomorphisms $\phi_\lambda, \phi_{\lambda'} : G \to G$

$$\phi_\lambda \circ \phi_{\lambda'} \neq \phi_{\lambda+\lambda'} .$$

Rather,

$$\phi_\lambda \circ \phi_{\lambda'} \xrightarrow{A} \phi_{\lambda+\lambda'}$$

for some isomorphism of bibundles A depending on λ, λ'. We get a so called *weak action* of S^1 on G.

The same thing happens when we try to integrate a vector field on G: we do not get a *flow* in the sense of an action of the reals. We get some sort of a weak flow. For the same reason the action of the Lie algebra $Lie(\Gamma)$ on a proper étale Lie groupoid G with the compact coarse moduli space G_0/G_1 will not integrate to the action of the Lie group Γ. It will only integrate to a weak action. This is somewhat embarrassing since in literature Lie groups routinely act on orbifolds.

There is another question that may be nagging the reader: are not groupoids supposed to be atlases on orbifolds, rather than being orbifolds themselves? There is a solution to both problems. It involves embedding the weak 2-category Bi into an even bigger gadget, the *2-category of stacks* St. Stacks form a strict 2-category. This is the subject of the next and last section. In particular in St the composition of 1-arrows is associative and strict group actions make perfectly good sense. Additionally there is a way of thinking of a groupoid as "coordinates" on a corresponding stack. Different choices of coordinates define Morita equivalent groupoids. And Morita equivalent groupoids define "the same" (isomorphic) stacks.

4. STACKS

In Section 3.2 we constructed a weak 2-category Bi whose objects are Lie groupoids, 1-arrows (morphisms) are bibundles and 2-arrows (morphisms between morphisms) are equivariant maps between bibundles. The goal of this section is to describe a particularly nice and concrete (?!) *strictification* of this weak 2-category. That is, we describe a strict 2-category St of stacks and a functor $B\colon \mathsf{Bi} \to \mathsf{St}$ which is an embedding of weak 2-categories (there is no established name in literature for this functor, so I made one up). The 2-category St of stacks is a sub-2-category of the category of categories Cat. Recall that the objects of Cat are categories, the 1-arrows are functors and the 2-arrows are natural transformations.

Here is a description of the 2-functor $B\colon \mathsf{Bi} \to \mathsf{Cat}$ (it will land in St once we define/explain what St is):

1. To a groupoid G assign the category BG, whose objects are principal G-bundles and morphisms are G-equivariant maps.

2. To a bibundle $P\colon G \to H$ assign a functor

$$BP\colon BG \to BH$$

as follows: A principal G-bundle Q on a manifold M is a bibundle from the groupoid $\{M \rightrightarrows M\}$ to G. Define

$$BP(Q) = P \circ Q \quad \text{(a composition of bibundles)}.$$

A G-equivariant map $\phi\colon Q_1 \to Q_2$ between two principal G-bundles $Q_1 \to M_1$, $Q_2 \to M_2$ induces an H-equivariant map $BP(\phi)\colon P \circ Q_1 \to P \circ Q_2$ between the corresponding principal H-bundles. It is not hard to check that BP is actually a functor.

3. To a G-H equivariant map $A\colon P \to P'$ assign a natural transformation $BA\colon BP \Rightarrow BP'$ as follows. Given a principal G-bundle Q, the map $A\colon P \to P'$ induces a G-H equivariant map $\widetilde{A}\colon Q \times_{G_0} P \to Q \times_{G_0} P'$ which descends to an H-equivariant diffeomorphism

$$BA(Q)\colon BP(Q) \equiv P \circ Q \equiv (Q \times_{G_0} P)/G \to (Q \times_{G_0} P')/G \equiv BP'(Q).$$

REMARK 4.1. The notation $B\{M \rightrightarrows M\}$ is quite cumbersome. Instead we will use the notation $\underline{M}$.

It follows from Example 3.28 that the category $\underline{M}$ has the following simple description. It objects are maps $Y \xrightarrow{f} M$ of manifolds into M. A morphism in $\underline{M}$ from $f\colon Y \to M$ to $f'\colon Y' \to M$ is a map of manifolds $h\colon Y \to Y'$ making the diagram $\begin{array}{ccc} Y & \overset{f}{\searrow} & \\ {\scriptstyle h}\downarrow & & M \\ Y' & \underset{f'}{\nearrow} & \end{array}$ commute. The category $\underline{M}$ is an example of a *slice* (or *comma*) category.

We now proceed to describe the image of the functor $B\colon \mathsf{Bi} \to \mathsf{Cat}$. More precisely we will describe a slightly larger 2-category of *geometric stacks* and the functor B will turn out to be an equivalence of weak 2-categories $B\colon \mathsf{Bi} \to$ geometric stacks. More precisely, we will see that every geometric stack is isomorphic to a stack of the form BG for some Lie groupoid G.

We define *geometric stacks* in several step. We first define *categories fibered in groupoids* (CFGs) over the category of manifolds Man. Next we define *stacks*. These are CFG's with sheaf-like properties. Then we single out *geometric stacks*. These are the stacks that have atlases. Finally any geometric stack is isomorphic (as a stack) to a stack of the form BG for some groupoid G.

4.1 Categories fibered in groupoids

DEFINITION 4.2. A *category fibered in groupoids (CFG)* over a category C is a functor $\pi\colon \mathsf{D} \to \mathsf{C}$ such that

(1) Given an arrow $f\colon C' \to C$ in C and an object $\xi \in \mathsf{D}$ with $\pi(\xi) = C$ there is an arrow $\widetilde{f}\colon \xi' \to \xi$ in D with $\pi(\widetilde{f}) = f$ (we think of ξ' as a *pullback* of ξ along f).

(2) Given a diagram $\begin{array}{ccc} \xi'' & \overset{f}{\searrow} & \\ & & \xi \\ \xi' & \underset{h}{\nearrow} & \end{array}$ in D and a commutative diagram $\begin{array}{ccc} \pi(\xi'') & \overset{\pi(f)}{\searrow} & \\ g\downarrow & & \pi(\xi) \\ \pi(\xi') & \underset{\pi(h)}{\nearrow} & \end{array}$ in C there is a unique arrow $\widetilde{g}\colon \xi'' \to \xi'$ in D making $\begin{array}{ccc} \xi'' & \overset{f}{\searrow} & \\ \widetilde{g}\downarrow & & \xi \\ \xi' & \underset{h}{\nearrow} & \end{array}$ commute and satisfying $\pi(\widetilde{g}) = g$. That is, there is a unique way to fill in the first diagram so that its image under π is the second diagram.

We will informally say that D is a category fibered in groupoids over C, with the functor π understood.

EXAMPLE 4.3. Fix a Lie groupoid G. I claim that the functor $\pi\colon BG \to \mathsf{Man}$ that sends a principal G-bundle to its base and a G-equivariant map between two principal G-bundles to the induced map between their bases makes the category BG into a category fibered in groupoids over the category Man of manifolds.

Indeed condition (1) of Definition 4.2 is easy to check. Given a map $f\colon N \to M$ between two smooth manifolds and a principal G-bundle $\xi \to M$

we have the pullback bundle $f^*\xi \to N$ and a G-equivariant map $\widetilde{f}\colon f^*\xi \to \xi$ inducing f on the bases of the bundles.

Note that if $\pi'\colon \xi' \to N$ is a principal G-bundle and $h\colon \xi' \to \xi$ is a G-equivariant map inducing $f\colon N \to M$ then there is a canonical G-equivariant map $\eta\colon \xi' \to f^*\xi$ which is given by $\eta(x) = (\pi'(x), h(x))$. By Corollary 3.21, the map η is a diffeomorphism.

To check condition (2) suppose that we have three principal G-bundles $\xi'' \to M''$, $\xi' \to M'$, $\xi \to M$, two G-equivariant maps $f\colon \xi'' \to \xi$, $h\colon \xi' \to \xi$ inducing $\bar{f}\colon M'' \to M$ and $\overline{h}\colon M' \to M$ respectively and a map $g\colon M'' \to M$ so that $\begin{array}{ccc} M'' & \overset{\bar{f}}{\searrow} & \\ {\scriptstyle g}\downarrow & & M \\ M' & \underset{\bar{h}}{\nearrow} & \end{array}$ commutes. We want to construct a G-equivariant map $\widetilde{g}\colon \xi'' \to \xi'$ with $h \circ \widetilde{g} = f$. By the preceding paragraph we may assume that $\xi'' = \overline{f}^*\xi = M'' \times_M \xi$ and $\xi' = \overline{h}^*\xi = M' \times_M \xi$. Define $\widetilde{g}\colon M'' \times_M \xi \to M' \times_M \xi$ by $\widetilde{g}(m, x) = (g(m), x)$. Hence $h \circ \widetilde{g} = f$, and we have verified that $\pi\colon BG \to \mathsf{Man}$ is a CFG.

DEFINITION 4.4 (Fiber of CFG). Let $\pi\colon \mathsf{D} \to \mathsf{C}$ be a category fibered in groupoids and $C \in \mathsf{C}_0$ an object. The *fiber* of D over C is the category $\mathsf{D}(C)$ with objects

$$\mathsf{D}(C)_0 := \{\xi \in \mathsf{D}_0 \mid \pi(\xi) = C\}$$

and arrows/morphisms

$$\mathsf{D}(C)_1 := \{(f\colon \xi' \to \xi) \in \mathsf{D}_1 \mid \xi, \xi' \in \mathsf{D}(C)_0 \text{ and } \pi(f) = id_C\}.$$

EXAMPLE 4.5. In the case of $\pi\colon BG \to \mathsf{Man}$ the fiber of BG over a manifold M is the category of principal G-bundles over M and *gauge transformations* (G-equivariant diffeomorphisms covering the identity map on the base).

REMARK 4.6. Let $\pi\colon \mathsf{D} \to \mathsf{C}$ be a CFG. Suppose $Y \xrightarrow{f} X$ is an arrow in C, $\xi \in \mathsf{D}(X)_0$, $\xi_1, \xi_2 \in \mathsf{D}(Y)_0$ and $h_i\colon \xi_i \to \xi$ $(i = 1, 2)$ are two arrows in D with $\pi(h_i) = f$. Then by Definition 4.2(2) there exist unique arrows $k\colon \xi_1 \to \xi_2$ and $\ell\colon \xi_2 \to \xi_1$ making the diagrams

$$\begin{array}{ccc} \xi_1 & \overset{h_1}{\searrow} & \\ {\scriptstyle k}\downarrow & & \xi \\ \xi_2 & \underset{h_2}{\nearrow} & \end{array} \quad \text{and} \quad \begin{array}{ccc} \xi_1 & \overset{h_1}{\searrow} & \\ {\scriptstyle \ell}\uparrow & & \xi \\ \xi_2 & \underset{h_2}{\nearrow} & \end{array}$$

commute, with $\pi(k) = \pi(\ell) = id_Y$.

Then, since $\pi(k \circ \ell) = id_Y$ and

$$\begin{array}{ccc} \xi_1 & \xrightarrow{h_1} & \\ {\scriptstyle \ell\circ k}\downarrow & & \xi \\ \xi_1 & \xrightarrow{h_1} & \end{array}$$

commutes, we must have $\ell \circ k = id_{\xi_1}$. Similarly, $k \circ \ell = id_{\xi_2}$. We conclude: *any two pullbacks of* ξ *along* $Y \xrightarrow{f} X$ *are isomorphic.*

CONVENTION. From now on, given a CFG $\pi\colon \mathsf{D} \to \mathsf{C}$ and $\xi \in \mathsf{D}(X)_0$ for each arrow $Y \xrightarrow{f} X \in \mathsf{C}_1$ we *choose* an arrow $\widetilde{f}$ in D with target ξ. We denote the source of $\widetilde{f}$ by $f^*\xi$ and refer to it as the *pullback of* ξ *by* f. We always choose $id^*\xi = \xi$.

Similarly we can define *pullbacks of arrows*: Suppose $(\xi_1 \xrightarrow{\gamma} \xi_2) \in \mathsf{D}(X)_1$ is an arrow in D and $(Y \xrightarrow{f} X)$ is an arrow in C. We then have a diagram in D:

$$\begin{array}{ccc} f^*\xi_1 & \xrightarrow{\widetilde{f}_1} & \xi_1 \\ & & \downarrow{\scriptstyle \gamma} \\ f^*\xi_2 & \xrightarrow[\widetilde{f}_2]{} & \xi_2 \end{array} \qquad (4.1)$$

By Definition 4.2 (2) applied to $\begin{array}{ccc} f^*\xi_1 & \xrightarrow{\gamma\circ\widetilde{f}_1} & \\ & & \xi_2 \\ f^*\xi_2 & \xrightarrow{\widetilde{f}_2} & \end{array}$ we get the unique arrow $f^*\gamma\colon f^*\xi_1 \to f^*\xi_2$ making (4.1) commute.

REMARK 4.7. Similar arguments show that a fiber $\mathsf{D}(C)$ of a category D fibered in groupoids over C is actually a groupoid. That is, all arrows in $\mathsf{D}(C)$ are invertible.

DEFINITION 4.8 (Maps of CFGs). Let $\pi_{\mathsf{D}}\colon \mathsf{D} \to \mathsf{C}$ and $\pi_{\mathsf{E}}\colon \mathsf{E} \to \mathsf{C}$ be two categories fibered in groupoids. A 1*-morphism* (or a 1*-arrow*) $F\colon \mathsf{D} \to \mathsf{E}$ of CFGs is a functor that commutes with the projections: $\pi_{\mathsf{E}} \circ F = \pi_{\mathsf{D}}$.

A 1-morphism $F\colon \mathsf{D} \to \mathsf{E}$ of CFGs is an *isomorphism* if it is an equivalence of categories.

Given two 1-morphisms $F, F'\colon \mathsf{D} \to \mathsf{E}$ of CFGs, a 2*-morphism* $\alpha\colon F \Rightarrow F'$ is a natural transformation from F to F'.

Thus the collection of all categories fibered in groupoids over a given category C is a strict 2-category. Note also that natural transformations between 1-arrows of CFGs are automatically invertible since the fibers of CFGs are groupoids. We note that for any two CFGs D and E over C, the collection of 1-arrows $\mathrm{Hom}(\mathsf{D}, \mathsf{E})$ forms a category. In fact, it is a groupoid.

4.2 DESCENT

To make sense of the next definition, consider how a principal G-bundle $P \to M$ (G a Lie groupoid) can be reconstructed from its restrictions to elements of an open cover $\{U_i\}$ of M and the gluing data[2]). We have restrictions $P_i = P|_{U_i}$ and isomorphisms $P_i|_{U_{ij}} \to P_j|_{U_{ij}}$ over double intersections $U_{ij} := U_i \cap U_j$ satisfying the cocycle conditions. Given a G-equivariant map $\phi\colon P' \to P$ of two principal G-bundles covering the identity map on the base, we have a collection of G-equivariant maps $\phi_i\colon P'_i \to P_i$ which agree on double intersections: $\phi_i|_{P_i|_{U_{ij}}} = \phi_j|_{P_j|_{U_{ij}}}$.

Conversely, given a collection of principal G-bundles $\{P_i \to U_i\}$ and isomorphisms $\theta_{ij}\colon P_i|_{U_{ij}} \to P_j|_{U_{ij}}$ satisfying the cocycle conditions, there is a principal G-bundle P over M with $P|_{U_i}$ isomorphic to P_i for all i.

Similarly, given two collections $(\{P'_i \to U_i\}, \{\theta'_{ij}\colon P_i|_{U_{ij}} \to P_j|_{U_{ij}}\})$, $(\{P_i \to U_i\}, \{\theta_{ij}\colon P_i|_{U_{ij}} \to P_j|_{U_{ij}}\})$ and a collection of principal G-bundle maps $\{\phi_i\colon P'_i \to P_i\}$ compatible with $\{\theta'_{ij}\}$ and $\{\theta'_{ij}\}$, there is a G-equivariant map $\phi\colon P' \to P$ which restricts to ϕ_i over U_i.

A succinct way of describing the above local-to-global correspondence is through the language of equivalences of categories. We have the category $BG(M)$ of principal G-bundles over M and G-equivariant maps covering id_M. We may think of it as the category $\mathsf{Bi}(\{M \rightrightarrows M\}, G)$ of bibundles from $\{M \rightrightarrows M\}$ to G. Given a cover $\mathcal{U} = \bigsqcup U_i \to M$, we have the cover groupoid $\mathcal{U} \times_M \mathcal{U} \rightrightarrows \mathcal{U}$. A collection $(\{P_i \to U_i\}, \{\theta_{ij}\colon P_i|_{U_{ij}} \to P_j|_{U_{ij}}\})$ of principal G-bundles is nothing but a bibundle from the cover groupoid to G. Similarly, a map between two such collections is an equivariant map between two bibundles. And the restriction map $P \mapsto \{P|_{U_i}\}$ induces a map between the two categories:

$$\Psi\colon \mathsf{Bi}(\{M \rightrightarrows M\}, G) \to \mathsf{Bi}(\{\mathcal{U} \times_M \mathcal{U} \rightrightarrows \mathcal{U}\}, G).$$

---

[2]) The reader may think of G as a Lie group to avoid getting bogged down in irrelevant technicalities.

Formally, on objects,

$$\Psi(Q) = Q \circ U,$$

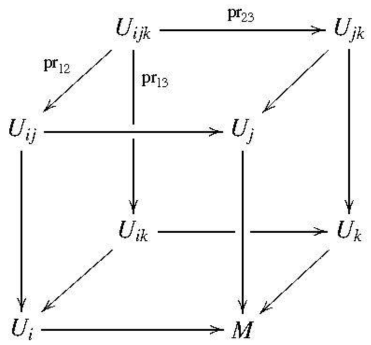
where $U\colon \{\mathcal{U} \times_M \mathcal{U} \rightrightarrows \mathcal{U}\} \to \{M \rightrightarrows M\}$ is the bibundle with the total space $\mathcal{U}$, left anchor the identity map and the right anchor the "embedding" $\mathcal{U} \to M$. Since a G-equivariant map $Q \to Q'$ induces a G-equivariant map $Q \circ U \to Q' \circ U$, Ψ is a functor. Moreover, since U is weakly invertible, Ψ is an equivalence of categories. One says that the principal G-bundles on the cover $\mathcal{U}$ satisfying the compatibility conditions *descend* to the principal G-bundles on M.

More generally, given a CFG $\pi\colon \mathsf{D} \to \mathsf{Man}$ and a cover $\mathcal{U} \to M$, one defines the *descent category* $\mathsf{D}(\mathcal{U} \to M)$. To do it properly, we need to correct one inaccuracy in the discussion above. We have taken advantage of the fact that one can restrict principal bundles to open sets. Furthermore if $\{U_i\}$ is a cover of a manifold M and $P \to M$ a principal G-bundle, then $(P|_{U_i})|_{U_{ij}} = P|_{U_{ij}} = (P|_{U_j})|_{U_{ij}}$ (here, again, $U_{ij} = U_i \cap U_j$). But if we want to think of $BG \to \mathsf{Man}$ abstractly, as a CFG, then restrictions should be replaced by pullbacks.

Now if $M'' \xrightarrow{g} M' \xrightarrow{f} M$ are maps of manifolds and ξ is an object of D over M, then we do not expect $(f \circ g)^*\xi$ to equal $g^*(f^*\xi)$; we only expect them to be canonically isomorphic. And indeed if $\mathsf{D} = BG$ so that ξ is a principal G-bundle, then the pullback $f^*(g^*P)$ is *not* the same as $(f \circ g)^*P$ even as a *set*! To talk about descent in general we need to replace restrictions by pull-backs: instead of $P|_{U_i}$ we should think ι_i^*P where $\iota_i\colon U_i \to M$ denotes the canonical inclusion. We will then discover that $\iota_{ij}^*\iota_i^*P$ is isomorphic but not equal to $\iota_{ji}^*\iota_j^*P$ (ι_{ij} and ι_{ji} denote the inclusions of the double intersection U_{ij} into U_i and U_j respectively), so the bookkeeping gets a bit more complicated. Let us now properly organize all this bookkeeping. We closely follow Vistoli [29].

Given an open covering $\{U_i \hookrightarrow M\}$ of a manifold M we think of the double intersections $U_{ij} = U_i \cap U_j$ as fiber products $U_i \times_M U_j$ and triple intersections U_{ijk} as fiber products $U_i \times_M \times U_j \times_M U_k$. Let $\mathrm{pr}_1\colon U_i \times_M U_j \to U_i$ and $\mathrm{pr}_2\colon U_i \times_M U_j \to U_j$ the first and second projection respectively. Similarly for any three indices i_1, i_2, i_3 we have projection $p_a\colon U_{i_1} \times_M \times U_{i_2} \times_M U_{i_3} \to U_{i_a}$, $a = 1, 2, 3$. We also have a commuting cube:

(4.2)



where pr_{12}, pr_{13} and pr_{23} denote the appropriate projections.

DEFINITION 4.9 (Descent category). Let $\pi\colon \mathsf{D} \to \mathsf{Man}$ be a category fibered in groupoids, M a manifold and $\{U_i\}$ an open cover of M. An *object with descent data* $(\{\xi_i\}, \{\phi_{ij}\})$ on M, is a collection of objects $\xi_i \in \mathsf{D}(U_i)$, together with isomorphisms $\phi_{ij}\colon \mathrm{pr}_2^* \xi_j \simeq \mathrm{pr}_1^* \xi_i$ in $\mathsf{D}(U_{ij}) = \mathsf{D}(U_i \times_M U_j)$, such that the following cocycle condition is satisfied: for any triple of indices i, j and k, we have the equality

$$\mathrm{pr}_{13}^* \phi_{ik} = \mathrm{pr}_{12}^* \phi_{ij} \circ \mathrm{pr}_{23}^* \phi_{jk}\colon \mathrm{pr}_3^* \xi_k \to \mathrm{pr}_1^* \xi_i$$

where pr_{ab} and pr_a are the projections discussed above. The isomorphisms ϕ_{ij} are called *transition isomorphisms* of the object with descent data.

An *arrow* between objects with descent data

$$\{\alpha_i\}\colon (\{\xi_i\}, \{\phi_{ij}\}) \to (\{\eta_i\}, \{\psi_{ij}\})$$

is a collection of arrows $\alpha_i\colon \xi_i \to \eta_i$ in $\mathsf{D}(U_i)$, with the property that for each pair of indices i, j, the diagram

$$\begin{array}{ccc} \mathrm{pr}_2^* \xi_j & \xrightarrow{\mathrm{pr}_2^* \alpha_j} & \mathrm{pr}_2^* \eta_j \\ \Big\downarrow{\scriptstyle \phi_{ij}} & & \Big\downarrow{\scriptstyle \psi_{ij}} \\ \mathrm{pr}_1^* \xi_i & \xrightarrow{\mathrm{pr}_1^* \alpha_i} & \mathrm{pr}_1^* \eta_i \end{array}$$

commutes.

There is an obvious way of composing morphisms, which makes objects with descent data the objects of a category, *the descent category of* $\{U_i \to M\}$. We denote it by $\mathsf{D}(\{U_i \to M\})$.

REMARK 4.10. As before let $\pi\colon \mathsf{D} \to \mathsf{Man}$ be a category fibered in groupoids, M a manifold and $\{U_i\}$ an open cover of M. We have a functor

$$\mathsf{D}(M) \to \mathsf{D}(\{U_i \to M\})$$

given by pullbacks.

We are now in a position to define stacks over manifolds.

DEFINITION 4.11 (Stack). A category fibered in groupoids $\pi\colon \mathsf{D} \to \mathsf{Man}$ is a *stack* if for any manifold M and any open cover $\{U_i \to M\}$ the pullback functor

$$\mathsf{D}(M) \to \mathsf{D}(\{U_i \to M\})$$

is an equivalence of categories.

EXAMPLE 4.12. The CFG $BG \to \mathsf{Man}$ is a stack for any Lie groupoid G.

EXAMPLE 4.13. Let Γ be a Lie group. The category $dB\Gamma$ with objects principal Γ-bundles *with connections* and morphisms connection preserving equivariant maps is a stack.

DEFINITION 4.14 (Maps of stacks). Let $\pi_{\mathsf{C}}\colon \mathsf{C} \to \mathsf{Man}$, $\pi_{\mathsf{D}}\colon \mathsf{D} \to \mathsf{Man}$ be two stacks. A functor $f\colon \mathsf{C} \to \mathsf{D}$ is a *map of stacks* (more precisely a 1-arrow in the 2-category St of stacks) if it is a map of CFGs (cf. Definition 4.8) — f commutes with the projections to Man:

$$\pi_{\mathsf{D}} \circ f = \pi_{\mathsf{C}}.$$

LEMMA 4.15. *Let M be a manifold, H a groupoid. Then any map of stacks $F : \underline{M} \to BH$ is naturally isomorphic to the functor BP induced by a principal H-bundle P over M.*

Proof. As we have seen in Remark 4.1, the objects of the CFG $\underline{M}$ are maps $Y \xrightarrow{f} M$. An arrow in $\underline{M}$ from $Y \xrightarrow{f} M$ to $Y' \xrightarrow{f'} M$ is a commuting triangle $\begin{array}{ccc} Y & \overset{f}{\searrow} & \\ h\downarrow & & M \\ Y' & \underset{f'}{\nearrow} & \end{array}$. The functor F assigns to each object $Y \xrightarrow{f} M$ of $\underline{M}$ a principal H-bundle $F(Y \xrightarrow{f} M)$ over M. Let $P = F(M \xrightarrow{id} M)$. Note that any map $f\colon Y \to M$ is *also an arrow in* $\underline{M}$: it maps $Y \xrightarrow{f} M$ to $M \xrightarrow{id} M$, since $\begin{array}{ccc} Y & \searrow^{f} & \\ {\scriptstyle f}\downarrow & & M \\ M & \nearrow_{id} & \end{array}$ commutes. Hence we get a map of principal H-bundles

$$F\left(\begin{array}{ccc} Y & \searrow^{f} & \\ {\scriptstyle f}\downarrow & & M \\ M & \nearrow_{id} & \end{array}\right) : F(Y \xrightarrow{f} M) \to P$$

projecting down to the map $f\colon Y \to M$ in Man. But $BH \to \mathsf{Man}$ is a CFG and $f^*P \to P$ is another arrow in BH projecting down to $f\colon Y \to M$. Consequently the principal H-bundle $F(Y \xrightarrow{f} M) \to Y$ is isomorphic to the bundle $f^*P \to Y$. Denote this isomorphism by $\alpha(f)$. Varying $f \in (\underline{M})_0$ we get a map

$$\alpha\colon (\underline{M})_0 \to (BH)_1;$$

it is a natural isomorphism of functors $\alpha\colon F \Rightarrow BP$.

COROLLARY 4.16. *Let M, M' be two manifolds. For any map $F\colon \underline{M} \to \underline{M'}$ of CFGs there is a unique map of manifolds $f\colon M \to M'$ defining F. That is, the functor* $\mathsf{Man} \to$ *CFG's over* Man, *$M \mapsto \underline{M}$ is an embedding of categories.*

Proof. Any two maps of CFGs from $\underline{M}$ to $\underline{M'}$ are equal since the only arrows in the fibers of $\underline{M'}$ are the identity arrows.

REMARK 4.17. Note a loss: if we think of smooth manifolds as stacks, we lose the way to talk about maps between manifolds that are *not* smooth.

REMARK 4.18. With a bit of work Lemma 4.15 above can be improved as follows:

Let G and H be two Lie groupoids. Then any map of stacks $F\colon BG \to BH$ is isomorphic to BP for some principal bibundle $P\colon G \to H$.

Indeed, let $P = F(G_1 \to G_0)$. It is an object of $BH(G_0)$, that is, a principal H-bundle over G_0. Since $G_1 \to G_0$ also has a left G-action and F is a functor, P also has a left G-action. A bit more work shows that BP is isomorphic to F.

4.3 2-YONEDA

Lemma 4.15 generalizes to arbitrary categories fibered in groupoids. The result is often referred to as 2-Yoneda lemma.

For any category C and any object $C \in \mathsf{C}_0$ there exists a CFG $\underline{C}$ over C defined as follows. The objects of $\underline{C}$ are maps $C' \xrightarrow{f} C \in \mathsf{C}_1$. A morphism from $C' \xrightarrow{f} C$ to $C'' \xrightarrow{g} C$ is a commuting triangle $\begin{array}{ccc} C' & & \\ & \searrow^{f} & \\ {\scriptstyle h}\downarrow & & C \\ & \nearrow_{g} & \\ C'' & & \end{array}$. There is an evident composition of such triangles (stick them together along the common side) making $\underline{C}$ into a category. There is also a functor $\pi_C \colon \underline{C} \to \mathsf{C}$:

$$\pi_C(C' \xrightarrow{f} C) = C' \text{ and } \pi_C\left(\begin{array}{ccc} C' & & \\ & \searrow^{f} & \\ {\scriptstyle h}\downarrow & & C \\ & \nearrow_{g} & \\ C'' & & \end{array} \right) = (h : C' \to C'').$$

LEMMA 4.19 (2-Yoneda). *Let $\mathsf{D} \to \mathsf{C}$ be a category fibered in groupoids. For any object $X \in \mathsf{C}$ there is an equivalence of categories*

$$\begin{aligned} \Theta \colon & \operatorname{Hom}_{\mathsf{CFG}}(\underline{X}, \mathsf{D}) \to \mathsf{D}(X), \\ & (F \colon \underline{X} \to \mathsf{D}) \mapsto F(X \xrightarrow{id} X), \\ & (\alpha \colon F \Rightarrow G) \mapsto (\alpha(X \xrightarrow{id} X) \colon F(X \xrightarrow{id} X) \to G(X \xrightarrow{id} X)), \end{aligned}$$

where $\operatorname{Hom}_{\mathsf{CFG}}(\underline{X}, \mathsf{D})$ denotes the category of maps of CFGs and natural transformations between them.

Proof. Suppose $F, G \colon \underline{X} \to \mathsf{D}$ are two functors with $F(id_X) = G(id_X) = \xi \in \mathsf{D}_0$. We argue that for any $Y \in \mathsf{C}$ and any $Y \xrightarrow{f} X \in \underline{X}(Y)_0$ there is a unique $\alpha(f) \in C(Y)_1$ with $G(f) \xrightarrow{\alpha(f)} F(f)$. Indeed, the diagram $\begin{array}{ccc} Y & & \\ & \searrow^{f} & \\ {\scriptstyle f}\downarrow & & X \\ & \nearrow_{id} & \\ X & & \end{array}$ in C defines an arrow in $\underline{X}$ from $(Y \xrightarrow{f} X) \in \underline{X}(Y)_0$ to $(X \xrightarrow{id} X) \in \underline{X}(X)_0$. Since $\pi_X \left(\begin{array}{ccc} Y & & \\ & \searrow^{f} & \\ {\scriptstyle f}\downarrow & & X \\ & \nearrow_{id} & \\ X & & \end{array} \right) = (Y \xrightarrow{f} X) \in \mathsf{C}_1$ and since F and G are maps of CFGs, we also have:

$$\pi_{\mathsf{D}}(F\left(\begin{array}{ccc} Y & & \\ & \searrow^{f} & \\ {\scriptstyle f}\downarrow & & X \\ & \nearrow_{id} & \\ X & & \end{array} \right)) = \pi_{\mathsf{D}}(G\left(\begin{array}{ccc} Y & & \\ & \searrow^{f} & \\ {\scriptstyle f}\downarrow & & X \\ & \nearrow_{id} & \\ X & & \end{array} \right)) = \left(Y \xrightarrow{f} X\right).$$

Hence we have a diagram

$$\begin{array}{ccc} G(f) & & \\ & \searrow^{f} & \\ & & \xi = G(id_X) = F(id_X) \\ & \nearrow_{id} & \\ F(f) & & \end{array}$$

in D. The functor $\pi_{\mathsf{D}} \colon \mathsf{D} \to \mathsf{C}$ takes the diagram above to the commuting diagram

$$\begin{array}{ccc} Y & & \\ {\scriptstyle id_Y}\downarrow & \searrow^{G(\triangleright)} & \\ & & X \\ & \nearrow_{F(\triangleright)} & \\ Y & & \end{array}, \qquad \text{where} \quad \triangleright := \begin{array}{ccc} Y & & \\ {\scriptstyle f}\downarrow & \searrow^{f} & \\ & & X \\ & \nearrow_{id} & \\ X & & \end{array}.$$

Therefore, by the axioms of CFG, there is a unique arrow $\alpha(f) \in \mathsf{D}(Y)_1$ with $\pi_{\mathsf{D}}(\alpha(f)) = id_Y$ making the diagram

$$\begin{array}{ccc} G(f) & & \\ {\scriptstyle \alpha(f)}\downarrow & \searrow^{G(\triangleright)} & \\ & & \xi \\ & \nearrow_{F(\triangleright)} & \\ F(f) & & \end{array}$$

commute. The map $\alpha \colon \underline{X}_0 \to \mathsf{D}_1$ is a natural transformation from G to F.

We now argue that Θ is essentially surjective and fully faithful. Let $\xi \in \mathsf{D}(X)_0$ be an object. Recall that for any arrow $(Y \xrightarrow{f} X) \in \mathsf{C}_1$ we have chosen a pullback $f^*\xi \in \mathsf{D}(Y)_0$. Define a functor $F_\xi \colon \underline{X} \to \mathsf{D}$ by

$$F_\xi(Y \xrightarrow{f} X) = f^*\xi,$$

$$F_\xi\left(\begin{array}{ccc} Y & & \\ {\scriptstyle h}\downarrow & \searrow^{f} & \\ & & X \\ & \nearrow_{g} & \\ Y' & & \end{array}\right) = \text{the unique arrow in } \mathsf{D} \text{ from } f^*\xi \text{ to } g^*\xi \text{ covering } Y' \xrightarrow{h} Y.$$

Note that $F_\xi(id_X) = id_X^*\xi = \xi$, so by the discussion above there is a natural transformation $\alpha \colon F \Rightarrow F_\xi$. Hence Θ is essentially surjective.

It remains to prove that Θ is fully faithful. Suppose $(\gamma \colon \xi' \to \xi) \in \mathsf{D}(X)_1$ is an arrow. We want to find a natural transformation $\alpha_\gamma \colon F_{\xi'} \Rightarrow F_\xi$ with $\Theta(\alpha_\gamma) = \gamma$ and prove that such a natural transformation is unique.

Given $(Y \xrightarrow{f} X) \in \underline{X}_0$ define

$$\alpha_\gamma(Y \xrightarrow{f} X) = (f^*\xi' \xrightarrow{f^*\gamma} f^*\xi).$$

Then α_γ is a natural transformation from $F_{\xi'}$ to F_ξ with $\alpha_\gamma(id_X) = id_X^*\gamma = \gamma$. Moreover α_γ is unique: if $\beta\colon \underline{X}_0 \to \mathsf{D}_1$ is another natural transformation from $F_{\xi'}$ to F_ξ then for any $(Y \xrightarrow{f} X) \in \underline{X}_0$ the diagram

$$\begin{array}{ccc} f^*\xi' = F_{\xi'}(f) & \longrightarrow & \xi' \\ \downarrow{\scriptstyle \beta(f)} & & {\scriptstyle \gamma}\downarrow \\ f^*\xi = F_\xi(f) & \longrightarrow & \xi \end{array} \tag{4.3}$$

commutes in D. Since $\beta(f) \in \mathsf{D}(Y)_1$, $\pi_{\mathsf{D}}(\beta(f)) = id_Y$. Therefore π_D takes the diagram (4.3) to

$$\begin{array}{ccc} Y & \longrightarrow & X \\ {\scriptstyle id_Y}\downarrow & & \downarrow{\scriptstyle id_X} \\ Y & \longrightarrow & X \end{array}.$$

By construction π_{D} also maps $\alpha_\gamma(f)\colon f^*\xi' \to f^*\xi$ to id_Y and makes

$$\begin{array}{ccc} f^*\xi' & \longrightarrow & \xi' \\ {\scriptstyle \alpha_\gamma(f)}\downarrow & & \downarrow{\scriptstyle \gamma} \\ f^*\xi & \longrightarrow & \xi \end{array}$$

commute. By (4.3) we must have $\alpha_\gamma(f) = \beta(f)$. Therefore Θ is fully faithful.

4.4 Atlases

One last idea that we would like to describe in this fast introduction to stacks is a way of determining a condition for a stack to be isomorphic to a stack BG for some Lie groupoid G. This involves the notion of an *atlas*, which, in turn, depends on a notion of a *fiber product* of categories fibered in groupoids.

DEFINITION 4.20. Let $\pi_X : X \to \mathsf{C}$, $\pi_Y : Y \to \mathsf{C}$ and $\pi_Z : Z \to \mathsf{C}$ be three categories fibered in groupoids over a category C. The 2-*fiber product* $Z \times_X Y$ of the diagram

$$\begin{array}{ccc} & & Y \\ & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow[g]{} & X \end{array}$$

is the category with objects

$$(Z \times_X Y)_0 = \left\{ (y, z, \alpha) \in Y_0 \times Z_0 \times X_1 \mid \pi_Y(y) = \pi_Z(z),\ f(y) \xrightarrow{\alpha} g(z) \right\}$$

and morphisms

$$\mathrm{Hom}_{Z \times_X Y}\big((z_1, y_1, \alpha_1), (z_2, y_2, \alpha_2)\big) =$$

$$\left\{ (z_1 \xrightarrow{v} z_2, y_1 \xrightarrow{u} y_2) \;\middle|\; \pi_Y(u) = \pi_Z(v) \in \mathsf{C}_1,\ \begin{array}{ccc} f(y_1) & \xrightarrow{f(u)} & f(y_2) \\ {\scriptstyle \alpha_1}\downarrow & \circlearrowleft & \downarrow{\scriptstyle \alpha_2} \\ g(z_1) & \xrightarrow[g(v)]{} & g(z_2) \end{array} \in X_1 \right\}$$

together with the functor $\pi : Z \times_X Y \to \mathsf{C}$ defined by

$$\pi((z, y, \alpha)) = \pi_Z(z) = \pi_Y(y), \quad \pi(v, u) = \pi_Z(v) = \pi_Y(u).$$

REMARK 4.21. It is not hard but tedious to check that $Z \times_X Y \to \mathsf{C}$ is a category fibered in groupoids.

REMARK 4.22. There are two evident maps of CFGs $\mathrm{pr}_1 : Z \times_X Y \to Z$ and $\mathrm{pr}_2 : Z \times_X Y \to Y$, but the diagram

$$\begin{array}{ccc} Z \times_X Y & \xrightarrow{\mathrm{pr}_2} & Y \\ \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle g} \\ Z & \xrightarrow[f]{} & X \end{array}$$

does not strictly speaking commute. Rather there is a natural isomorphism $g \circ \mathrm{pr}_2 \Rightarrow f \circ \mathrm{pr}_1$ which need not be the identity.

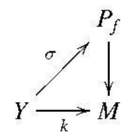
REMARK 4.23. The fiber product $Z \times_{f,X,g} Y$ is characterized by the following universal property: For any category fibered in groupoids W, there is a natural equivalence of categories

$$\mathrm{Hom}(W, Z \times_X Y) \to$$
$$\{(u, v, \alpha) \mid u : W \to Z, v : W \to Y \text{ functors}, \alpha : u \Rightarrow v \text{ natural isomorphism}\};$$

it sends a functor $h : W \to Z \times_X Y$ to the pair of functors $h \circ \mathrm{pr}_1$, $h \circ \mathrm{pr}_2$ and the natural isomorphism between them.

EXAMPLE 4.24. Let G be a groupoid and $p\colon \underline{G_0} \to BG$ be the map of CFGs defined by the canonical principal G-bundle $t\colon G_1 \to G_0$ (G acts on G_1 by multiplication on the right). Then for any map $f\colon \underline{M} \to BG$ from (the stack defined by) a manifold M to the stack BG, the fiber product $\underline{M} \times_{f,BG,p} \underline{G_0}$ is (isomorphic to) $\underline{P_f}$, where $P_f \to M$ is the principal G-bundle corresponding to the map f by 2-Yoneda.

Proof. We sort out what the objects of $\underline{M} \times_{f,BG,p} \underline{G_0}$ are, leaving the morphism as an exercise to the reader. Fix a manifold Y. The objects of the fiber $\underline{M} \times_{f,BG,p} \underline{G_0}(Y)$ are triples (z, y, α), where $z \in \underline{M}(Y)_0$, $y \in \underline{G_0}(Y)$ and α is an arrow in $BG(Y)$ from $f(z)$ to $p(y)$. The objects of $\underline{M}(Y)$ are maps of manifolds $Y \xrightarrow{k} M$. The image $f(Y \xrightarrow{k} M)$ of such an object is a principal G-bundle over Y. By 2-Yoneda this bundle is k^*P_f (recall that $P_f = f(id_M) \in BG(M)$). Similarly $p(Y \xrightarrow{\ell} G_0) = \ell^*(G_1 \to G_0)$. Finally $\alpha\colon f(Y \xrightarrow{k} M) \to p(Y \xrightarrow{\ell} G_0)$ is an arrow in the category $BG(Y)$. That is, $\alpha\colon k^*P_f \to \ell^*(G_1 \to G_0)$ is an isomorphism of two principal G-bundles over Y. Note that since $G_1 \to G_0$ has a global section, the pullback $\ell^*(G_1 \to G_0)$ also has a global section. And the isomorphism $\alpha^{-1}\colon \ell^*(G_1 \to G_0) \to k^*P_f$ is uniquely determined by the image of this global section. Hence the objects of $\underline{M} \times_{f,BG,p} \underline{G_0}(Y)$ are pairs (pullback to Y of $P_f \to M$, global section of the pullback). A global section of $k^*P_f \to Y$ uniquely determines a map $\sigma\colon Y \to P_f$ making the diagram



commute. Therefore objects of $\underline{M} \times_{f,BG,p} \underline{G_0}(Y)$ "are" maps from Y to P_f.

Unpacking the definitions further one sees that $\underline{M} \times_{f,BG,p} \underline{G_0}$ is isomorphic to $\underline{P_f}$ as a category fibered in groupoids, where by "isomorphic" we mean "equivalent as a category".

REMARK 4.25. The map of manifolds $P_f \to M$ in the construction above is a surjective submersion. Therefore we may think of $\underline{G_0} \xrightarrow{p} BG$ as a surjective submersion.

REMARK 4.26. To keep the notation from getting out of control we now drop the distinction between a manifold M and the associated stack $\underline{M}$. We will also drop the distinction between stacks isomorphic to manifolds and manifolds. Thus, in the example above we would say that for any Lie groupoid G, any manifold M and any map $M \to BG$ the fiber product $M \times_{BG} G_0$ is a manifold.

DEFINITION 4.27 (Atlas of a stack). Let $\mathsf{D} \to \mathsf{Man}$ be a stack over the category manifolds. An *atlas* for D is a manifold X and a map $p\colon X \to \mathsf{D}$ such that for any map $f\colon M \to \mathsf{D}$ from a manifold M the fiber product $M \times_{f,D,p} X$ is a manifold and the map $\mathrm{pr}_1\colon M \times_{f,D,p} X \to M$ is a surjective submersion.

REMARK 4.28. A stack over manifolds which possesses an atlas is alternatively referred to as a *geometric* stack, a *differentiable* stack or an *Artin* stack.

EXAMPLE 4.29. Let M be a manifold and let $\mathcal{U} = \bigsqcup U_i \to M$ be a cover by coordinate charts. Then the map of stacks $p\colon \underline{\mathcal{U}} \to \underline{M}$ is an atlas.

EXAMPLE 4.30. For any Lie groupoid G the canonical map $p\colon G_0 \to BG$ sending id_{G_0} to the principal G-bundle $G_1 \to G_0$ is an atlas.

PROPOSITION 4.31. *Given a stack with an atlas $p\colon X \to \mathsf{D}$ there is a Lie groupoid G such that D is isomorphic to BG. Moreover we may take $G_0 = X$ and $G_1 = X \times_{p,\mathsf{D},p} X$. In other words any geometric stack D is BG for some Lie groupoid G.*

It is relatively easy to produce the groupoid G out of the atlas $p\colon X \to \mathsf{D}$. It is more technical to define a map of stacks $\psi\colon \mathsf{D} \to BG$ and to prove that it is an isomorphism of stacks (that is, prove that ψ is an equivalence of categories commuting the projections $\pi_{BG}\colon BG \to \mathsf{Man}$ and $\pi_{\mathsf{D}}\colon \mathsf{D} \to \mathsf{Man}$). We will only sketch its construction and refer the reader to stacks literature for a detailed proof. The reader may consult, for example, [19, Proposition 70].

Sketch of proof of Proposition 4.31. We first construct a Lie groupoid out of an atlas on a stack. Let D be a stack over manifolds and $p\colon G_0 \to \mathsf{D}$ an atlas. Then the stack $G_0 \times_{p,\mathsf{D},p} G_0$ is a manifold; call it G_1. We want to produce the five structure maps: source, target $s, t\colon G_1 \to G_0$, unit $u\colon G_0 \to G_1$, inverse $i\colon G_1 \to G_1$ and multiplication $m\colon G_1 \times_{G_0} G_1 \to G_1$ satisfying the appropriate identities. We will produce five maps of stacks. By Corollary 4.16 this is enough. We take as source and target the projection maps $\mathrm{pr}_1, \mathrm{pr}_2 \colon G_0 \times_{p,\mathsf{D},p} G_0 \to G_0$. Since the diagram

$$\begin{array}{ccc} G_0 & \xrightarrow{id} & G_0 \\ {\scriptstyle id}\downarrow & & \downarrow{\scriptstyle p} \\ G_0 & \xrightarrow[p]{} & \mathsf{D} \end{array}$$

commutes, there is a unique map of stacks $u \colon G_0 \to G_0 \times_{p,\mathsf{D},p} G_0$. Concretely, on objects, it sends $x \in G_0$ to $(x, x, id_{p(x)})$. We also have the multiplication functor

$$m \colon (G_0 \times_{p,\mathsf{D},p} G_0) \times_{G_0} (G_0 \times_{p,\mathsf{D},p} G_0) \to (G_0 \times_{p,\mathsf{D},p} G_0),$$

which on objects is given by composition:

$$m((x_1, x_2, \alpha), (x_2, x_3, \beta)) = (x_1, x_3, \beta\alpha).$$

It is easy to see that the multiplication is associative. Finally the inverse map

$$inv \colon G_0 \times_{p,\mathsf{D},p} G_0 \to G_0 \times_{p,\mathsf{D},p} G_0$$

is given, on objects, by

$$inv(x_1, x_2, \alpha) = (x_2, x_1, \alpha^{-1}).$$

Note that the construction above does not use the descent properties of D. That is, we could have just as well defined an atlas for a category fibered in groupoids. The construction would then still produce a Lie groupoid.

Next we sketch a construction of a map $\psi \colon \mathsf{D} \to BG$ of CFGs. It will turn out to be a fully faithful functor. We will only need the fact that D is a stack to prove that ψ is essentially surjective.

By 2-Yoneda, an object of D over a manifold M is a map of CFGs $f \colon M \to \mathsf{D}$. Since $p \colon X \to \mathsf{D}$ is an atlas, the fiber product $M \times_{\mathsf{D}} X$ is a manifold and the map $\mathrm{pr}_1 \colon M \times_{\mathsf{D}} X \to M$ is a surjective submersion. There is a free and transitive action of G on the fibers of pr_1 with respect to the anchor map $\mathrm{pr}_2\, M \times_{\mathsf{D}} X \to X = G_0$ (once again we identify manifolds with the corresponding stacks). The right action of G is given by the "composition"

$$\begin{gathered}(M \times_{\mathsf{D}} X) \times_X (X \times_{\mathsf{D}} X) \to M \times_{\mathsf{D}} X \\ ((x_1, x_2, \alpha), (x_2, x_3, \beta)) \mapsto (x_1, x_3, \beta\alpha)\end{gathered}$$

(following the tradition in the subject we only wrote out the map on objects). It is free and transitive since the map

$$(M \times_{\mathsf{D}} X) \times_X (X \times_{\mathsf{D}} X) \to (M \times_{\mathsf{D}} X) \times_M (M \times_{\mathsf{D}} X)$$
$$((x_1, x_2, \alpha), (x_2, x_3, \beta)) \mapsto ((x_1, x_2, \alpha), (x_2, x_3, \beta\alpha))$$

is an isomorphism of stacks. Thus

$$\psi(f\colon M \to D) = (\mathrm{pr}_1\colon M \times_{f,\mathsf{D},p} X \to M)\,.$$

Next we define ψ on arrows. An arrow from $f_1\colon M_1 \to \mathsf{D}$ to $f_2\colon M_2 \to \mathsf{D}$ is a 2-commuting triangle

$$\begin{array}{ccc} M_1 & \xrightarrow{f_1} & \\ {\scriptstyle h}\downarrow & \Nearrow & \mathsf{D} \\ M_2 & \xrightarrow{f_2} & \end{array}$$

(this can be proved more or less the same way as we proved 2-Yoneda). Since the diagram

$$\begin{array}{ccc} M_1 \times_{\mathsf{D}} X & \longrightarrow & X \\ {\scriptstyle h\circ \mathrm{pr}_1}\downarrow & \Nearrow & \downarrow{\scriptstyle p} \\ M_1 & \xrightarrow[f_1]{} & \mathsf{D} \end{array}$$

2-commutes, we get, by the universal property of the 2-fiber product, a map

$$\widetilde{h}\colon M_1 \times_{\mathsf{D}} X \to M_2 \times_{\mathsf{D}} X$$

making the diagram

$$\begin{array}{ccc} M_1 \times_{\mathsf{D}} X & \xrightarrow{\widetilde{h}} & M_2 \times_{\mathsf{D}} X \\ \downarrow & & \downarrow \\ M_1 & \xrightarrow[h]{} & M_2 \end{array}$$

2-commute. And since all the objects in the diagram are manifolds, it actually commutes on the nose. It is not hard to check that $\widetilde{h}$ is compatible with the action of G. This defines ψ on arrows and gives us a functor

$$\psi\colon \mathsf{D} \to BG\,.$$

One checks that ψ is fully faithful (I am waving my hands here).

Next we argue that the full subcategory BG_{triv} of BG consisting of the trivial bundles is in the image of ψ. A trivial G-bundle on a manifold M is the pull back of the unit G-bundle $G_1 \to G_0 = X$ by a map $k\colon M \to X$. The diagram

$$\begin{array}{ccc} & \overset{k}{\nearrow} & X \\ & & \downarrow{\scriptstyle p} \\ M & \xrightarrow[p\circ k]{} & \mathsf{D} \end{array}$$

commutes by definition. Hence $\mathrm{pr}_1\colon M \times_{\mathsf{D}} X \to M$ has a global section σ with $\mathrm{pr}_2 \circ \sigma = k$. Therefore $M \times_{\mathsf{D}} X \to M$ is isomorphic to $k^*(G_1 \to G_0) \to M$. That is,

$$\psi(p \circ k) \simeq k^*(G_1 \to G_0).$$

Similarly if

$$\begin{array}{ccc} M_1 & \xrightarrow{k_1} & \\ {\scriptstyle h}\downarrow & & X \\ M_2 & \xrightarrow{k_2} & \end{array}$$

is a commuting diagram of maps of manifolds, then

$$\begin{array}{ccc} M_1 & \xrightarrow{p\circ k_1} & \\ {\scriptstyle h}\downarrow & & \mathsf{D} \\ M_2 & \xrightarrow{p\circ k_2} & \end{array}$$

is a commuting triangle of maps of CFGs, i.e., a map between two objects in D. One checks that $\psi\left(\begin{array}{ccc} M_1 & \searrow & \\ {\scriptstyle h}\downarrow & \Nearrow & \mathsf{D} \\ M_2 & \nearrow & \end{array}\right)$ is the map $\widetilde{h}\colon k_1^*(G_1 \to G_0) \to k_2^*(G_1 \to G_0)$. Thus the image of ψ includes the full subcategory BG_{triv} of trivial bundles.

Finally we use the fact that D is a stack to argue that ψ is essentially surjective. If $P \to M$ is a principal G-bundle, then M has an open cover $\{U_i \to M\}$ so that the restrictions $P|_{U_i}$ have global sections. Then for each i there is $\xi_i \in D(U_i)_0$ with $\psi(\xi_i)$ isomorphic to $P|_{U_i}$. The cover also defines descent data $(\{P|_{U_i}\}, \{\phi_{ij}\})$. These descent data really live in BG_{triv}. Hence, since the image of ψ contains BG_{triv} and since ψ is fully faithful, $(\{P|_{U_i}\}, \{\phi_{ij}\})$ defines descent data $(\{\xi_i\}, \{\psi^{-1}(\phi_{ij})\})$ in D. Since D is a stack, these descent data define an object ξ of $\mathsf{D}(M)$. Since ψ is a functor, $\psi(\xi)$ is isomorphic to P. We conclude that $\psi\colon \mathsf{D} \to BG$ is essentially surjective. □

REMARK 4.32. Atlases of geometric stacks are not unique. For example, if $p\colon X \to \mathsf{D}$ is an atlas and $f\colon Y \to X$ is map of manifolds which is a surjective submersion, then $p \circ f\colon Y \to \mathsf{D}$ is also an atlas. However, if $p\colon G_0 \to \mathsf{D}$ and $q\colon H_0 \to \mathsf{D}$ are two atlases, then by Proposition 4.31, the stacks BG and BH are isomorphic. It is not hard to construct an invertible bibundle $P\colon G \to H$ explicitly: P is the fiber product $G_0 \times_{p,\mathsf{D},q} H_0$. The actions of G and H are defined as in the proof of Proposition 4.31 and they are both principal.

It is useful to think of these two atlases and of the two corresponding Lie groupoids as two choices of "coordinates" on the stack D.

REMARK 4.33. In the light of the above remark it makes sense to say that *a geometric stack* $\mathsf{D} \to \mathsf{Man}$ is *an orbifold* if there is an atlas $p: X \to \mathsf{D}$ so that the corresponding groupoid $X \times_{\mathsf{D}} X \rightrightarrows X$ is a proper étale Lie groupoid.

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