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THE HESSE PENCIL OF PLANE CUBIC CURVES

by Michela ARTEBANI *) and Igor DOLGACHEV ‡)

ABSTRACT. This is a survey of the classical geometry of the Hesse configuration of 12 lines in the projective plane with relation to the inflection points of a plane cubic curve. We also study two $K3$ surfaces with Picard number 20 which arise naturally in connection with this configuration.

1. INTRODUCTION

In this paper we discuss some old and new results about the widely known *Hesse configuration* of 9 points and 12 lines in the projective plane $\mathbf{P}^2(k)$: each point lies on 4 lines and each line contains 3 points, giving an abstract configuration $(12_3, 9_4)$. Through most of the paper we will assume that k is the field of complex numbers $\mathbf{C}$ although the configuration can be defined over any field containing three cubic roots of unity. The Hesse configuration can be realized by the 9 inflection points of a nonsingular projective plane curve of degree 3. This discovery is attributed to C. Maclaurin (1698–1746) (see [46], p. 384), however the configuration [1]) is named after O. Hesse who was the first to study its properties in [24], [25]. In particular, he proved that the nine inflection points of a plane cubic curve form one orbit with respect to the projective group of the plane and can be taken as common inflection

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[1]) Not to be confused with another Hesse configuration $(12_4, 16_3)$, also related to plane cubic curves, see [15].

points of a pencil of cubic curves generated by the curve and its Hessian curve. In appropriate projective coordinates the *Hesse pencil* is given by the equation

$$\lambda(x^3 + y^3 + z^3) + \mu xyz = 0\,.$$

The pencil was classically known as the *syzygetic pencil* [2]) of cubic curves (see [9], p. 230 or [16], p. 274), the name attributed to L. Cremona. We do not know who is responsible for renaming the pencil, but apparently the new terminology is widely accepted in modern literature (see, for example, [4]).

Recently Hesse pencils have become popular among number-theorists in connection with computational problems in the arithmetic of elliptic curves (see, for example, [51]), and also among theoretical physicists, for example in connection with homological mirror symmetry for elliptic curves (see [56]).

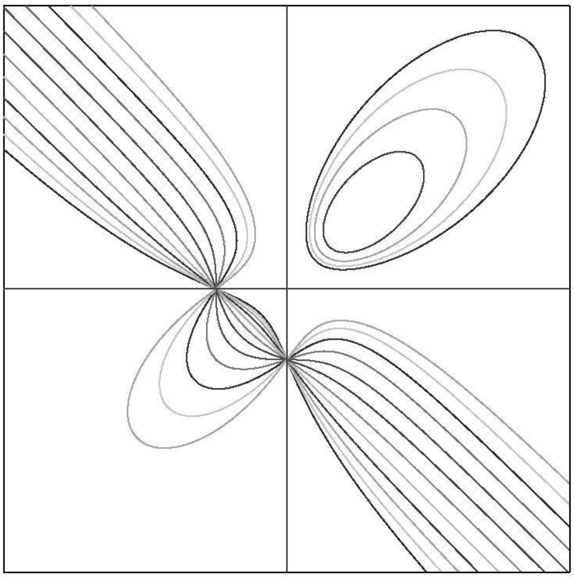

FIGURE 1
The Hesse pencil

The group of projective automorphisms which transform the Hesse pencil into itself is a group G_{216} of order 216 isomorphic to the group of affine transformations with determinant 1 of the projective plane over the field $\mathbf{F}_3$.

[2]) The term "syzygy" was used in astronomy to describe the alignment of three celestial bodies along a straight line. Sylvester adopted this word to express a linear relation between the covariants of a form. We will see later that the pencil contains the Hesse covariant of each of its members.

This group was discovered in 1878 by C. Jordan [31], who called it the *Hessian group*. Its invariants were described in 1889 by H. Maschke [36]. A detailed historical account and the first figure of the Hesse pencil can be found in [21].

The projective action of the Hessian group comes from a linear action of a complex reflection group $\overline{G}_{216}$ of order 648 (no. 25 in the Shephard-Todd list [48]) whose set of reflection hyperplanes consists of the 12 inflection lines of the Hesse configuration. The algebra of invariant polynomials of the group $\overline{G}_{216}$ is freely generated by three polynomials of degrees 6, 9, 12 (see [36]). An invariant polynomial of degree 6 defines a nonsingular plane sextic curve C_6. The double cover of the plane branched along the sextic curve C_6 is a $K3$ surface X on which G_{216} acts as a group of automorphisms. Its subgroup $H = \mathbf{F}_3^2 \rtimes Q_8$, where Q_8 is the Sylow 2-subgroup of $\mathrm{SL}(2, \mathbf{F}_3)$ isomorphic to the quaternion group of order 8, acts on the surface as a group of symplectic automorphisms. In fact, the group $\mathbf{F}_3^2 \rtimes Q_8$ can be found in Mukai's list [41] of finite groups which can be realized as maximal finite groups of symplectic automorphisms of a complex $K3$ surface.

The linear system of plane sextics with double points at 8 inflection points of a plane cubic is of projective dimension 3. The stabilizer H of the ninth remaining inflection point in G_{216} is isomorphic to $\mathrm{SL}(2, \mathbf{F}_3)$ and acts on this space by projective transformations. There is a unique invariant sextic C_6' for this action, having cuspidal singularities at the inflection points. The double cover of the plane branched along C_6' is birational to another $K3$ surface X' and the action of H can be lifted to X'. We show that X' is birationally isomorphic to the quotient of X by the subgroup $\mathbf{F}_3^2$ and that the induced action of the quotient group $G_{216}/\mathbf{F}_3^2 \cong \mathrm{SL}(2, \mathbf{F}_3)$ coincides with the action of H on X'. Both $K3$ surfaces X and X' are singular in the sense of Shioda, i.e. the subgroup of algebraic cycles in the second cohomology group is of maximal possible rank, equal to 20. We compute the intersection form defined by the cup-product on these subgroups.

The invariant sextic C_6 cuts out a set of 18 points on each nonsingular member of the pencil. We explain its geometric meaning, a result which we were unable to find in the classical literature.

It is a pleasure to thank Bert van Geemen who kindly provided us with his informal notes on this topic and made many useful comments on our manuscript. We thank Noam Elkies and Matthias Schuett for their help in the proof of Theorem 7.10. We are also indebted to Thierry Vust for his numerous suggestions for improving the exposition of the paper.

2. The Hesse pencil

Let k be an algebraically closed field of characteristic different from 3 and E be a nonsingular cubic in the projective plane $\mathbf{P}^2(k)$ defined by a homogeneous equation $F(x,y,z)=0$ of degree 3. The *Hessian curve* $\mathrm{He}(E)$ of E is the plane cubic curve defined by the equation $\mathrm{He}(F)=0$, where $\mathrm{He}(F)$ is the determinant of the matrix of the second partial derivatives of F. The nine points in $E\cap \mathrm{He}(E)$ are the *inflection points* of E. Fixing one of the inflection points p_0 defines a commutative group law $\oplus$ on E with p_0 equal to the zero: $p\oplus q$ is the unique point r such that p_0, r and the third point of intersection in $\overline{p,q}\cap E$ lie on a line. It follows from this definition of the group law that each inflection point is a 3-torsion point and that the group $E[3]$ of 3-torsion points on E is isomorphic to $(\mathbf{Z}/3\mathbf{Z})^2$. Any line $\overline{p,q}$ through two inflection points intersects E at another inflection point r such that p,q,r form a coset with respect to some subgroup of $E[3]$. Since we have 4 subgroups of order 3 in $(\mathbf{Z}/3\mathbf{Z})^2$ we find 12 lines, each containing 3 inflection points. They are called the *inflection lines* (or the *Maclaurin lines* [16]) of E. Since each element in $(\mathbf{Z}/3\mathbf{Z})^2$ is contained in 4 cosets, we see that each inflection point is contained in four inflection lines. This gives the famous *Hesse configuration* $(12_3, 9_4)$ of 12 lines and 9 points in the projective plane. It is easy to see that this configuration is independent of the choice of the point p_0.

The *Hesse pencil* is the one-dimensional linear system of plane cubic curves given by

$$E_{t_0,t_1}: \quad t_0(x^3+y^3+z^3)+t_1xyz=0, \quad (t_0,t_1)\in\mathbf{P}^1. \tag{1}$$

We use the affine parameter $\lambda = t_1/t_0$ and denote $E_{1,\lambda}$ by E_λ; the curve $xyz=0$ is denoted by E_∞. Since the pencil is generated by the Fermat cubic E_0 and its Hessian, its nine base points are in the Hesse configuration. In fact, they are the inflection points of any smooth curve in the pencil. In coordinates they are:

$$\begin{aligned}
&p_0=(0,1,-1), && p_1=(0,1,-\epsilon), && p_2=(0,1,-\epsilon^2),\\
&p_3=(1,0,-1), && p_4=(1,0,-\epsilon^2), && p_5=(1,0,-\epsilon),\\
&p_6=(1,-1,0), && p_7=(1,-\epsilon,0), && p_8=(1,-\epsilon^2,0),
\end{aligned}$$

where ϵ denotes a primitive third root of 1.

If we fix the group law by choosing the point p_0 to be the zero point, then the set of inflection points is the group of 3-torsion points of each member of the Hesse pencil. Hence we can define an isomorphism

$$\alpha\colon E_\lambda[3]_{p_0} \longrightarrow (\mathbf{Z}/3\mathbf{Z})^2$$

by sending the point $p_1 = (0,1,-\epsilon)$ to $(1,0)$ and the point $p_3 = (1,0,-1)$ to $(0,1)$. Under this isomorphism we can identify the nine base points with elements of $(\mathbf{Z}/3\mathbf{Z})^2$ as follows:

$$\begin{matrix} p_0 & p_1 & p_2 \\ p_3 & p_4 & p_5 \\ p_6 & p_7 & p_8 \end{matrix} \quad = \quad \begin{matrix} (0,0) & (1,0) & (2,0) \\ (0,1) & (1,1) & (2,1) \\ (0,2) & (1,2) & (2,2)\,. \end{matrix} \tag{2}$$

It is now easy to see that any triple of base points which represents a row, a column, or a term in the expansion of the determinant of matrix (2) spans an inflection line (cf. [38], p. 335).

The existence of an isomorphism α not depending on the member of the pencil can be interpreted by saying that the Hesse pencil is a family of elliptic curves together with a 3-level structure (i.e. a basis in the subgroup of 3-torsion points). In fact, in the following lemma we will prove that any smooth plane cubic is projectively isomorphic to a member of the Hesse pencil. It follows (see [4]) that its parameter space can be naturally identified with a smooth compactification of the fine moduli space $\mathcal{A}_1(3)$ of elliptic curves with a 3-level structure (when $k = \mathbf{C}$ this is the *modular curve* $X(3)$ *of principal level* 3).

LEMMA 2.1. *Any nonsingular cubic in* $\mathbf{P}^2(k)$ *is projectively equivalent to a member of the Hesse pencil, i.e. it admits a Hesse*[3]) *canonical form:*

$$x^3 + y^3 + z^3 + \lambda xyz = 0\,.$$

Proof. We will follow the arguments from [55]. Let E be a nonsingular plane cubic. Given two inflection tangent lines for E we can choose projective coordinates such that their equations are $x = 0$ and $y = 0$. Then it is easy to see that the equation of E can be written in the form

$$F(x,y,z) = xy(ax + by + cz) + dz^3 = 0\,, \tag{3}$$

where $ax + by + cz = 0$ is a third inflection tangent line. Suppose $c = 0$, then $ab \neq 0$ since otherwise the curve would be singular. Since a binary form of degree 3 with no multiple roots can be reduced, by a linear change of variables, to the form $x^3 + y^3$, the equation takes the form $x^3 + y^3 + dz^3 = 0$.

[3]) Called the *second canonical form* in [47], the first one being the Weierstrass form.

After scaling the coordinate z, we arrive at a Hesse equation. So we may assume that $c \neq 0$ and, after scaling the coordinate z, that $c = 3$. Let ϵ be a primitive 3rd root of unity and define new coordinates u, v by the formulae

$$ax + z = \epsilon u + \epsilon^2 v, \qquad by + z = \epsilon^2 u + \epsilon v.$$

Then

$$\begin{aligned} abF(x,y,z) &= (\epsilon u + \epsilon^2 v - z)(\epsilon^2 u + \epsilon v - z)(-u - v + z) + dz^3 \\ &= -u^3 - v^3 + (d+1)z^3 - 3uvz = 0. \end{aligned}$$

Since the curve is nonsingular we have $d \neq -1$. Therefore, after scaling the coordinate z, we get a Hesse equation for E:

$$x^3 + y^3 + z^3 + \lambda xyz = 0.$$

Assume additionally that the characteristic of the field k is not equal to 2. Recall that a plane nonsingular cubic also admits the Weierstrass canonical form

$$y^2 z = x^3 + axz^2 + bz^3, \qquad 4a^3 + 27b^2 \neq 0.$$

Projecting from the point p_0 we exhibit each curve of the Hesse pencil as a double cover of $\mathbf{P}^1$ branched at 4 points. By a standard procedure, this allows one to compute the Weierstrass form of any curve from the Hesse pencil:

$$y^2 z = x^3 + A(t_0,t_1)\,xz^2 + B(t_0,t_1)\,z^3, \tag{4}$$

where

$$\begin{aligned} A(t_0,t_1) &= 12u_1(u_0^3 - u_1^3), \\ B(t_0,t_1) &= 2(u_0^6 - 20u_0^3u_1^3 - 8u_1^6), \end{aligned} \tag{5}$$

and $(t_0,t_1) = (u_0, 6u_1)$. The discriminant of the cubic curve given by (4) is

$$\Delta = 4A^3 + 27B^2 = 2^2 3^3 u_0^3 (u_0^3 + 8u_1^3)^3,$$

its zeros describe the singular members of the pencil. The zeros of the binary form $A(t_0,t_1)$ define the curves from the Hesse pencil which admit an automorphism of order 6 with a fixed point (*equianharmonic cubics*). For example, the Fermat curve $E_0 : x^3 + y^3 + z^3 = 0$ is one of them. The zeros of the binary form $B(t_0,t_1)$ define the curves from the Hesse pencil which admit an automorphism of order 4 with a fixed point (*harmonic cubics*). The map

$$j\colon \mathbf{P}^1 \longrightarrow \mathbf{P}^1, \qquad (t_0,t_1) \longmapsto (4A^3, 4A^3 + 27B^2)$$

coincides (up to a scalar factor) with the map assigning to the elliptic curve E_λ its j-invariant, which distinguishes the projective equivalence classes of cubic curves.

The Hesse pencil naturally defines a rational map

$$\mathbf{P}^2 \dashrightarrow \mathbf{P}^1, \qquad (x,y,z) \longmapsto (xyz, x^3+y^3+z^3)$$

which is not defined at the nine base points. Let

$$\pi\colon S(3) \longrightarrow \mathbf{P}^2$$

be the blowing up of the base points. This is a rational surface such that the composition of rational maps $S(3) \longrightarrow \mathbf{P}^2 \dashrightarrow \mathbf{P}^1$ is a regular map

$$\phi\colon S(3) \longrightarrow \mathbf{P}^1 \tag{6}$$

whose fibres are isomorphic to the members of the Hesse pencil. The map ϕ defines a structure of a minimal elliptic surface on $S(3)$. Here and later we refer to [5], [18], [39] or [10] for the theory of elliptic fibrations on algebraic surfaces. The surface $S(3)$ is a special case of an *elliptic modular surface* $S(n)$ of level n (see [4], [49]), isomorphic to the universal family of elliptic curves with an n-level.

There are four singular members in the Hesse pencil, each is the union of three lines:

$$\begin{array}{ll}
E_\infty: & xyz = 0,\\
E_{-3}: & (x+y+z)(x+\epsilon y+\epsilon^2 z)(x+\epsilon^2 y+\epsilon z) = 0,\\
E_{-3\epsilon}: & (x+\epsilon y+z)(x+\epsilon^2 y+\epsilon^2 z)(x+y+\epsilon z) = 0,\\
E_{-3\epsilon^2}: & (x+\epsilon^2 y+z)(x+\epsilon y+\epsilon z)(x+y+\epsilon^2 z) = 0.
\end{array}$$

We will call these singular members the *triangles* and denote them by $T_1, \ldots, T_4$, respectively. The singular points of the triangles will be called the *vertices* of the triangles. They are

$$\begin{array}{lll}
v_0 = (1,0,0), & v_1 = (0,1,0), & v_2 = (0,0,1),\\
v_3 = (1,1,1), & v_4 = (1,\epsilon,\epsilon^2), & v_5 = (1,\epsilon^2,\epsilon),\\
v_6 = (\epsilon,1,1), & v_7 = (1,\epsilon,1), & v_8 = (1,1,\epsilon),\\
v_9 = (\epsilon^2,1,1), & v_{10} = (1,\epsilon^2,1), & v_{11} = (1,1,\epsilon^2).
\end{array} \tag{7}$$

The 12 lines forming the triangles are the inflection lines of the Hesse configuration. If we fix a point p_i as the origin in the group law of a nonsingular member of the pencil, then the side of a triangle T_i passing through p_i contains 3 base points forming a subgroup of order 3, while the other sides of T_i contain the cosets with respect to this subgroup. The triangles obviously give four singular fibres of Kodaira's type I_3 of the elliptic fibration ϕ.

REMARK 2.2. The Hesse pencil makes sense over a field of any characteristic. It is popular in number-theory and cryptography for finding explicit algorithms to compute the number of points of an elliptic curve over a finite field of characteristic 3 (see [22], [51]). We are grateful to Kristian Ranestad for this comment.

The proof of the existence of a Hesse equation for an elliptic curve E over a field of characteristic 3 goes through if we assume that E is an ordinary elliptic curve with rational 3-torsion points. We find equation (3) and check that it defines a nonsingular curve only if $abc \neq 0$. By scaling the variables we may assume that $a = b = -1$, $c = 1$. Next we use the variable change $z = u + x + y$ to transform the equation to the Hesse form

$$xyu + d(u + x + y)^3 = xyu + d(u^3 + x^3 + y^3) = 0\,.$$

The Hesse pencil (1) in characteristic 3 has two singular members: $(x+y+z)^3 = 0$ and $xyz = 0$. It has three base points $(1,-1,0)$, $(0,1,-1)$, $(1,0,-1)$, each of multiplicity 3, which are the inflection points of all nonsingular members of the pencil. Blowing up the base points, including infinitely near base points, we get a rational elliptic surface. It has two singular fibres of Kodaira's types IV^* and I_2. The fibre of type IV^* has the invariant δ of wild ramification equal to 1. This gives an example of a rational elliptic surface in characteristic 3 with finite Mordell-Weil group of sections (these surfaces are classified in [35]). The Mordell-Weil group of our surface is of order 3.

The Hesse configuration of 12 lines with 9 points of multiplicity 4 can also be defined over a finite field of 9 elements (see [26], Lemma 20.3.7). It is formed by four reducible members of a pencil of cuspidal cubics with 9 base points. The blow-up of the base points defines a rational quasi-elliptic surface in characteristic 3 with 4 singular fibres of Kodaira's type III.

3. THE HESSIAN AND THE CAYLEYAN OF A PLANE CUBIC

The first polar of a plane curve E with equation $F = 0$ with respect to a point $q = (a, b, c) \in \mathbf{P}^2$ is the curve $P_q(E)$ defined by $aF'_x + bF'_y + cF'_z = 0$. It is easy to see that the Hessian curve $\mathrm{He}(E)$ of a plane cubic E coincides with the locus of points q such that the polar conic $P_q(E) = 0$ is reducible.

If E_λ is a member of the Hesse pencil, we find that $\mathrm{He}(E_\lambda)$ is the member $E_{\mathfrak{h}(\lambda)}$ of the Hesse pencil, where

$$\mathfrak{h}(\lambda) = -\frac{108 + \lambda^3}{3\lambda^2}\,. \tag{8}$$

Let $p_i = (a, b, c)$ be one of the base points of the Hesse pencil. By computing the polar $P_{p_i}(E_\lambda)$ we find that it is equal to the union of the inflection tangent line $\mathbf{T}_{p_i}(E_\lambda)$ to the curve at the point p_i and the line $L_i : ax + by + cz = 0$. The lines $L_0, \dots, L_8$ are called the *harmonic polars*. It follows easily from the known properties of the first polars (which can be checked directly in our case) that the line L_i intersects the curve E_λ at 3 points q_j such that the tangent to the curve at q_j contains p_i. Together with p_i they form the group of 2-torsion points in the group law on the curve in which the origin is chosen to be the point p_i.

The harmonic polars, considered as points in the dual plane $\check{\mathbf{P}}^2$, give the set of base points of a Hesse pencil in $\check{\mathbf{P}}^2$. Its inflection lines are the lines dual to the vertices of the inflection triangles given in (7). If we identify the plane with its dual by means of the quadratic form $x^2 + y^2 + z^2$, the equation of the dual Hesse pencil coincides with the equation of the original pencil. For any nonsingular member of the Hesse pencil its nine tangents at the inflection points, considered as points in the dual plane, determine uniquely a member of the dual Hesse pencil.

REMARK 3.1. In the theory of line arrangements, the Hesse pencil defines two different arrangements (see [6] and [27]). The *Hesse arrangement* consists of 12 lines (the inflection lines), it has 9 points of multiplicity 4 (the base points) and no other multiple points. The second arrangement is the dual of the Hesse arrangement, denoted by $A_3^0(3)$. It consists of 9 lines (the harmonic polars) and has 12 multiple points of multiplicity 3. Together these two arrangements form an abstract configuration $(12_3, 9_4)$ which is a special case of a modular configuration (see [15]). In [27] Hirzebruch constructs certain finite covers of the plane with abelian Galois groups ramified over the lines of the Hesse configuration or its dual configuration. One of them, for each configuration, is a surface of general type with universal cover isomorphic to a complex ball.

PROPOSITION 3.2. *Let E_λ be a nonsingular member of the Hesse pencil. Let $L_i \cap E_\lambda = \{q_1, q_2, q_3\}$ and let E_{λ_j}, $j = 1, 2, 3$, be the curve from the Hesse pencil whose tangent at p_i contains q_j. Then* $\mathrm{He}(E_\mu) = E_\lambda$ *if and only if $\mu \in \{\lambda_1, \lambda_2, \lambda_3\}$.*

Proof. It is a straightforward computation. Because of the symmetry of the Hesse configuration, it is enough to consider the case when $i = 0$, i.e. $p_i = (0, 1, -1)$. We have that $L_0 : y - z = 0$ and $L_0 \cap E_\lambda$ is equal to the set of points $q_j = (1, y_j, y_j)$ satisfying $1 + 2y_j^3 + \lambda y_j^2 = 0$. The line $\overline{p_0, q_j}$ has the equation $-2y_j x + y + z = 0$. The curve E_μ from the Hesse pencil is tangent to this line at the point $(0, 1, -1)$ if and only if $(-\mu, 3, 3) = (-2y_j, 1, 1)$, i.e. $y_j = \mu/6$. Thus

$$\lambda = -\frac{1 + 2y_j^3}{y_j^2} = -\frac{108 + \mu^3}{3\mu^2}.$$

Comparing with formula (8), we see that $\mathfrak{h}(\mu) = \lambda$. This proves the assertion.

Let E be a smooth plane cubic curve which is not equianharmonic. Then $\mathrm{He}(E)$ is smooth and, for any $q \in \mathrm{He}(E)$, the polar conic $P_q(E)$ has one isolated singular point s_q. In fact, s_q lies on $\mathrm{He}(E)$ and the map $q \mapsto s_q$ is a fixed point free involution on $\mathrm{He}(E)$ (see, for example, [14]). If we fix a group law on $\mathrm{He}(E)$ with zero at p_i, then the map $q \mapsto s_q$ is the translation by a non-trivial 2-torsion point η. In the previous proposition this 2-torsion point is one of the intersection points of the harmonic polar L_i with $\mathrm{He}(E)$ such that E is tangent to the line connecting this point with the inflection point p_i.

The quotient $\mathrm{He}(E)/\langle\eta\rangle$ is isomorphic to the cubic curve in the dual plane $\check{\mathbf{P}}^2$ parametrizing the lines $\overline{q, s_q}$. This curve is classically known as the *Cayleyan curve* of E. One can show that the Cayleyan curve also parametrizes the line components of reducible polar conics of E. In fact, the line $\overline{q, s_q}$ is a component of the polar conic $P_a(E)$, where a is the intersection point of the tangents of $\mathrm{He}(E)$ at q and s_q.

PROPOSITION 3.3. *If $E = E_\lambda$ is a member of the Hesse pencil, then its Cayleyan curve $\mathrm{Ca}(E_\lambda)$ is the member of the dual Hesse pencil corresponding to the parameter*

$$\mathfrak{c}(\lambda) = \frac{54 - \lambda^3}{9\lambda}. \tag{9}$$

Proof. To see this, following [9], p. 245, we write the equation of the polar conic $P_q(E_{6\mu})$ with respect to a point $q = (u, v, w)$:

$$u(x^2 + 2\mu yz) + v(y^2 + 2\mu xz) + w(z^2 + 2\mu xy) = 0.$$

It is a reducible conic if the equation decomposes into linear factors, say

$$u(x^2 + 2\mu yz) + v(y^2 + 2\mu xz) + w(z^2 + 2\mu xy) = (ax + by + cz)(\alpha x + \beta y + \gamma z).$$

This happens if and only if

$$\begin{pmatrix} u & 2\mu w & 2\mu v \\ 2\mu w & v & 2\mu u \\ 2\mu v & 2\mu u & w \end{pmatrix} = \begin{pmatrix} a\alpha & a\beta + b\alpha & a\gamma + c\alpha \\ a\beta + b\alpha & b\beta & c\beta + b\gamma \\ a\gamma + c\alpha & c\beta + b\gamma & c\gamma \end{pmatrix}.$$

Considering this as a system of linear equations in the variables u, v, w, a, b, c we get the condition of solvability as the vanishing of the determinant

$$\begin{vmatrix} -1 & 0 & 0 & \alpha & 0 & 0 \\ 0 & -1 & 0 & 0 & \beta & 0 \\ 0 & 0 & -1 & 0 & 0 & \gamma \\ -2\mu & 0 & 0 & 0 & \gamma & \beta \\ 0 & -2\mu & 0 & \gamma & 0 & \alpha \\ 0 & 0 & -2\mu & \beta & \alpha & 0 \end{vmatrix} = \mu(\alpha^3 + \beta^3 + \gamma^3) + (1 - 4\mu^3)\alpha\beta\gamma = 0.$$

If we take (α, β, γ) as the coordinates in the dual plane, this equation represents the equation of the Cayleyan curve because the line $\alpha x + \beta y + \gamma z$ is an irreducible component of a singular polar conic. Setting $\mu = \lambda/6$, we get (9).

Note that the Cayleyan curve $\mathrm{Ca}(E_\lambda) = \mathrm{He}(E_\lambda)/\langle\eta\rangle$ comes with a distinguished nontrivial 2-torsion point, which is the image of the nontrivial coset of 2-torsion points on $\mathrm{He}(E_\lambda)$. This shows that $\mathrm{Ca}(E_\lambda) = \mathrm{He}(E'_\mu)$ for a uniquely defined member E'_μ of the dual Hesse pencil. The map $\alpha\colon \mathbf{P}^1 \to \mathbf{P}^1$, $\lambda \mapsto \mu$ gives an isomorphism between the spaces of parameters of the Hesse pencil and of its dual pencil such that $\mathfrak{h}(\alpha(\lambda)) = \mathfrak{c}(\lambda)$. One checks that

$$\mathfrak{h}(-18/\lambda) = \mathfrak{c}(\lambda).$$

REMARK 3.4. The Hesse pencil in the dual plane should not be confused with the (non-linear) pencil of the dual curves of members of the Hesse pencil. The dual curve of a nonsingular member $E_m = E_{m_0,3m_1}$ of the Hesse pencil is a plane curve of degree 6 with 9 cusps given by the equation

$$\begin{aligned} (10)\quad & m_0^4(X_0^6 + X_1^6 + X_2^6) - m_0(2m_0^3 + 32m_1^3)(X_0^3X_1^3 + X_0^3X_2^3 + X_2^3X_1^3) \\ & - 24m_0^2m_1^2X_0X_1X_2(X_0^3 + X_1^3 + X_2^3) - (24m_0^3m_1 + 48m_1^4)X_0^2X_1^2X_2^2 = 0. \end{aligned}$$

This equation defines a surface V in $\mathbf{P}^1 \times \check{\mathbf{P}}^2$ of bi-degree $(4,6)$, the universal family of the pencil. The projection to the first factor has fibres isomorphic to the dual curves of the members of the Hesse pencil, where the dual of a triangle becomes a triangle taken with multiplicity 2. The base points p_i of the Hesse pencil define 9 lines ℓ_{p_i} in the dual plane and each of the 9 cusps of an irreducible member from (10) lies on one of these lines. The unique cubic passing through the nine cusps is the Cayleyan curve of the dual cubic. If $(m,x) \in V$, then the curve E_m has the line ℓ_x (dual to x) as its tangent line. For a general point x, there will be 4 curves in the Hesse pencil tangent to this line, in fact the degree of the second projection $V \to \check{\mathbf{P}}^2$ is equal to 4. Each line ℓ_{p_i} lies in the branch locus of this map and its preimage in V has an irreducible component $\overline{\ell}_{p_i}$ contained in the ramification locus. The surface V is singular along the curves $\overline{\ell}_{p_i}$ and at the points corresponding to the vertices of the double triangles. One can show that a nonsingular minimal relative model of the elliptic surface $V \to \mathbf{P}^1$ is a rational elliptic surface isomorphic to $S(3)$. Thus, the dual of the Hesse pencil is the original Hesse pencil in disguise.

REMARK 3.5. The iterations of the maps $\mathfrak{h}\colon \mathbf{P}^1 \to \mathbf{P}^1$ and $\mathfrak{c}\colon \mathbf{P}^1 \to \mathbf{P}^1$ given by (8) and (9) were studied in [28]. They give interesting examples of complex dynamics in one complex variable. The critical points of $\mathfrak{h}$ are the four equianharmonic cubics and its critical values correspond to the four triangles. Note that the set of triangles is invariant under this map. The set of critical points of $\mathfrak{c}$ is the set of triangles and it coincides with the set of critical values. The equianharmonic cubics are mapped to critical points. This shows that both maps are critically finite maps in the sense of Thurston (see [37]).

4. THE HESSIAN GROUP

The *Hessian group* is the subgroup G_{216} of $\mathrm{Aut}(\mathbf{P}^2) \cong \mathrm{PGL}(3,\mathbf{C})$ preserving the Hesse pencil [4]). The Hessian group acts on the space $\mathbf{P}^1$ of parameters of the Hesse pencil, hence defines a homomorphism

$$\alpha\colon G_{216} \longrightarrow \mathrm{Aut}(\mathbf{P}^1)\,. \tag{11}$$

Its kernel K is generated by the transformations

$$g_0(x,y,z) = (x,z,y)\,,$$
$$g_1(x,y,z) = (y,z,x)\,,$$
$$g_2(x,y,z) = (x,\epsilon y,\epsilon^2 z)\,,$$

[4]) Not to be confused with the Hesse group isomorphic to $\mathrm{Sp}(6,\mathbf{F}_2)$ which is related to the 28 bitangents of a plane quartic.

and contains a normal subgroup of index 2

$$\Gamma = \langle g_1, g_2 \rangle \cong (\mathbf{Z}/3\mathbf{Z})^2 .$$

If we use the group law with zero p_0 on a nonsingular member of the pencil, then g_1 induces the translation by the 3-torsion point p_3 and g_2 that by the point p_1.

The image of the homomorphism (11) is clearly contained in a finite subgroup of $\mathrm{Aut}(\mathbf{P}^1)$ isomorphic to the permutation group S_4. Note that it leaves invariant the zeros of the binary forms $A(t_0, t_1)$, $B(t_0, t_1)$ from (5). It is known that the group S_4 acts on $\mathbf{P}^1$ as an octahedral group, with orbits of cardinalities $24, 12, 8, 6$, so it cannot leave invariant the zeros of a binary form of degree 4. However, its subgroup A_4 acts as a tetrahedral group with orbits of cardinalities $12, 6, 4, 4$. This suggests that the image of (11) is indeed isomorphic to A_4. In order to see that it is, it suffices to exhibit transformations from G_{216} which are mapped to generators of A_4 of orders 2 and 3. They are

$$g_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \epsilon & \epsilon^2 \\ 1 & \epsilon^2 & \epsilon \end{pmatrix}, \qquad g_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & \epsilon \end{pmatrix}.$$

The group generated by g_0, g_3, g_4 is a central extension of degree two of A_4. It is isomorphic to the binary tetrahedral group and to the group $\mathrm{SL}(2, \mathbf{F}_3)$. Note that $g_3^2 = g_0$ so

$$G_{216} = \langle g_1, g_2, g_3, g_4 \rangle .$$

It is clear that the order of G_{216} is equal to the order of K multiplied by that of A_4, making it equal to 216. Hence the notation.

PROPOSITION 4.1. *The Hessian group G_{216} is isomorphic to the semi-direct product*

$$\Gamma \rtimes \mathrm{SL}(2, \mathbf{F}_3),$$

where $\mathrm{SL}(2, \mathbf{F}_3)$ acts on $\Gamma \cong (\mathbf{Z}/3\mathbf{Z})^2$ via the natural linear representation.

The Hessian group clearly acts on the set of nine points p_i, giving a natural homomorphism from G_{216} to $\mathrm{Aff}_2(3)$, the affine group of $\mathbf{F}_3^2$. In fact, the Hessian group is the subgroup of index 2 of $\mathrm{Aff}_2(3)$ of transformations with linear part of determinant equal to 1. In this action the group G_{216} is realized as a 2-transitive subgroup of the permutation group S_9 on $\{0, 1, \dots, 8\}$ generated by permutations

$$T = (031)(475)(682) \quad \text{and} \quad U = (147)(285)$$

(see [11], 7.7). The stabilizer subgroup of the point p_0 is generated by U and $TUT^{-1} = (354)(678)$, and coincides with $\langle g_3, g_4 \rangle$.

REMARK 4.2. The group $\mathrm{Aff}_2(3)$ of order 432 that contains G_{216} as a subgroup of order 2 is isomorphic to the Galois group of the equation of degree 9 defining the first coordinates of the inflection points of a cubic with general coefficients in the affine plane [13], [55].

The Hessian group G_{216}, considered as a subgroup of $\mathrm{PGL}(3,\mathbf{C})$, admits two different extensions to a subgroup of $\mathrm{GL}(3,\mathbf{C})$ generated by complex reflections. The first group $\overline{G}_{216}$ is of order 648 and is generated by reflections of order 3 (no. 25 in Shephard-Todd's list [48]). The second group $\overline{G}'_{216}$ is of order 1296 and is generated by reflections of order 3 and reflections of order 2 (no. 26 in Shephard-Todd's list). The images of the reflection hyperplanes of $\overline{G}_{216}$ in the projective plane are the inflection lines, while the images of the reflection hyperplanes of $\overline{G}'_{216}$ are the inflection lines and the harmonic polars.

The algebra of invariants of $\overline{G}_{216}$ is generated by three polynomials of degrees 6, 9 and 12 (see [36], [52]):

$$\begin{aligned}
\Phi_6 &= x^6 + y^6 + z^6 - 10(x^3y^3 + x^3z^3 + y^3z^3),\\
\Phi_9 &= (x^3 - y^3)(x^3 - z^3)(y^3 - z^3),\\
\Phi_{12} &= (x^3 + y^3 + z^3)[(x^3 + y^3 + z^3)^3 + 216x^3y^3z^3].
\end{aligned}$$

Note that the curve $\Phi_9 = 0$ is the union of the nine harmonic polars L_i and that the curve $\Phi_{12} = 0$ is the union of the four equianharmonic members of the pencil. The union of the 12 inflection lines is obviously invariant with respect to G_{216}, however the corresponding polynomial Φ'_{12} of degree 12 is not an invariant but a relative invariant (i.e. the elements of $\overline{G}_{216}$ transform the polynomial to a constant multiple).

The algebra of invariants of the second complex reflection group $\overline{G}'_{216}$ is generated by Φ_6, Φ'_{12} and a polynomial of degree 18,

$$\Phi_{18} = (x^3 + y^3 + z^3)^6 - 540x^3y^3z^3(x^3 + y^3 + z^3)^3 - 5832x^6y^6z^6.$$

The curve $\Phi_{18} = 0$ is the union of the six harmonic cubics in the pencil. Later we will give a geometric meaning to the 18 intersection points of the curve defined by $\Phi_6 = 0$ with nonsingular members of the pencil.

A third natural linear extension of the group G_{216} is the preimage G'_{216} of the group under the projection $\mathrm{SL}(3,\mathbf{C}) \to \mathrm{PGL}(3,\mathbf{C})$. This is a group of order 648 isomorphic to the central extension $3G_{216}$ of G_{216}, but it is not isomorphic to $\overline{G}_{216}$. The preimage of the subgroup Γ in $3G_{216}$ is a non-abelian group of order 27 isomorphic to the *Heisenberg group* $\mathcal{H}_3(3)$ of unipotent 3×3-matrices with entries in $\mathbf{F}_3$. The group G'_{216} is then isomorphic to the semi-direct product $\mathcal{H}_3(3) \rtimes \mathrm{SL}(2, \mathbf{F}_3)$ and is generated by g_1, g_2, $\frac{1}{\epsilon - \epsilon^2} g_3$, $e^{2\pi i/9} g_4$ considered as linear transformations.

REMARK 4.3. Classical geometers used to define a projective transformation as a pair consisting of a nondegenerate quadric in the projective space and a nondegenerate quadric in the dual projective space. If $\mathbf{P}^n = \mathbf{P}(V)$, then the first quadric is given by a quadratic form on V which defines a linear map $\phi: V \to V^*$. The second quadric defines a linear map $V^* \to V$ and the composition with the first one is a linear map $V \to V$. In [21] the Hessian group is given by a set of 36 conics which are identified with conics in the dual plane $\check{\mathbf{P}}^2$ by means of an isomorphism $\mathbf{P}^2 \to \check{\mathbf{P}}^2$ defined by the conic $x_0^2 + x_1^2 + x_2^2 = 0$. These conics are the polars of four equianharmonic cubics in the pencil with respect to the 12 vertices of the inflection triangles. The 12 of them which are double lines have to be omitted.

It is known that the simple group $G = \mathrm{PSp}(4, \mathbf{F}_3)$ of order 25,920 has two maximal subgroups of index 40. One of them is isomorphic to the complex reflection group $\overline{G}_{216}$ of order 648. It has the following beautiful realization in terms of complex reflection groups in dimensions 4 and 5.

It is known that the group $\mathbf{Z}/3\mathbf{Z} \times \mathrm{Sp}(4, \mathbf{F}_3)$ is isomorphic to a complex reflection group in $\mathbf{C}^4$ with 40 reflection hyperplanes of order 3 (no. 32 in Shephard-Todd's list [48]). This defines a projective representation of G in $\mathbf{P}^3$ and the stabilizer subgroups of the reflection projective planes are isomorphic to $\overline{G}_{216}$. The reflection planes cut out on each fixed reflection plane the extended Hesse configuration of 12 inflection lines and 9 harmonic polars ([36], p. 334).

It is also known that the group $\mathbf{Z}/2\mathbf{Z} \times G \not\cong \mathrm{Sp}(4, \mathbf{F}_3)$ is isomorphic to a complex reflection group in $\mathbf{C}^5$ with 45 reflection hyperplanes of order 2 (no. 33 in Shephard-Todd's list [48]). This defines a projective representation of G in $\mathbf{P}^4$. The algebra of invariant polynomials with respect to the complex reflection group $\mathbf{Z}/2\mathbf{Z} \times G$ was computed by Burkhardt [8]. The smallest degree invariant is of degree 4. Its zero locus in $\mathbf{P}^4$ is the famous Burkhardt quartic hypersurface with 45 nodes where 12 reflection hyperplanes meet. There are 40 planes forming one orbit, each containing 9 nodes. Each such plane contains 12 lines cut out by the reflection hyperplanes. They form the Hesse configuration with the 9 points equal to the set of base points of the Hesse pencil.

One can find an excellent exposition of the results of Maschke and Burkhardt in [29]. There is also a beautiful interpretation of the geometry of the two complex reflection groups in terms of the moduli space $\mathcal{A}_2(3)$ of principally polarized abelian surfaces with some 3-level structure (see [17], [20]). For example, one can identify $\mathcal{A}_2(3)$ with an open subset of the Burkhardt quartic whose complement is equal to the union of the 40 planes.

5. The quotient plane

Consider the blowing up $\pi\colon S(3) \longrightarrow \mathbf{P}^2$ of the base points p_i of the Hesse pencil and the elliptic fibration (6):

$$\phi\colon S(3) \longrightarrow \mathbf{P}^1, \qquad (x, y, z) \longmapsto (xyz, x^3 + y^3 + z^3).$$

The action of the group Γ on $\mathbf{P}^2$ lifts to an action on $S(3)$. Fixing one section of ϕ (i.e. one point p_i), the group Γ is identified with the Mordell-Weil group of the elliptic surface and its action with the translation action. Let

$$\overline{\phi}\colon S(3)/\Gamma \longrightarrow \mathbf{P}^1, \qquad \overline{\pi}\colon S(3)/\Gamma \longrightarrow \mathbf{P}^2/\Gamma,$$

be the morphisms induced by ϕ and π, respectively.

PROPOSITION 5.1. *The quotient surfaces $\mathbf{P}^2/\Gamma$ and $S(3)/\Gamma$ have 4 singular points of type A_2 given by the orbits of the vertices (7). The minimal resolution of singularities are isomorphic to a Del Pezzo surface S of degree 1 and to $S(3)$, respectively. Up to these resolutions, $\overline{\phi}$ is isomorphic to ϕ and $\overline{\pi}$ is the blowing up of S in one point, the Γ-orbit of the points p_i.*

Proof. The group Γ preserves each singular member of the Hesse pencil and any of its subgroups of order 3 leaves invariant the vertices of one of the triangles. Without loss of generality we may assume that the triangle is $xyz = 0$. Then the subgroup of Γ stabilizing its vertices is generated by the transformation g_2, which acts locally at the point $y = z = 0$ by the formula

$(y, z) \mapsto (\epsilon y, \epsilon^2 z)$. It follows that the orbits of the vertices give 4 singular points of type A_2 in $\mathbf{P}^2/\Gamma$ and $S(3)/\Gamma$, locally given by the equation $uv+w^3 = 0$.

Let E be an elliptic curve with a group law and let $[n]\colon E \to E$ be the map $x \mapsto nx$. It is known that this map is a surjective map of algebraic groups with kernel equal to the group of n-torsion points. Its degree is n^2 if n is coprime to the characteristic. In our case the quotient map by Γ acts on each member of the Hesse pencil as the map $[3]$. This implies that the quotient of the surface $S(3)$ by the group Γ is isomorphic to $S(3)$ over the open subset $U = \mathbf{P}^1 \setminus \{\Delta(t_0, t_1) = 0\}$.

The map $\overline{\phi}\colon S(3)/\Gamma \to \mathbf{P}^1$ induced by the map ϕ has four singular fibres. Each fibre is an irreducible rational curve with a double point which is a singular point of the surface of type A_2. Let $\sigma\colon S(3)' \to S(3)/\Gamma$ be a minimal resolution of the four singular points of $S(3)/\Gamma$. The composition $\overline{\phi} \circ \sigma\colon S(3)' \to \mathbf{P}^1$ is an elliptic surface isomorphic to $\phi\colon S(3) \to \mathbf{P}^1$ over the open subset U of the base $\mathbf{P}^1$. Moreover, $\overline{\phi} \circ \sigma$ and ϕ have singular fibres of the same types, thus $S(3)'$ is a minimal elliptic surface. Since it is known that a birational isomorphism of minimal elliptic surfaces is an isomorphism, this implies that $\overline{\phi} \circ \sigma$ is isomorphic to ϕ.

The minimal resolution S of $\mathbf{P}^2/\Gamma$ contains a pencil of cubic curves intersecting in one point q_0, the orbit of the points p_i. Hence it easily follows (see for example [10]) that S is isomorphic to a Del Pezzo surface of degree one and $\overline{\pi}$ is the blowing up of the point q_0.

Let $\pi'\colon S(3)' \to \mathbf{P}^2$ be the contraction of the 9 sections $E_0, \dots, E_8$ of the elliptic fibration $\overline{\phi} \circ \sigma$ to the points $q_0, \dots, q_8$ in $\mathbf{P}^2$, the base points of the Hesse pencil in the second copy of $\mathbf{P}^2$.

By Proposition 5.1 the following diagram is commutative:

$$
\begin{tikzcd}
S(3) \arrow[r, "r"] \arrow[d, "\pi"'] & S(3)/\Gamma \arrow[d, "\overline{\pi}"'] & S(3)' \arrow[l, "\sigma"'] \arrow[d, "\alpha"'] \arrow[dr, "\pi'"] & \\
\mathbf{P}^2 \arrow[r, "p"] & \mathbf{P}^2/\Gamma & S \arrow[l, "\beta"'] \arrow[r, "\gamma"] & \mathbf{P}^2 .
\end{tikzcd}
\tag{12}
$$

Here p is the quotient map by Γ, β is a minimal resolution of singularities of the orbit space $\mathbf{P}^2/\Gamma$, α is the blow-up of the point q_0 on S, and γ is the blow-up of $q_1, \dots, q_8$ (see the notation in the proof of Proposition 5.1).

PROPOSITION 5.2. *The curves* $B_i = p^{-1}(\beta(\alpha(E_i)))$, $i = 1, \dots, 8$, *are plane cubic curves with equations*

$$\begin{array}{llll} B_1: & x^3+\epsilon y^3+\epsilon^2 z^3=0\,, & B_5: & x^3+\epsilon^2 y^3+\epsilon z^3=0\,,\\ B_2: & x^2y+y^2z+z^2x=0\,, & B_6: & x^2z+y^2x+z^2y=0\,,\\ B_3: & x^2y+\epsilon^2y^2z+\epsilon z^2x=0\,, & B_7: & x^2z+\epsilon y^2x+\epsilon^2z^2y=0\,,\\ B_4: & x^2y+\epsilon y^2z+\epsilon^2z^2x=0\,, & B_8: & x^2z+\epsilon^2y^2x+\epsilon z^2y=0\,. \end{array}$$

The union of the eight cubics B_i *cuts out on each nonsingular member of the Hesse pencil the set of points of order* 9 *in the group law with the point* p_0 *as the origin.*

Each of them has one of the triangles of the Hesse pencil as inflection triangle and is inscribed and circumscribed to the other three triangles (i.e. is tangent to one side of the triangle at each vertex).

Proof. Recall that the sections $E_1, \dots, E_8$ on $S(3)'$ are non-trivial 3-torsion sections (the zero section is equal to E_0). The preimage $\bar{B}_i$ of E_i under the map $r^{-1}\circ\sigma$ cuts out on each nonsingular fibre the Γ-orbit of a point of order 9. Thus the image B_i of $\bar{B}_i$ in $\mathbf{P}^2$ is a plane cubic cutting out the Γ-orbit of a point of order 9 on each nonsingular member of the Hesse pencil.

Let E be a nonsingular member of the Hesse pencil. Take a point $p \in E$ and let $q \neq p$ be the intersection of E with the tangent line at p. Let $r \neq q$ be the intersection of E with the tangent line at q. Finally, let $s \neq r$ be the intersection of E with the tangent line at r. It follows from the definition of the group law that we have $2p \oplus q = 2q \oplus r = 2r \oplus s = 0$. This immediately implies that $9p = 0$ if and only if $p = s$ (this explains why the classical authors called a point of order 9 a *coincidence point*). The triangle formed by the lines $\overline{p,q}$, $\overline{q,r}$, $\overline{r,p}$ is inscribed and circumscribed to E. Following Halphen [23], we will use this observation to find the locus of points of order 9.

The tangent line of E at $p = (x_0, y_0, z_0)$ has the equation

$$(x_0^2+ty_0z_0)x+(y_0^2+tx_0z_0)y+(z_0^2+tx_0y_0)z=0\,,$$

where we assume that $E = E_{3t}$. The point $q = (x_0, \epsilon y_0, \epsilon^2 z_0)$ lies on E because $(x_0, y_0, z_0) \in E$; it also lies on the tangent line at p if $p = (x_0, y_0, z_0)$ satisfies the equation

$$B_1\colon x^3+\epsilon y^3+\epsilon^2 z^3=0\,. \tag{13}$$

If p satisfies this equation, then q also satisfies it, hence $r = (x_0, \epsilon^2 y_0, \epsilon z_0)$ lies on the tangent at q and again satisfies (13). If we repeat this procedure we return to the original point p. Hence we see that any point in $B_1 \cap E$ is a point of order 9. Now we apply the elements of the Hessian group to the curve B_1 in order to get the remaining cubic curves $B_2, \dots, B_8$. Notice that the stabilizer of B_1 in the Hessian group is generated by Γ and g_4. It is a Sylow 3-subgroup of the Hessian group isomorphic to a semi-direct product $\Gamma \rtimes \mathbf{Z}/3\mathbf{Z}$.

To check the last assertion it is enough, using the G_{216}-action, to consider one of the curves B_i. For example, we see that the triangle T_1 of equation $xyz = 0$ is an inflection triangle of the curve B_1 and that the triangles T_2, T_3, T_4 are inscribed and circumscribed to B_1. More precisely we have the following configuration:

i) B_i and B_{i+4} have T_i as a common inflection triangle and they intersect in the 9 vertices of the other triangles;

ii) B_i and B_j, $i \neq j$, $i, j \leq 4$, intersect in the 3 vertices of a triangle T_k and are tangent in the 3 vertices of T_ℓ with $k, \ell \notin \{i, j\}$;

iii) B_i and B_{j+4}, $i \neq j$, $i, j \leq 4$, intersect similarly with k and ℓ interchanged.

For example, B_1 and B_2 intersect in the vertices of T_3 and are tangent in the vertices of T_4, while B_1 and B_6 intersect transversally on T_4 and are tangent on T_3. □

We will call the cubics B_i the *Halphen cubics*. Observe that the element g_0 from the Hessian group sends B_i to B_{i+4}. We will call the pairs $(B_i, B_i' = B_{i+4})$ the *pairs of Halphen cubics* and we will denote by q_i, $q_i' = q_{i+4}$ the corresponding pairs of points in $\mathbf{P}^2$.

It can easily be checked that the projective transformations g_3, g_4 act on the Halphen cubics as follows (with an obvious notation):

$$g_3: \quad (121'2')(434'3'), \qquad g_4: \quad (243)(2'4'3').$$

REMARK 5.3. The linear representation of Γ on the space of homogeneous cubic polynomials decomposes into the sum of one-dimensional eigensubspaces. The cubic polynomials defining B_i together with the polynomials xyz, $x^3 + y^3 + z^3$ form a basis of eigenvectors. Moreover, note that the cubics B_i are equianharmonic cubics. In fact, they are all projectively equivalent to B_1, which is obviously isomorphic to the Fermat cubic. We refer to [2], [3] where the Halphen cubics play a role in the construction of bielliptic surfaces in $\mathbf{P}^4$.

REMARK 5.4. According to G. Halphen [23], the rational map

$$\gamma \circ \beta^{-1} \circ p\colon \mathbf{P}^2 \dashrightarrow \mathbf{P}^2$$

can be given explicitly by

$$(x, y, z) \mapsto (P_2'P_3'P_4',\ P_2P_3P_4,\ xyz\,P_1P_1'),$$

where P_i, P_i' are the polynomials defining B_i, B_i' as in Proposition 5.2. His paper [23], besides many other interesting results, describes the locus of m-torsion points of nonsingular members of the Hesse pencil (see [19] for a modern treatment of this problem).

REMARK 5.5. In characteristic 3 the cyclic group of projective transformations generated by g_1 acts on nonsingular members of the Hesse pencil as translation by 3-torsion points with the zero point taken to be $(1, -1, 0)$. The polynomials

$$(X, Y, Z, W) = (x^2y + y^2z + z^2x,\ xy^2 + yz^2 + zx^2,\ x^3 + y^3 + z^3,\ xyz)$$

are invariant with respect to g_1 and map $\mathbf{P}^2$ onto a cubic surface in $\mathbf{P}^3$ given by the equation (see [22], (3.1))

$$X^3 + Y^3 + Z^2W = XYZ. \tag{14}$$

Among the singular points of the cubic surface, $(0, 0, 0, 1)$ is a rational double point of type $E_6^{(1)}$ in Artin's notation [1]. The image of the member E_λ of the Hesse pencil is the plane section $Z + \lambda W = 0$. Substituting in equation (14), we find that the image of this pencil of plane sections under the projection from the singular point is the Hesse pencil. The parameter λ of the original pencil and the new parameter λ' are related by $\lambda = \lambda'^3$.

6. THE 8-CUSPIDAL SEXTIC

Let C_6 be the sextic curve with equation $\Phi_6 = 0$, where Φ_6 is the degree six invariant of the Hessian group. This is a smooth curve and one immediately verifies that it does not contain the vertices of the inflection triangles $T_1, \dots, T_4$ given in (7) or the base points of the Hesse pencil.

This shows that the preimage $\overline{C}_6 = \pi^{-1}(C_6)$ of C_6 in the surface $S(3)$ is isomorphic to C_6 and that the group Γ acts on $\overline{C}_6$ freely. The orbit space $\overline{C}_6/\Gamma$ is a smooth curve of genus 2 in $S(3)/\Gamma$ which does not pass through the singular points and does not contain the orbit of the section $\pi^{-1}(p_0)$. Its preimage under σ is a smooth curve $\overline{C}_6'$ of genus 2 in $S(3)'$ that intersects a general fibre of the Hesse pencil at 2 points. Observe that the curve C_6 is tangent to each Halphen cubic B_i, B_i' at a Γ-orbit of 9 points. In fact, it is enough to check that C_6 is tangent to one of them, say B_1, at some point. We have

$$\begin{aligned} x^6 + y^6 + z^6 &- 10(x^3y^3 + x^3z^3 + y^3z^3) \\ &= (x^3 + y^3 + z^3)^2 - 12(x^3y^3 + x^3z^3 + y^3z^3) \\ &= -3(x^3 + y^3 + z^3)^2 + 4(x^3 + \epsilon y^3 + \epsilon^2 z^3)(x^3 + \epsilon^2 y^3 + \epsilon z^3). \end{aligned}$$

This shows that the curves B_1 and B_1' are tangent to C_6 at the points where C_6 intersects the curve $E_0 : x^3 + y^3 + z^3 = 0$. The map $\pi' \colon S(3)' \to \mathbf{P}^2$ blows down the curves E_i, $i = 1, \dots, 8$, to the base points $q_1, \dots, q_8$, of the Hesse pencil. Hence the image C_6' of $\overline{C}_6'$ in $\mathbf{P}^2$ is a curve of degree 6 with cusps at the points $q_1, \dots, q_8$.

PROPOSITION 6.1. *The 8-cuspidal sextic C_6' is projectively equivalent to the sextic curve defined by the polynomial*

$$\Phi_6'(x,y,z) = (x^3+y^3+z^3)^2 - 36y^3z^3 + 24(z^4y^2+z^2y^4) - 12(z^5y+zy^5) - 12x^3(z^2y+zy^2).$$

Proof. In an appropriate coordinate system the points q_i have the same coordinates as the p_i's. By using the action of the group Γ, we may assume that the sextic has cusps at $p_1, \dots, p_8$. Let V be the vector space of homogeneous polynomials of degree 6 vanishing at $p_1, \dots, p_8$ with multiplicity ≥ 2. If S is the blowing-up of $q_1, \dots, q_8$ and K_S is its canonical bundle, then $\mathbf{P}(V)$ can be identified with the linear system $|-2K_S|$. It is known that the linear system $|-2K_S|$ is of dimension 3 (see [12]) and defines a regular map of degree 2 from S to $\mathbf{P}^3$ with the image a singular quadric.

A basis of V can be found by considering the product of six lines among the 12 inflection lines. In this way one finds the following sextic polynomials

$$\begin{aligned} (15)\quad A_1 &= yz(x + \epsilon y + z)(x + y + \epsilon z)(x + \epsilon^2 y + z)(x + y + \epsilon^2 z), \\ A_2 &= yz(x + \epsilon y + \epsilon^2 z)(x + \epsilon^2 y + \epsilon z)(x + y + \epsilon z)(x + \epsilon y + z), \\ A_3 &= yz(x + \epsilon^2 y + z)(x + y + \epsilon^2 z)(x + \epsilon y + \epsilon^2 z)(x + \epsilon^2 y + \epsilon z), \\ A_4 &= (x + \epsilon y + \epsilon^2 z)(x + \epsilon^2 y + \epsilon z)(x + y + \epsilon z) \\ &\qquad \times (x + \epsilon y + z)(x + \epsilon^2 y + z)(x + y + \epsilon^2 z). \end{aligned}$$

A polynomial $P(x,y,z)$ defining the curve C_6' is invariant with respect to the linear representation of the binary tetrahedral group $\bar{T} \cong \mathrm{SL}(2, \mathbf{F}_3)$ in V.

This representation decomposes into the direct sum of the 3-dimensional representation isomorphic to the second symmetric power of the standard representation of $2.A_4$ in $\mathbf{C}^2$ and a one-dimensional representation spanned by $P(x,y,z)$. Applying g_4 we find that

$$(A_1, A_2, A_3, A_4) \mapsto (\epsilon^2 A_2, \epsilon^2 A_3, \epsilon^2 A_1, A_4).$$

Thus $P(x,y,z) = \lambda(A_1 + \epsilon^2 A_2 + \epsilon A_3) + \mu A_4$ for some constants λ, μ. Now we apply g_3 and find λ, μ such that $P(x,y,z)$ is invariant. A simple computation gives the equation of C_6'.

REMARK 6.2. The geometry of the surface S, the blow-up of $\mathbf{P}^2$ at $q_1, \ldots, q_8$, is well-known. We now present several birational models of this surface and relations between them.

The surface S is a Del Pezzo surface of degree 1 and admits a birational morphism $\psi \colon S \to \bar{S}$ onto a surface in the weighted projective space $\mathbf{P}(1,1,2,3)$ given by an equation

$$-u_3^2 + u_2^3 + A(u_0, u_1)u_2 + B(u_0, u_1) = 0, \tag{16}$$

where (u_0, u_1, u_2, u_3) have weights $1,1,2,3$ (see [12]). The morphism ψ is an isomorphism outside of the union of the 8 lines $\ell_1, \ldots, \ell_8$ which correspond to factors of the polynomials $A_1, \ldots, A_4$ from (15). In fact, the map ψ is a resolution of indeterminacy points of the rational map $\overline{\psi} \colon \mathbf{P}^2 \dashrightarrow \mathbf{P}(1,1,2,3)$. It is given by the formulae

$$(x,y,z) \mapsto (u_0, u_1, u_2, u_3) = (-xyz,\ x^3 + y^3 + z^3,\ \Phi_6'(x,y,z),\ P_9(x,y,z)),$$

where $P_9(x,y,z) = 0$ is the union of the line $\ell_0 : y - z = 0$ and the 8 lines $\ell_1, \ldots, \ell_8$. Explicitly,

$$P_9(x,y,z) = yz(y-z)(x^6 + x^3(2y^3 - 3y^2z - 3yz^2 + 2z^3)) + (y^3 - yz + z^2)^3.$$

Up to some constant factors, the polynomials A, B are the same as in (5). The 8 lines are blown down to singular points of the surface.

The composition of $\overline{\psi}$ with the projection $(u_0, u_1, u_2, u_3) \mapsto (u_0^2, u_0u_1, u_1^2, u_2)$ gives the rational map $\mathbf{P}^2 \dashrightarrow \mathbf{P}^3$ defined by

$$(x,y,z) \mapsto (u_0, u_1, u_2, u_3) = (x^2y^2z^2, xyz(x^3 + y^3 + z^3),\ (x^3 + y^3 + z^3)^2,\ \Phi_6').$$

This is a $2.A_4$-equivariant map of degree 2 onto the quadric cone $u_0u_2 - u_1^2 = 0$. The ramification curve is the line $y - z = 0$ and the branch curve is the intersection of the quadric cone and a cubic surface. This is a curve W of degree 6 with 4 ordinary cuspidal singularities lying on the hyperplane $u_3 = 0$.

Consider the rational map $\phi = \gamma \circ \beta^{-1} \circ p\colon \mathbf{P}^2 \dashrightarrow \mathbf{P}^2$ from diagram (12). It follows from the description of the maps in the diagram that the preimage of the Hesse pencil is a Hesse pencil, the preimage of the curve C_6' is the curve C_6, and the preimage of the union of the lines $\ell_0, \ell_1, \ldots, \ell_8$ is the union of harmonic polars. This shows that the composition $\psi \circ \phi\colon \mathbf{P}^2 \dashrightarrow \mathbf{P}(1,1,2,3)$ can be given by the formulae

$$(x,y,z) \mapsto (xyz,\ x^3+y^3+z^3,\ \Phi_6(x,y,z),\ \Phi_9(x,y,z)),$$

where $\Phi_9(x,y,z)$ is the invariant of degree 9 for the group $\overline{G}_{216}$ given in §4. This agrees with a remark of van Geemen in [53] that the polynomials xyz, $x^3+y^3+z^3$, $\Phi_6(x,y,z)$, and $\Phi_9(x,y,z)$ satisfy the same relation (16) as the polynomials xyz, $x^3+y^3+z^3$, $\Phi_6'(x,y,z)$, and $P_9(x,y,z)$. Using the standard techniques of invariant theory of finite groups one can show that the polynomials xyz, $x^3+y^3+z^3$, $\Phi_6(x,y,z)$, and $\Phi_9(x,y,z)$ generate the algebra of invariants of the Heisenberg group $\mathcal{H}_3(3)$, the preimage of Γ in $\mathrm{SL}(3,\mathbf{F}_3)$. The equations of S with respect to different sets of generators were given in [7] and [54].

Finally, we explain the geometric meaning of the intersection points of the sextic curve C_6 with a nonsingular member E_λ of the Hesse pencil. This set of intersection points is invariant with respect to the translation group Γ and the involution g_0, thus its image in $C_6' = C_6/\Gamma$ consists of two points on the curve E_λ. These points lie on the line through the point p_0 because they differ by the negation involution g_0 on E_λ in the group law with the zero point p_0.

PROPOSITION 6.3. *The curves C_6' and E_λ intersect at two points p, q outside the base points $p_1, \ldots, p_8$. These points lie on a line through p_0 which is the tangent line to the Hessian cubic* $\mathrm{He}(E_\lambda)$ *at p_0. The 18 points in $C_6 \cap E_\lambda$ are the union of the two Γ-orbits of p and q.*

Proof. This is checked by a straightforward computation. By using MAPLE® we find that the curves C_6', E_λ and the tangent line to $E_{\mathfrak{h}(\lambda)}$ at p_0 have two intersection points.

7. A $K3$ SURFACE WITH AN ACTION OF G_{216}

In the previous sections we introduced two plane sextics, C_6 and C_6', which are naturally related to the Hesse configuration. The double cover of $\mathbf{P}^2$ branched along any of these curves is known to be birationally isomorphic to a *K3 surface*, i.e. a simply connected compact complex surface with trivial canonical bundle. This follows from the formula for the canonical sheaf of a double cover $f\colon Y \to \mathbf{P}^2$ of the projective plane branched along a plane curve of degree $2d$

$$\omega_Y = f^*(\omega_{\mathbf{P}^2} \otimes \mathcal{O}_{\mathbf{P}^2}(d))$$

and the fact that the singular points of Y are rational double points, i.e. they can be characterized by the condition $\pi^*(\omega_Y) \cong \omega_X$, where $\pi\colon X \to Y$ is a minimal resolution of singularities.

In the following sections we will study the geometry of the $K3$ surfaces associated to C_6 and C_6'; in particular we will show how the symmetries of the Hesse configuration can be lifted to the two surfaces. We start by presenting some basic properties of $K3$ surfaces and their automorphisms (see for example [5] and [40]).

Since the canonical bundle is trivial, the vector space $\Omega^2(X)$ of holomorphic 2-forms on a $K3$ surface X is one-dimensional. Moreover, the cohomology group $L = H^2(X, \mathbf{Z})$ is known to be a free abelian group of rank 22. The cup-product equips L with a structure of a *quadratic lattice*, i.e. a free abelian group together with an integral quadratic form. The quadratic form is unimodular and its signature is $(3, 19)$. The sublattice $S_X \subset L$ generated by the fundamental cocycles of algebraic curves on X is called the *Picard lattice* and has signature equal to $(1, k)$. Its orthogonal complement T_X in L is the *transcendental lattice* of X.

Any automorphism g of X clearly acts on $\Omega^2(X)$ and also induces an isometry g^* on L which preserves S_X and T_X. An automorphism g that acts identically on $\Omega^2(X)$ is called *symplectic*. We recall here a result proved in [43].

THEOREM 7.1. *Let g be an automorphism of finite order on a $K3$ surface X.*

i) *If g is symplectic then g^* acts trivially on T_X and its fixed locus is a finite union of points. The quotient surface $X/(g)$ is birational to a $K3$ surface.*

ii) *If g is not symplectic then g^* acts on $\Omega^2(X)$ as the multiplication by a primitive r-th root of unity and its eigenvalues on $T_X \otimes \mathbf{C}$ are the primitive r-th roots of unity. Moreover, if the fixed locus is not empty, then the quotient $X/(g)$ is a rational surface.*

Let $q\colon X \dashrightarrow \mathbf{P}^2$ be the double cover branched along C_6. We now prove that the action of the Hessian group on the projective plane lifts to an action on X. We denote by Q_8 the 2-Sylow subgroup of $\mathrm{SL}(2, \mathbf{F}_3)$, isomorphic to the quaternion group.

PROPOSITION 7.2. *The Hessian group G_{216} is isomorphic to a group of automorphisms of the K3 surface X. Under this isomorphism, any automorphism in the normal subgroup $H_{72} = \Gamma \rtimes Q_8$ is symplectic.*

Proof. The double cover $q\colon X \to \mathbf{P}^2$ branched along the curve C_6 can be defined by the equation

$$w^2 + \Phi_6(x, y, z) = 0,$$

considered as a weighted homogeneous polynomial with weights $(1, 1, 1, 3)$. Thus we can consider X as a hypersurface of degree 6 in the weighted projective space $\mathbf{P}(1, 1, 1, 3)$.

Let G'_{216} be the preimage of G_{216} in $\mathrm{SL}(3, \mathbf{C})$ considered in Section 4 and let g'_i $(i = 1, \ldots, 4)$ be the lifts of the generators g_i in G'_{216}. It is checked immediately that the generators g'_1, g'_2, g'_3 leave the polynomial Φ_6 invariant and g'_4 multiplies Φ_6 by ϵ^2. Thus the group G'_{216} acts on X by the formula

$$g_i(x, y, z, w) = (g'_i(x, y, z), w) \quad \text{for } i \neq 4, \qquad g_4(x, y, z, w) = (g'_4(x, y, z), \epsilon w).$$

The kernel of $G'_{216} \to G_{216}$ is generated by the scalar matrix $(\epsilon, \epsilon, \epsilon)$, which acts as the identity transformation on X. Then it is clear that the induced action of G_{216} on X is faithful.

The subgroup H_{72} of G_{216} is generated by the transformations g_1, g_2, g_3, $g_4 g_3 g_4^{-1}$. To check that it acts symplectically on X we recall that the space of holomorphic 2-forms on a hypersurface $F(x_0, \ldots, x_n)$ of degree d in $\mathbf{P}^n$ is generated by the residues of the meromorphic n-forms on $\mathbf{P}^n$ of the type

$$\omega = \frac{P}{F} \sum_{i=0}^{n} (-1)^i x_i \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n,$$

where P is a homogeneous polynomial of degree $d - n - 1$. This is easily generalized to the case of hypersurfaces in a weighted projective space $\mathbf{P}(q_0, \ldots, q_n)$. In this case the generating forms are

$$\omega = \frac{P}{F} \sum_{i=0}^{n} (-1)^i q_i x_i \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n,$$

where $\deg P = d - q_0 - \cdots - q_n$.

In our case $d = q_0 + q_1 + q_2 + q_3 = 6$, hence there is only one form, up to proportionality. It is given by

$$\omega = \frac{x\,dy \wedge dz \wedge dw - y\,dx \wedge dz \wedge dw + z\,dx \wedge dy \wedge dw - 3w\,dx \wedge dy \wedge dz}{w^2 + \Phi_6(x, y, z)}.$$

It is straightforward to check that the generators of H_{72} leave this form invariant (cf. [41], p. 193).

REMARK 7.3. The action of $\Gamma \rtimes Q_8$ appears as Example 0.4 in the paper of S. Mukai [41] containing the classification of maximal finite groups of symplectic automorphisms of complex $K3$ surfaces.

Let P_i, P'_i $(i = 1, \dots, 4)$ be the polynomials defining the cubics B_i, B_{i+4} as given in Section 5 and F_i be the equations of the equianharmonic cubics in the Hesse pencil:

$$F_i(x, y, z) = x^3 + y^3 + z^3 + \alpha_i xyz \qquad (i = 1, \dots, 4),$$

where $\alpha_1 = 0$ and $\alpha_i = 6\epsilon^{2-i}$ for $i = 2, 3, 4$ (see Section 2).

PROPOSITION 7.4. *The $K3$ surface X is isomorphic to the hypersurface of bidegree $(2, 3)$ in $\mathbf{P}^1 \times \mathbf{P}^2$ with equation*

$$u^2 P_i(x, y, z) + v^2 P'_i(x, y, z) + \sqrt{3}uvF_i(x, y, z) = 0 \tag{17}$$

for any $i = 1, \dots, 4$.

Proof. As noticed in the previous section we can write

$$\Phi_6 = \det \begin{pmatrix} 2P_1 & \sqrt{3}F_1 \\ \sqrt{3}F_1 & 2P'_1 \end{pmatrix} = -3F_1^2 + 4P_1P'_1 .$$

The $K3$ surface Y given by the bihomogeneous equation of bidegree $(2, 3)$ in $\mathbf{P}^1 \times \mathbf{P}^2$

$$u^2 P_1(x, y, z) + v^2 P'_1(x, y, z) + \sqrt{3}uvF_1(x, y, z) = 0 \tag{18}$$

is a double cover of $\mathbf{P}^2$ with respect to the projection to the second factor and its branch curve is defined by $\Phi_6 = 0$. Thus Y is isomorphic to the $K3$ surface X. By acting on equation (18) with the Hessian group G_{216} we find analogous equations for X in $\mathbf{P}^1 \times \mathbf{P}^2$ in terms of the polynomials P_i, P'_i and F_i for $i = 2, 3, 4$.

An important tool for understanding the geometry of $K3$ surfaces is the study of their elliptic fibrations. We recall that the fibration is called *jacobian* if it has a section.

PROPOSITION 7.5. *The $K3$ surface X has 4 pairs of elliptic fibrations*

$$\mathfrak{h}_i, \mathfrak{h}_i' : X \longrightarrow \mathbf{P}^1 \qquad (i = 1, \dots, 4)$$

with the following properties:

a) *$\mathfrak{h}_i$ and $\mathfrak{h}_i'$ are exchanged by the covering involution of q and G_{216} acts transitively on $\mathfrak{h}_1, \dots, \mathfrak{h}_4$;*

b) *the j-invariant of any smooth fibre of $\mathfrak{h}_i$ or $\mathfrak{h}_i'$ is equal to zero;*

c) *each fibration has 6 reducible fibres of Kodaira's type IV, i.e. the union of three smooth rational curves intersecting at one point. The singular points in the reducible fibres of $\mathfrak{h}_i$ and $\mathfrak{h}_i'$ are mapped by q to the vertices of the triangle T_i;*

d) *each fibration is jacobian.*

Proof. Consider the equations (17) for X in $\mathbf{P}^1 \times \mathbf{P}^2$. The projections on the first factor $\mathfrak{h}_i : X \to \mathbf{P}^1$, $i = 1, \dots, 4$ are elliptic fibrations on X since the fibre over a generic point (u, v) is a smooth plane cubic. A second set of elliptic fibrations on X is given by $\mathfrak{h}_i' = \mathfrak{h}_i \circ \sigma$, where σ is the covering involution of q. Since all these fibrations are equivalent modulo the group generated by σ and G_{216}, it will be enough to prove properties b), c) and d) for $\mathfrak{h}_1$.

The fibre of $\mathfrak{h}_1$ over a point (u, v) is isomorphic to the plane cubic defined by equation (18). This equation can be also written in the form

$$(u^2 + v^2 + \sqrt{3}uv)x^3 + (\epsilon u^2 + \epsilon^2 v^2 + \sqrt{3}uv)y^3 + (\epsilon^2 u^2 + \epsilon v^2 + \sqrt{3}uv)z^3 = 0.$$

Hence it is clear that all smooth fibres of $\mathfrak{h}_1$ are isomorphic to a Fermat cubic i.e. they are equianharmonic cubics. This system of plane cubics contains exactly 6 singular members corresponding to the vanishing of the coefficients at x^3, y^3 and z^3. Each of them is equal to the union of three lines meeting at one point and defines six singular fibres of type IV of the elliptic fibration $\mathfrak{h}_1$. The singular points of these reducible fibres are the inverse images of the vertices v_0, v_1, v_2 of the triangle T_1 under the map q (see (7) in Section 2). This proves assertions b) and c).

It remains to show that the elliptic fibration $\mathfrak{h}_1$ has a section. We thank N. Elkies for explicitly finding such a section. It is given by

$$(x, y, z) = \big((1 - \epsilon)u + d_0 v, (1 - \epsilon)u + d_2 v, (1 - \epsilon)u + d_1 v\big),$$

where $(d_0, d_1, d_2) = ig_3(-\sqrt[3]{4}, \epsilon + 1, \epsilon\sqrt[3]{2})$.

REMARK 7.6. Consider the map

$$\mathbf{P}^2 \longrightarrow \mathbf{P}^2, \qquad (x,y,z) \longmapsto (x^3,y^3,z^3).$$

The image of the curve C_6 is a conic T and the preimages of the tangent lines to T are plane cubics that are everywhere tangent to C_6. The map induces a degree 9 morphism from X to $\mathbf{P}^1 \times \mathbf{P}^1$ isomorphic to the double cover of $\mathbf{P}^2$ branched along T. The projections to the two factors give the fibrations $\mathfrak{h}_1$ and $\mathfrak{h}_1'$.

Note that each family of everywhere tangent cubics to C_6 corresponds to an even theta characteristic θ on C_6 with $h^0(\theta) = 2$.

Let $\pi\colon Y \to \mathbf{P}^1$ be a jacobian elliptic fibration on a $K3$ surface Y. The fibre of π over the generic point η is an elliptic curve Y_η over the field of rational functions K of $\mathbf{P}^1$. The choice of a section E of π fixes a K-rational point on Y_η and hence allows one to find a birational model of Y_η given by a Weierstrass equation $y^2 - x^3 - ax - b = 0$, where $a, b \in K$. The construction of the Weierstrass model can be "globalized" to obtain the following birational model of Y (see [10]).

PROPOSITION 7.7. *There exists a birational morphism $f\colon Y \to W$, where W is a hypersurface in the weighted projective space $\mathbf{P}(1,1,4,6)$ given by an equation of degree* 12

$$y^2 - x^3 - A(u,v)x - B(u,v) = 0,$$

with $A(u,v), B(u,v)$ binary forms of degrees 8 *and* 12 *respectively. Moreover:*

1. *The image of the section E is the point $p = (0,0,1,1) \in W$. The projection $(u,v,x,y) \mapsto (u,v)$ from p gives an elliptic fibration $\pi'\colon W' \to \mathbf{P}^1$ on the blow-up W' of W with center at p. It has a section defined by the exceptional curve E' of the blow-up.*

2. *The map f extends to a birational morphism $f'\colon Y \to W'$ over $\mathbf{P}^1$ which maps E onto E' and blows down irreducible components of fibres of π which are disjoint from E to singular points of W'.*

3. *Each singular point of W' is a rational double point of type A_n, D_n, E_6, E_7 or E_8. A singular point of type A_n corresponds to a fibre of π of Kodaira type I_{n+1}, III (if $n = 1$), or IV (if $n = 2$). A singular point of type D_n corresponds to a fibre of type I_{n+4}^*. A singular point of type E_6, E_7, E_8 corresponds to a fibre of type IV^*, III^*, II^* respectively.*

The elliptic surface W is determined uniquely, up to isomorphism, by the elliptic fibration on Y. It is called the *Weierstrass model* of the elliptic fibration π.

It is easy to find the Weierstrass model of the elliptic fibration $\mathfrak{h}_1\colon X \to \mathbf{P}^1$ on our surface X.

LEMMA 7.8. *The Weierstrass model of the elliptic fibration* $\mathfrak{h}_1$ *is given by the equation*

$$y^2 - x^3 - (u^6 + v^6)^2 = 0\,.$$

Proof. We know from Proposition 7.5 that the j-invariant of a general fibre of $\mathfrak{h}_1$ is equal to zero. This implies that the coefficient $A(u,v)$ in the Weierstrass equation is equal to zero. We also know that the fibration has 6 singular fibres of type IV over the zeros of the polynomial

$$(u^2 + v^2 + \sqrt{3}uv)(\epsilon u^2 + \epsilon^2 v^2 + \sqrt{3}uv)(\epsilon^2 u^2 + \epsilon v^2 + \sqrt{3}uv) = u^6 + v^6\,.$$

Since each of the fibres is of Kodaira type IV, the singularity of W over a root of $u^6 + v^6$ must be a rational double point of type A_3, locally isomorphic to the singularity $y^2+x^3+z^2$. This easily implies that the binary form $B(u,v)$ is equal to $(u^6+v^6)^2$ up to a scalar factor which does not affect the isomorphism class of the surface.

LEMMA 7.9. *Let Y be a $K3$ surface with Picard number* 20 *having a non-symplectic automorphism of order* 3. *Then the intersection matrix of* T_Y *with respect a suitable basis is given by*

$$A_2(-m) = \begin{pmatrix} 2m & m \\ m & 2m \end{pmatrix}, \tag{19}$$

for some $m \in \mathbf{Z}$, $m > 0$.

Proof. Let f be a non-symplectic automorphism of order 3 on Y. By Theorem 7.1 ii), f^* acts on $T_Y \otimes \mathbf{C}$ with eigenvalues ϵ, ϵ^2. Let $x \in T_Y$, $x \neq 0$, then

$$0 = (x + f^*(x) + (f^*)^2(x), f^*(x)) = 2(x, f^*(x)) + x^2\,.$$

Note that $x^2 = 2m$ for some positive integer m because the lattice T_Y is even and positive definite. Then the intersection matrix of T_Y with respect to the basis $x, -f^*(x)$ is $A_2(-m)$. See also Lemma 2.8 in [45].

The proof of the following theorem follows a suggestion of M. Schuett.

THEOREM 7.10. *The intersection matrix of the transcendental lattice of the $K3$ surface X with respect to a suitable basis is*

$$A_2(-6) = \begin{pmatrix} 12 & 6 \\ 6 & 12 \end{pmatrix}.$$

Proof. Consider the automorphism σ of order 6 of X that acts on the Weierstrass model by the formula $(u, v, x, y) \mapsto (\eta u, v, \eta^2 x, \eta^3 y)$, where $\eta = e^{\pi i/3}$. It is easy to see that σ acts freely outside the union of the two nonsingular fibres F_0, F_∞ over the points $(u, v) = (1, 0)$ and $(0, 1)$. The action of the cyclic group $G = \langle\sigma\rangle$ on each of the fibres is an automorphism of order 6 such that G has one fixed point, $\langle\sigma^3\rangle$ has 4 fixed points and $\langle\sigma^2\rangle$ has 3 fixed points.

Let X/G be the orbit space. The images $\bar{F}_0$ and $\bar{F}_\infty$ of F_0 and F_∞ in X/G are smooth rational curves and X/G has 3 singular points on each of these curves, of types A_5, A_3 and A_2. A minimal resolution of X/G is a $K3$ surface Y. The elliptic fibration $\mathfrak{h}_1$ on X defines an elliptic fibration $\rho\colon Y \to \mathbf{P}^1$ with two fibres of type II^*, equal to preimages of $\bar{F}_0$ and $\bar{F}_\infty$ on Y, and one fibre of type IV, the orbit of the six singular fibres of $\mathfrak{h}_1$.

It is easy to compute the Picard lattice S_Y of Y. Its sublattice generated by irreducible components of fibres and a section of ρ is isomorphic to $U \oplus E_8 \oplus E_8 \oplus A_2$, where U is generated by a general fibre and a section. It follows from the Shioda-Tate formula in [49] that this sublattice coincides with S_Y and that the discriminant of its quadratic form is equal to -3. Since the transcendental lattice T_Y is equal to the orthogonal complement of S_Y in the unimodular lattice $L = H^2(X, \mathbf{Z})$, this easily implies that T_Y is a rank 2 positive definite even lattice with discriminant equal to 3. There is only one isomorphism class of such a lattice and it is given by $A_2(-1)$.

The transcendental lattices of the surfaces X and Y are related in the following way. By Proposition 5 of [50], there is an isomorphism of abelian groups $(T_Y) \otimes \mathbf{Q} \cong (T_X)^G \otimes \mathbf{Q}$, defined by taking the inverse transform of transcendental cycles under the rational map $X \dashrightarrow Y$. Since G acts symplectically on X, we have $(T_X)^G = T_X$. Under this map the intersection form is multiplied by the degree of the map, equal to 6. This implies that T_X has rank two and contains $T_Y(6) \cong A_2(-6)$ as a sublattice of finite index.

Note that the automorphism $g_4(x, y, z, w) = (x, \epsilon y, \epsilon z, \epsilon w)$ of X clearly fixes the curve $\{x = 0\}$ pointwise. Hence g_4 is non-symplectic by Theorem 7.1. It follows from Lemma 7.9 and the previous remarks that $T_X \cong A_2(-m)$. Hence we only need to determine the integer m. As we saw above, T_X contains a sublattice isomorphic to $A_2(-6)$, hence $m \in \{1,2,3,6\}$. We now exclude all possibilities except the last one.

The $K3$ surface with $T_Y \cong A_2(-1)$ was studied in [44], in particular all jacobian elliptic fibrations on Y are classified in Theorem 3.1. Since none of these fibrations has the same configuration of singular fibres as $\mathfrak{h}_1$ (see Proposition 7.5), this excludes the case $m = 1$.

The $K3$ surface with $T_Y \cong A_2(-2)$ is isomorphic to the Kummer surface from Theorem 8.6 below. All its jacobian fibrations are described in [44], Theorem 3.1 (Table 1.1) and, as in the previous case, none of them has 6 fibres of type IV. This excludes the case $m = 2$.

Finally, a direct computation shows that $A_2(-3)$ does not contain a sublattice isomorphic to $A_2(-6)$. In fact, since the equation $x^2+y^2+xy = 2$ has no integral solutions, then $A_2(-3)$ does not contain any element with self-intersection 12. This completes the proof of our theorem.

We conclude this section by giving another model for the surface X.

PROPOSITION 7.11. *The $K3$ surface X is birational to the double cover of $\mathbf{P}^2$ branched along a sextic with 8 nodes which admits a group of linear automorphisms isomorphic to A_4.*

Proof. The lift $\widetilde{g}_0$ of the involution g_0 to the cover $X = \{w^2 + \Phi_6(x,y,z) = 0\}$ given by $\widetilde{g}_0(x,y,z,w) = (x,z,y,w)$ is a non-symplectic involution. The fixed locus of $\widetilde{g}_0$ is the genus two curve $\widetilde{L}_0$ which is the double cover of the harmonic polar $L_0 = \{y-z = 0\}$ branched along $L_0 \cap C_6$. The quotient surface $R = X/(\widetilde{g}_0)$ is a Del Pezzo surface of degree 1, the double cover of $\mathbf{P}^2/(g_0) \cong Q$, where Q is the quadratic cone with vertex equal to the orbit of the fixed point $p_0 = (0,1,-1)$ of g_0. We denote by B the image of $\widetilde{L}_0$ in R.

Let $b\colon R \dashrightarrow \mathbf{P}^2$ be the blowing-down of 8 disjoint (-1)-curves on R to points $s_1,\dots,s_8$ in $\mathbf{P}^2$. The pencil of cubic curves through the eight points is the image of the elliptic pencil $|-K_R|$ on R. Note that the stabilizer of the point p_0 in the Hessian group is isomorphic to $2\,.A_4$ with center equal to (g_0), thus the group A_4 acts naturally on R and on the elliptic pencil $|-K_R|$. The curve $B \in |-2K_R|$ is an A_4-invariant member of the linear system $|-2K_R|$ and $b(B)$ is a plane sextic with 8 nodes at the points $s_1,\dots,s_8$. Thus we see that X admits 9 isomorphic models as a double cover of the plane branched along a 8-nodal sextic with a linear action of A_4.

REMARK 7.12. In [34] the authors study a $K3$ surface birationally isomorphic to the double cover of $\mathbf{P}^2$ branched along the union of two triangles from the Hesse pencil. This surface has transcendental lattice of rank 2 with intersection matrix $\left(\begin{smallmatrix} 6 & 0 \\ 0 & 6 \end{smallmatrix}\right)$ and it admits a group of automorphisms isomorphic to $A_6 \rtimes \mathbf{Z}/4\mathbf{Z}$.

8. A $K3$ SURFACE WITH AN ACTION OF $\mathrm{SL}(2, \mathbf{F}_3)$

We now study the $K3$ surface which is birational to the double cover of $\mathbf{P}^2$ branched along the sextic C_6' defined by $\Phi_6' = 0$.

We recall that C_6' has 8 cusps in the base points $q_1, \ldots, q_8$ of the Hesse pencil. The double cover of $\mathbf{P}^2$ branched along C_6' is locally isomorphic to $z^2 + x^2 + y^3 = 0$ over each cusp of C_6', hence it has 8 singular points of type A_2 (see [5]). It is known that the minimal resolution of singularities of this surface is a $K3$ surface and that the exceptional curve over each singular point of type A_2 is the union of two rational curves intersecting in one point (see for example [40], § 2).

In this section we will study the properties of this $K3$ surface, which will be denoted by X'.

PROPOSITION 8.1. *The $K3$ surface X' is birationally isomorphic to the quotient of the $K3$ surface X by the subgroup Γ of G_{216}. In particular, the group $\mathrm{SL}(2, \mathbf{F}_3)$ is isomorphic to a group of automorphisms of X'.*

Proof. The minimal resolution of the double cover of $\mathbf{P}^2$ branched along C_6' can be obtained by first resolving the singularities of C_6' through the morphism $\gamma\colon S \to \mathbf{P}^2$ from diagram (12) and then taking the double cover $\bar{q}'\colon X' \to S$ branched over the proper transform $\overline{C}_6'$ of C_6' ([5]). Since $\beta^{-1}(p(C_6)) = \overline{C}_6'$ we have the commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{\bar{r}} & X/\Gamma & \xleftarrow{\bar{\sigma}} & X' & & \\ {\scriptstyle q}\downarrow & & \downarrow & & {\scriptstyle \bar{q}'}\downarrow & \searrow{\scriptstyle q'} & \\ \mathbf{P}^2 & \xrightarrow{p} & \mathbf{P}^2/\Gamma & \xleftarrow{\beta} & S & \xrightarrow{\gamma} & \mathbf{P}^2, \end{array} \tag{20}$$

where q is the double cover branched along C_6, q' is the minimal resolution of the double cover branched along C_6', $\bar{r}$ and p are the natural quotient maps, $\bar{\sigma}$ is a minimal resolution of singularities and the bottom maps are as in diagram (12). This gives the first statement. The second one follows from the isomorphism $G_{216}/\Gamma \cong \mathrm{SL}(2, \mathbf{F}_3)$.

REMARK 8.2. The points in X with nontrivial stabilizer for the action of Γ are exactly the 24 preimages by q of the vertices of the triangles in the Hesse pencil. In fact these points belong to 8 orbits for the action of Γ and give 8 singular points of type A_2 in the quotient surface X/Γ (see Proposition 5.1).

We now describe some natural elliptic fibrations on the surface X'.

PROPOSITION 8.3. *The pencil of lines through each of the cusps of C_6' induces a jacobian fibration on X' with 3 singular fibres of Kodaira's type I_6 and one of type I_3 (i.e. cycles of 6 and 3 rational curves respectively).*

Proof. Let p be a cusp of C_6' and h_p be the pencil of lines through p. The generic line in the pencil intersects C_6' in p and 4 other distinct points, hence its preimage in X' is an elliptic curve. Thus h_p induces an elliptic fibration $\tilde{h}_p$ on X'.

The pencil h_p contains 3 lines through 3 cusps and one line through 2 cusps of C_6', since the cusps of C_6' are the base points of the Hesse pencil. The proper transform of a line containing 3 cusps is a disjoint union of two smooth rational curves. Together with the preimages of the cusps, the full preimage of such a line in X' gives a fibre of $\tilde{h}_p$ of Kodaira's type I_6, described by the affine Dynkin diagram $\widetilde{A}_5$. Similarly, the preimage of a line containing 2 cusps gives a fibre of $\tilde{h}_p$ of type I_3 (in this case the proper transform of the line does not split). Thus $\tilde{h}_p$ has three fibres of type I_6 and one of type I_3.

The exceptional divisor over the cusp p splits into two rational curves e_1, e_2 on X' and each of them intersects each fibre of $\tilde{h}_p$ in one point, i.e. it is a section of $\tilde{h}_p$.

PROPOSITION 8.4. *The elliptic fibrations $\mathfrak{h}_i, \mathfrak{h}_i'$, $i = 1, \ldots, 4$, on X induce 8 elliptic fibrations $\overline{\mathfrak{h}}_i, \overline{\mathfrak{h}}_i'$ on X' such that*

a) *$\overline{\mathfrak{h}}_i$ and $\overline{\mathfrak{h}}_i'$ are exchanged by the covering involution of q' and* $\mathrm{SL}(2, \mathbf{F}_3)$ *acts transitively on $\overline{\mathfrak{h}}_1, \ldots, \overline{\mathfrak{h}}_4$;*

b) *the j-invariant of a smooth fibre of the elliptic fibration $\overline{\mathfrak{h}}_i$ or $\overline{\mathfrak{h}}_i'$ is equal to zero;*

c) *each fibration has two fibres of Kodaira's type IV^* (i.e. 7 rational curves in the configuration described by the affine Dynkin diagram $\widetilde{E}_6$) and two of type IV.*

Proof. It will be enough to study the fibration $\mathfrak{h}_1$, since all other fibrations are projectively equivalent to this one by the action of G_{216} and σ.

Let g_1, g_2 be the generators of Γ as in Section 4. The polynomials P_1, P_1' and F_1 are eigenvectors for the action of Γ (Remark 5.3), hence it is clear from equation (18) that Γ preserves the elliptic fibration $\mathfrak{h}_1$. In fact, g_1 acts on the basis of the fibration as an order three automorphism and fixes exactly the two fibres E_1, E_1' such that $q(E_1) = B_1$ and $q(E_1') = B_1'$. The automorphism g_2 preserves each fibre of $\mathfrak{h}_1$ and acts on it as an order 3 automorphism without fixed points. Hence it follows that the image of the elliptic fibration $\mathfrak{h}_1$ by the map $\tilde{\sigma}^{-1}\tilde{r}$ in diagram (20) is an elliptic fibration on X'. We will denote it by $\overline{\mathfrak{h}}_1$.

Now statements a), b), c) are easy consequences of the analogous statements in Proposition 7.5.

According to Proposition 5.2 the cubics B_1 and B_1' contain the 9 vertices of the triangles T_2, T_3, T_4 in the Hesse pencil. Hence the fibres E_1, E_1' each contain 9 points in the preimage of the 9 vertices by q. It follows from Remark 8.2 that the images of E_1 and E_1' in X/Γ each contain 3 singular points of type A_2. The preimage of one of these fibres in the minimal resolution X' is a fibre of type IV^* in the elliptic fibration $\overline{\mathfrak{h}}_1$ on X' (the union of 3 exceptional divisors of type A_2 and the proper transform of E_1 or E_1').

It can easily be seen that the 6 singular fibres of $\mathfrak{h}_1$ of type IV belong to two orbits for the action of Γ. In fact, the singular points in each of these fibres are the preimages by q of the vertices of T_1 (see Proposition 7.5). The image of a singular fibre of type IV in X/Γ is a rational curve containing a singular point of type A_2 and its preimage in X' is again a fibre of type IV. Hence $\overline{\mathfrak{h}}_1$ has two fibres of type IV.

REMARK 8.5. It can be proved that the image of any of these fibrations by the cover q' is a one-dimensional family of curves of degree 9 in $\mathbf{P}^2$ with 8 triple points in $q_1, \ldots, q_8$ and 3 cusps on C_6'. In fact, q' sends the fibre $\tilde{\sigma}^{-1}\tilde{r}(E_1)$ of $\overline{\mathfrak{h}}_1$ to the union of the 6 inflection lines through q_1 and q_1' not containing q_0, where the 3 lines through q_1 are double. Clearly, the analogous statement is true for E_1' (the lines through p_1' are now double). Hence the image of a fibre of $\overline{\mathfrak{h}}_1$ is a plane curve D of degree 9 with 8 triple points at $q_1, \ldots, q_8$. Moreover, the curve D intersects the sextic C_6' in 6 more points and since its inverse image in X' has genus one, then D must also have three cusps at smooth points of C_6' which are resolved in the double cover q'.

THEOREM 8.6. *The K3 surface X' is birationally isomorphic to the Kummer surface* $\mathrm{Kum}(E_\epsilon \times E_\epsilon)$, *where E_ϵ is the elliptic curve with fundamental periods* $1, \epsilon$. *Its transcendental lattice has rank* 2 *and its intersection matrix with respect to a suitable basis is*

$$A_2(-2) = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}.$$

Proof. We will consider one of the jacobian fibrations on X' described in Proposition 8.3. Let M be the lattice generated by the two sections e_1, e_2, the components of the 3 singular fibres of type I_6 not intersecting e_2 and the components of the fibre of type I_3. The intersection matrix of M has determinant $-2^2 \cdot 3^5$, hence $\mathrm{rank}\, M = \mathrm{rank}\, S_{X'} = 20$ and $\mathrm{rank}\, T_{X'} = 2$.

The non-symplectic automorphism g_4 of order 3 on X induces an automorphism g_4' on X'. Recall that g_4 fixes the curve $R = \{x = 0\}$ on X, hence g_4' fixes the proper transform of $\tilde{r}(R)$ on X'. Thus by Theorem 7.1, g_4' is a non-symplectic automorphism of order three on X'. This implies, as in the proof of Theorem 7.10, that the intersection matrix of $T_{X'}$ is of the form (19) with respect to an appropriate choice of generators; in particular its discriminant group $A_{T_{X'}} = T_{X'}^*/T_{X'}$ is isomorphic to $\mathbf{Z}/3\mathbf{Z} \oplus \mathbf{Z}/3m\mathbf{Z}$.

A direct computation of M^* shows that the discriminant group A_M is isomorphic to $\mathbf{Z}/3\mathbf{Z}^3 \oplus \mathbf{Z}/6\mathbf{Z}^2$. Since M is a sublattice of finite index of $S_{X'}$, the discriminant group $A_{T_{X'}} \cong A_{S_{X'}}$ is isomorphic to a quotient of a subgroup of A_M. This implies that $m \leq 2$.

By Theorem 3.1 (Table 1.1) in [44], the unique $K3$ surface with transcendental lattice as in (19) with $m = 1$ has no jacobian elliptic fibration as in Proposition 8.3. Hence $m = 2$ and by [30], X' is isomorphic to the Kummer surface of the abelian surface $E_\epsilon \times E_\epsilon$.

REMARK 8.7. i) In [32] it is proved that all elliptic fibrations on the Kummer surface $\mathrm{Kum}(E_\epsilon \times E_\epsilon)$ are jacobian. All these fibrations and their Mordell-Weil groups are described in [44]. In particular it is proved that the Mordell-Weil group of the elliptic fibration in Proposition 8.3 is isomorphic to $\mathbf{Z} \oplus \mathbf{Z}/3\mathbf{Z}$ and that those of the 8 elliptic fibrations in Proposition 8.4 are isomorphic to $\mathbf{Z}^2 \oplus \mathbf{Z}/3\mathbf{Z}$ (see Theorem 3.1, Table 1.3, No. 19, 30).

ii) The full automorphism group of X' has been computed in [33], but the full automorphism group of X is not known at present.

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