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35

SELF-SIMILAR CONTRACTING GROUPS

by Rostislav I. GRIGORCHUK and Volodymyr NEKRASHEVYCH

Let X be a finite set (alphabet) and let X^* be the free monoid generated by it. We imagine X^* as a rooted tree with the root equal to the empty word and a word v connected to every word of the form vx for $x \in X$.

DEFINITION 35.1. A *self-similar group* is a group G acting faithfully on X^* such that for every $g \in G$ and $x \in X$ there exist $h \in G$ and $y \in X$ such that

$$g(xv) = yh(v)$$

for all $v \in X^*$.

It follows that for every $g \in G$ and $u \in X^*$ there exists $h \in G$ such that

$$g(uv) = g(u)h(v)$$

for all $v \in X^*$. The element h is denoted $g|_v$ and is called the *restriction of* g in v.

DEFINITION 35.2. A self-similar group G acting on X^* is called *contracting* if there exists a finite set $N \subset G$ such that for every $g \in G$ there exists $n \in \mathbf{N}$ such that $g|_v \in N$ for all $v \in X^*$ of length $|v| \geq n$.

CONJECTURE 35.3. *Finitely generated contracting groups have solvable conjugacy problem.*

CONJECTURE 35.4. *Finitely generated contracting groups have solvable membership problem.*

REMARK 35.5. It is known that the word problem in contracting groups is solvable in polynomial time.

The next three conjectures are ordered by their strength (the last one is the strongest).

CONJECTURE 35.6. *Contracting groups have no non-abelian free subgroups.*

CONJECTURE 35.7. *Contracting groups are amenable.*

CONJECTURE 35.8. *A simple random walk on a contracting group has zero entropy.*

CONJECTURE 35.9. *The group generated by the transformations a and b of $\{0,1\}^*$ defined by*

$$a(0w) = 1w, \quad a(1w) = 0a(w), \quad b(0w) = 0b(w), \quad b(1w) = 1a(w)$$

is amenable.

REMARK 35.10. This group is not contracting; however, it is known (due to a result of S. Sidki [4]) that this group does not contain a free subgroup. The graphs of the action of this group on the boundary of the tree X^* have intermediate growth.

The next conjecture was suggested by Zoran Šunić.

CONJECTURE 35.11. *The group H_k generated by the transformations a_{ij} of $\{1,2,\ldots,k\}^*$, for $1 \le i < j \le k$, defined by*

$$a_{ij}(iw) = jw, \quad a_{ij}(jw) = iw, \quad a_{ij}(kw) = ka_{ij}(w) \qquad \text{for } k \neq i,j$$

is non-amenable for $k \ge 4$.

REMARK 35.12. The group H_k models the "Hanoi tower game" on k pegs. The graph of its action on the nth level of the tree $\{1,\ldots,k\}^*$ coincides with the graph of the game with n discs [3]. It is also not contracting, but the graphs of its action on the boundary of the tree are of intermediate growth. The group H_3 is amenable and the graphs of the action on the boundary have polynomial growth.

Following Furstenberg we say that a group is *Tychonoff* if it has a fixed ray for any affine action on a convex cone with compact base [1]. A definition of branch groups can be found in [2]. Every proper quotient of a branch group is virtually abelian.

CONJECTURE 35.13. *A branch group G is Tychonoff if and only if G is indicable*[4]) *and every proper non-trivial quotient is Tychonoff.*

ADDED IN PROOF. Conjecture 35.6 is now a theorem by the second author. See V. NEKRASHEVYCH. 'Free subgroups in groups acting on rooted trees'. Preprint arXiv: math.GR/0802.2554 (2008).

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[4]) An infinite group is called *indicable* if it admits a homomorphism onto **Z**.