

# The Friedlander-Milnor conjecture

Autor(en): **Friedlander, Eric M.**

Objektyp: **Article**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **54 (2008)**

Heft 1-2

PDF erstellt am: **22.09.2024**

Persistenter Link: <https://doi.org/10.5169/seals-109901>

## **Nutzungsbedingungen**

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern. Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden. Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

## **Haftungsausschluss**

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

## 32

### THE FRIEDLANDER–MILNOR CONJECTURE

by Eric M. FRIEDLANDER

The conjecture of the title of this note has resisted 40 years of effort and remains not only unsolved but also lacking in a plausible means of either proof or counter-example.

The original form of this conjecture is one I struggled with during my days at Princeton in the early 1970's:

**CONJECTURE 32.1.** *Let  $G(\mathbf{C})$  be a complex reductive algebraic group and let  $G(\mathbf{C})^\delta$  denote this group viewed as a discrete group. Then the map on classifying spaces of the continuous (identity) group homomorphism*

$$i: G(\mathbf{C})^\delta \rightarrow G(\mathbf{C})$$

*induces an isomorphism in cohomology with finite coefficients  $\mathbf{Z}/n$  for any  $n \geq 0$ :*

$$i^*: H^*(BG(\mathbf{C}), \mathbf{Z}/n) \xrightarrow{\simeq} H^*(G(\mathbf{C})^\delta, \mathbf{Z}/n).$$

Conjecture 32.1 is easily seen to be true for a torus (i.e.,  $G = \mathbf{G}_m^{\times r}$  for some  $r > 0$ ), but even the simplest non-trivial case (that of  $G = \mathrm{SL}_2$ ) remains inaccessible.

Guido and I published 5 papers together, all in some sense connected with this conjecture. We used the integral form  $G_{\mathbf{Z}}$  of a complex reductive algebraic group (which is a group scheme over  $\mathrm{Spec} \mathbf{Z}$ ) in order to form the group  $G(F)$  of points of  $G$  with values in a field  $F$ . Most of our joint work investigated various relations between  $G(\mathbf{C})$  and  $G(F)$ , the case  $F = \overline{\mathbf{F}}_p$  (the algebraic closure of a prime field  $\mathbf{F}_p$ ) being of special interest.

One knows from considerations of étale cohomology that the cohomology of  $BG(\mathbf{C})$  with  $\mathbf{Z}/n$  coefficients is naturally isomorphic to that of the étale

homotopy classifying space of the algebraic group  $G_F$  for  $F$  algebraically closed of characteristic  $p \geq 0$ :

$$H^*(BG(\mathbf{C}), \mathbf{Z}/n) \simeq H^*((BG_F)_{\text{et}}, \mathbf{Z}/n), \quad \text{provided that } (p, n) = 1.$$

This enables one to construct a map  $H^*(BG(\mathbf{C}), \mathbf{Z}/n) \rightarrow H^*(G(F), \mathbf{Z}/n)$  relating the cohomology with mod- $n$  coefficients of the classifying space of  $G(\mathbf{C})$  with the cohomology with mod- $n$  coefficients of the discrete group  $G(F)$  for any field  $F$ .

The following is a generalization of Conjecture 32.1, one that appears likely to be true if and only if Conjecture 32.1 is valid.

**CONJECTURE 32.2.** *Let  $G(\mathbf{C})$  be a complex reductive algebraic group, let  $n > 0$  be a positive integer, and let  $p$  denote either 0 or a prime which does not divide  $n$ . Then for any algebraically closed field  $F$  of characteristic  $p$ , the comparison of the cohomology of  $BG(\mathbf{C})$  and  $G(F)$  determines an isomorphism*

$$H^*(G(F), \mathbf{Z}/n) \simeq H^*(BG(\mathbf{C}), \mathbf{Z}/n).$$

In our first paper together [1], Guido and I began our investigation of “locally finite approximations” of Lie groups. We also formulated the following conjecture and proved it equivalent to Conjecture 32.2.

**CONJECTURE 32.3.** *Let  $F$  be an algebraically closed field of characteristic  $p \geq 0$  and let  $n > 0$  be a positive integer not divisible by  $p$  if  $p > 0$ . Then Conjecture 32.2 is valid for  $G(F)$  if and only for every  $0 \neq x \in H^*(G(F), \mathbf{Z}/n)$ , there exists some finite subgroup  $\pi \subset G(F)$  such that  $x$  restricts non-trivially to  $H^*(\pi, \mathbf{Z}/n)$ .*

The most familiar form of the “Friedlander–Milnor Conjecture” is that formulated by John Milnor in [2]. In that paper, Milnor verifies this conjecture for solvable groups.

**CONJECTURE 32.4.** *Let  $G$  be a Lie group with finitely many components and let  $G^\delta$  denote the same group now viewed as a discrete group. Then for any integer  $n > 0$ , the continuous (identity) map  $i: G^\delta \rightarrow G$  induces an isomorphism on cohomology with mod- $n$  coefficients:*

$$i^*: H^*(BG, \mathbf{Z}/n) \xrightarrow{i^*} H^*(G^\delta, \mathbf{Z}/n).$$

We remark that the most substantial progress to date on these conjectures is due to Andrei Suslin, who proves a “stable” version of Conjectures 32.1 and 32.2 in [3].

## REFERENCES

- [1] FRIEDLANDER, E. and G. MISLIN. Cohomology of classifying spaces of complex Lie groups and related discrete groups. *Comment. Math. Helv.* 59 (1984), 347–361.
- [2] MILNOR, J. On the homology of Lie groups made discrete. *Comment. Math. Helv.* 58 (1983), 72–85.
- [3] SUSLIN, A. On the  $K$ -theory of local fields. *J. Pure Appl. Algebra* 34 (1984), 301–318.

E. Friedlander

Department of Mathematics  
Northwestern University  
Evanston, IL 60208  
USA  
*e-mail*: eric@math.northwestern.edu