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USING GAUSS MAPS TO DETECT INTERSECTIONS

by Frederico XAVIER

To Andréa, with love and admiration

ABSTRACT. A family of compact submanifolds with boundary has a non-empty stable interior intersection in $\mathbf{R}^n$ provided a certain geometric estimate holds. As a global application we establish a sharp intersection criterion for a family of properly embedded m_j-dimensional submanifolds $M_j \subset \mathbf{R}^n$ that is based solely on their Gauss maps $\mathcal{G}_j \colon M_j \to G(n-m_j, n)$, $\mathcal{G}_j(p) = [T_p M_j]^\perp$.

1. INTRODUCTION

For $1 \leq j \leq k \leq n$, $2 \leq k \leq n$, let $M_j \subset \mathbf{R}^n$ be a smooth compact submanifold with boundary, $1 \leq \dim M_j \leq n-1$, and $f_j \colon M_j \to \mathbf{R}^n$ a smooth immersion. In this note we are interested in formulating geometric conditions under which the intersection

$$f_1(\operatorname{int} M_1) \cap \cdots \cap f_k(\operatorname{int} M_k)$$

is non-empty and stable. In order to ensure stability under perturbations, and since one does not know *a priori* where the intersections are going to lie, we require that for *all* choices of $p_j \in \operatorname{int} M_j$ the normal spaces

$$[df(p_1)T_{p_1}M_1]^\perp, \ldots, [df(p_k)T_{p_k}M_k]^\perp$$

are in direct sum. Starting from this natural assumption, we prove that the above intersection is non-empty provided the conditions are such that a certain geometric estimate holds. The inequality in question involves three features:

the intrinsic sizes of the submanifolds, a weighed measure of the effect of the Euclidean translations, and the distortion of the aforementioned configurations of normal spaces. This is the content of Theorem 2, our main result.

In the simplest case when M_1 and M_2 are compact hypersurfaces with boundary, Theorem 2 reads as follows. Let $\alpha \in [0, \frac{\pi}{2}]$ be the infimum of the angles formed by all normal spaces (lines) corresponding to arbitrary points in $\text{int}M_1$ and $\text{int}M_2$, one point from each hypersurface. If $\alpha > 0$ and

$$\frac{\sqrt{2}}{2}\cot\left(\frac{\alpha}{2}\right)\csc\left(\frac{\alpha}{2}\right) < \sup_{q_1 \in M_1,\, q_2 \in M_2} \frac{\min\left\{d_1(q_1, \partial M_1),\ d_2(q_2, \partial M_2)\right\}}{|f_1(q_1) - f_2(q_2)|},$$

then the intersection $f(\text{int}M_1) \cap f_2(\text{int}M_2)$ is non-empty.

The result below is a sharp global consequence of our work on the intersection of compact manifolds with boundary:

THEOREM 1. *Let $M_1, \ldots, M_k \subset \mathbf{R}^n$ be connected, properly embedded non-compact smooth submanifolds without boundary,* $1 \le \dim M_j = m_j < n$, $\text{codim}\, M_1 + \cdots + \text{codim}\, M_k \le n$. *Let $\mathcal{G}_j\colon M_j \to G(n-m_j, n)$, $\mathcal{G}_j(p) = [T_pM_j]^\perp$, be the Grassmanian-valued Gauss map of M_j. Assume that for all points E_j in the closure of $\mathcal{G}_j(M_j)$, $1 \le j \le k$, the subspaces $E_1, \ldots, E_k$ of $\mathbf{R}^n$ are in direct sum. Then $M_1 \cap \cdots \cap M_k$ is non-empty. Moreover, this intersection reduces to a single point if* $\text{codim}\, M_1 + \cdots + \text{codim}\, M_k = n$.

COROLLARY 1. *Let $M_1, M_2 \subset \mathbf{R}^n$ be properly embedded connected smooth hypersurfaces without boundary. If $\overline{\mathcal{G}_1(M_1)} \cap \overline{\mathcal{G}_2(M_2)} = \varnothing$ then $M_1 \cap M_2 \neq \varnothing$.*

COROLLARY 2. *Let $M_1, \ldots, M_n \subset \mathbf{R}^n$ be properly embedded connected smooth hypersurfaces without boundary. If every hyperplane in $\mathbf{RP}^{n-1} \cong G(1,n)$ intersects at most $n-1$ of the sets $\overline{\mathcal{G}_1(M_1)}, \ldots, \overline{\mathcal{G}_n(M_n)}$, then $M_1 \cap \cdots \cap M_n$ consists of a single point.*

Corollary 2 generalizes to properly embedded hypersurfaces the observation that n affine hyperplanes with linearly independent normals intersect at a single point. In the linear case the Gaussian image $\mathcal{G}_j(M_j)$ is trivially closed, since it reduces to a single point. However, elementary examples show that Theorem 1 fails already at the simple level of Corollary 1 if one does not take the closures of the Gaussian images into account. To see this, consider an open circular cone K in $\mathbf{R}^3$ with central angle $< \pi$. Inside K we take a rotationally symmetric complete non-compact surface M_1 of positive curvature that is asymptotic to ∂K. Let M_2 be any plane tangent to $K - \{0\}$. One can see that Corollary 1 does not apply. Indeed, $M_1 \cap M_2 = \varnothing$ and $\mathcal{G}_1(M_1) \cap \mathcal{G}_2(M_2) = \varnothing$, but $\mathcal{G}_2(M_2) = \overline{\mathcal{G}_2(M_2)} \subset \overline{\mathcal{G}_1(M_1)}$.

One can also regard Corollary 2 as an existence and uniqueness result for solutions of certain systems of non-linear equations. From this standpoint the subject matter of this paper relates to the broader question of deciding when a locally invertible map admits a global inverse. The latter topic has been the object of intense research over the years, with applications and connections to many different areas of mathematics. Recently, new mechanisms of global inversion have been discovered, mostly of a topological nature. The interested reader may want to consult [1]-[4], [6]-[14] and the references therein. We mention explicitly some of the more recent developments. In [9] and [10] algebraic methods and surgery theory were used to give a generic solution, in a suitable sense, of the Jacobian Conjecture. A special case of the main result of [8] states that a surjective local biholomorphism $f : \mathbf{C}^n \to \mathbf{C}^n$ is bijective if and only if the pre-image of every complex line is connected and simply-connected (the proof uses geodesics and the fact that the Hopf map has no continuous sections). The main result in [14] is a necessary and sufficient analytic condition for an injective holomorphic self-map of $\mathbf{C}^n$ to be the identity. This is potentially of interest in the study of Aut $(\mathbf{C}^n)$, $n \geq 2$. In [1] invertible maps were characterized by a homological condition: a local diffeomorphism of $\mathbf{R}^n$ into itself is bijective if and only if the pre-image of every affine hyperplane is non-empty and acyclic. This result is more general than the one in [7], which in turn improves in finite dimensions the classical global inverse function theorem of Hadamard.

As to the organization of this paper, Theorem 1 is established in §4, in a more general form (Theorem 3), as an application of Theorem 2 and the ideas that go into its proof. Theorem 2 itself is stated in §2 and proved in §3. Our arguments have a clear dynamical interpretation and are self-contained.

2. Statement of the main result

Recall from the introduction that our basic problem is to formulate conditions to have the immersions $f_j \colon M_j \to \mathbf{R}^n$ satisfy $\bigcap_{j=1}^{k} f_j(\mathrm{int} M_j) \neq \varnothing$. In order to ensure that an eventual intersection persists under perturbations, we require that for all choices of $p_j \in \mathrm{int} M_j$ the normal spaces $[df(p_1)T_{p_1}M_1]^{\perp}, \ldots, [df(p_k)T_{p_k}M_k]^{\perp}$ are in direct sum. Besides this qualitative condition, our estimate in Theorem 2 below captures what seem to be the three essential quantitative features of the problem:

i) The sizes of the submanifolds $\operatorname{int} M_j$, relative to the metric induced by f_j.

ii) A numerical way to measure the effect of translations.

iii) The deviation from orthogonality of all sums $[df_1(p_1)T_{p_1}M_1]^\perp \oplus \cdots \oplus [df_k(p_k)T_{p_k}M_k]^\perp$.

Before we can state our results we need to discuss iii) above and introduce a quantity that measures how distorted a direct sum of subspaces is, relative to orthogonal sums. This can be formalized as follows.

Given $n \geq 2$ and positive integers $\alpha_1, \dots, \alpha_k$ such that $\alpha_1 + \cdots + \alpha_k \leq n$, we say that a continuous $\mathcal{D}\colon G(\alpha_1, n) \times \cdots \times G(\alpha_k, n) \to [1, \infty]$ is a *distortion function* if the following two conditions are met:

a) $(E_1, \dots, E_k) \in \mathcal{D}^{-1}(\infty)$ if and only if the subspaces $E_1, \dots, E_k$ are not in direct sum.

b) $(E_1, \dots, E_k) \in \mathcal{D}^{-1}(1)$ if and only if the subspaces $E_1, \dots, E_k$ are in direct sum and the decomposition $E_1 \oplus \cdots \oplus E_k$ is orthogonal.

The idea of considering the distortion of a direct sum (deviation from orthogonality) is a natural one, and it comes up in other contexts as well (e.g., the Oseledec multiplicative ergodic theorem in dynamical systems [5]). The particular choice of distortion function that we make below reflects the flow of the estimates in the proof of Theorem 2.

Consider proper subspaces $E_1, \dots, E_k$ of $\mathbf{R}^n$ which are in direct sum and set $\alpha_j = \dim E_j$. In particular, $m := \dim\Big(\sum_{j=1}^k E_j\Big) = \sum_{j=1}^k \alpha_j$. We say that an ordered set of unit vectors $\mathcal{B} = \{e_1, \dots, e_m\}$ is an adapted basis of $E := E_1 \oplus \cdots \oplus E_k$ if $\{e_1, \cdots, e_{\alpha_1}\}$ is a basis of E_1, $\{e_{\alpha_1+1}, \dots, e_{\alpha_1+\alpha_2}\}$ is a basis for E_2, and so on. Let $M = M_{\mathcal{B}}$ be the symmetric matrix given by $M_{ij} = \langle e_i, e_j \rangle$, $1 \leq i, j \leq m$. Since $\big|\sum_{i=1}^m y_i e_i\big|^2 > 0$ whenever $\sum_{i=1}^m y_i^2 > 0$, the eigenvalues of $M_{\mathcal{B}}$ are strictly positive. Denote by $\Lambda_{\mathcal{B}}$ and $\lambda_{\mathcal{B}}$ the largest and smallest eigenvalues of $M_{\mathcal{B}}$, respectively. From $\det\big(\langle e_i, e_j \rangle\big) = \big|\det\big(\langle e_i, e_j \rangle\big)\big| = |e_1 \wedge \cdots \wedge e_m| \leq |e_1| \cdots |e_m| = 1$ and $\operatorname{tr} M = m$, one has

$$m > \Lambda_{\mathcal{B}} \geq \sqrt{\Lambda_{\mathcal{B}}} \geq 1 \geq \lambda_{\mathcal{B}} > 0\,. \tag{1}$$

Given an adapted basis $\mathcal{B}$ of E, we set

$$\operatorname{DIS}(\mathcal{B}) = \frac{\sqrt{\Lambda_{\mathcal{B}}}}{\lambda_{\mathcal{B}}}\,.$$

We define a distortion function as follows. If $(E_1, \ldots, E_k)$ are subspaces in direct sum, set

$$\mathrm{DIS}(E_1, \ldots, E_k) = \inf_{\mathcal{B}} \mathrm{DIS}(\mathcal{B}),$$

where the infimum is taken over all adapted bases of the decomposition. Of course, to comply with a) we define $\mathrm{DIS}(E_1, \ldots, E_k) = \infty$ if $E_1, \ldots, E_k$ are not in direct sum. We shall see shortly that DIS is a continuous function.

As an example, let E_1 and E_2 be two one-dimensional subspaces in $\mathbf{R}^n$ making an angle $\alpha \in (0, \frac{\pi}{2}]$. If $\mathcal{B} = \{e_1, e_2\}$ is an adapted basis of $E_1 \oplus E_2$ with $\langle e_1, e_2 \rangle = \cos\alpha$ the eigenvalues of $M_{\mathcal{B}}$ are $1 + \cos\alpha$ and $1 - \cos\alpha$, so that

$$\mathrm{DIS}(E_1, E_2) = \frac{\sqrt{2}}{2} \cot\left(\frac{\alpha}{2}\right) \csc\left(\frac{\alpha}{2}\right).$$

It follows from (1) that if $\mathrm{DIS}(\mathcal{B}) = 1$ the adapted basis $\mathcal{B}$ is as straight as possible that is, it is orthonormal. Another easy consequence of (1) is that the infimum in the definition of $\mathrm{DIS}(E_1, \ldots, E_k)$ is actually attained by some adapted basis. In particular, if $\mathrm{DIS}(E_1, \ldots, E_k) = 1$ any adapted basis that realizes the infimum must be orthonormal, and therefore the summands must be pairwise orthogonal. This shows that DIS satisfies b) in the definition of distortion function.

As before, let $E_1, \ldots, E_k$ be subspaces in direct sum. Taking the normalization of the orthogonal projection of an admissible basis that realizes $\mathrm{DIS}(E_1, \ldots, E_k)$ onto a nearby configuration $F_1, \ldots, F_k$ in direct sum (and vice versa), and using the fact that the (largest and smallest) eigenvalues of a matrix vary continuously with its entries, it is easy to argue that DIS is a continuous function on the open dense subset of $G(\alpha_1, n) \times \cdots \times G(\alpha_k, n)$ formed by all subspaces which are in direct sum. Hence DIS is locally bounded, as long as the summands are in direct sum. We thus have:

LEMMA 1. *For* $1 \leq l \leq k$, *let* $\{E_l^j\}_{j=1}^{\infty} \subset G(\alpha_l, n)$ *be a sequence converging to* E_l *such that the subspaces* $E_1^j, \ldots, E_k^j$ *are in direct sum for every* j. *If* $\lim_{j\to\infty} \mathrm{DIS}(E_1^j, \ldots, E_k^j) = \infty$, *then the subspaces* $E_1, \ldots, E_k$ *are not in direct sum.*

Similar arguments show that the function DIS is also continuous at those $(E_1, \ldots, E_k)$ for which $E_1, \ldots, E_k$ are not in direct sum, but this will not be used in our proof. Hence, the larger the distortion of a direct sum, the closer the decomposition comes to not being a direct sum.

We can now state the main result of this paper, an intersection criterion for compact submanifolds with boundary:

THEOREM 2. *For* $1 \leq j \leq k \leq n$, $2 \leq k \leq n$, *let* M_j *be a connected compact manifold with boundary,* $m_j = \dim M_j \leq n-1$. *Let* $f_j\colon M_j \to \mathbf{R}^n$ *be a smooth immersion,* d_j *the riemannian distance in* $\operatorname{int} M_j$ *induced by* f_j *and* $\mathcal{G}_j\colon \operatorname{int} M_j \to G(n-m_j, n)$, $\mathcal{G}_j(p) = [df_j(p)T_pM_j]^{\perp} \subset \mathbf{R}^n$, *the associated Gauss map. Suppose that for all choices of points* $p_j \in \operatorname{int} M_j$, $1 \leq j \leq k$, *the subspaces* $\mathcal{G}_1(p_1), \ldots, \mathcal{G}_k(p_k)$ *of* $\mathbf{R}^n$ *are in direct sum. Assume also that there are points* $q_j \in M_j$, $1 \leq j \leq k$, *such that*

$$\sup_{\substack{p_i \in \operatorname{int} M_i \\ 1 \leq i \leq k}} \operatorname{DIS}(\mathcal{G}_1(p_1), \ldots, \mathcal{G}_k(p_k)) < \min_{1 \leq j \leq k} \frac{d_j(q_j, \partial M_j)}{\left[\sum_{1 \leq i \leq k} (n-m_i)\,|f_i(q_i) - f_j(q_j)|^2\right]^{\frac{1}{2}}}. \tag{2}$$

Then $f_1(\operatorname{int} M_1) \cap \cdots \cap f_k(\operatorname{int} M_k)$ *is non-empty.*

REMARK. We may assume that the right hand side of (2) is finite, otherwise one already has $f_1(q_1) = \cdots = f_k(q_k) \in f_1(\operatorname{int} M_1) \cap \cdots \cap f_k(\operatorname{int} M_k)$.

The estimate in Theorem 2 assumes a particularly simple form in the case of two compact hypersurfaces. In the next corollary we use the above formula for $\operatorname{DIS}(E_1, E_2)$.

COROLLARY 3. *For* $j = 1, 2$, *let* $f_j\colon M_j \to \mathbf{R}^n$ *define a smooth connected immersed compact hypersurface with boundary. Let* $\alpha \in [0, \frac{\pi}{2}]$ *be the infimum of the angles formed by all pairs of normal spaces (lines) corresponding to arbitrary points in* $\operatorname{int} M_1$ *and* $\operatorname{int} M_2$, *one point from each hypersurface. Suppose that* $\alpha > 0$ *and there are points* $q_1 \in M_1$, $q_2 \in M_2$ *such that*

$$\frac{\sqrt{2}}{2} \cot\left(\frac{\alpha}{2}\right) \csc\left(\frac{\alpha}{2}\right) < \frac{\min\left\{d_1(q_1, \partial M_1),\ d_2(q_2, \partial M_2)\right\}}{|f_1(q_1) - f_2(q_2)|}. \tag{3}$$

Then the intersection $f(\operatorname{int} M_1) \cap f_2(\operatorname{int} M_2)$ *is non-empty.*

The estimate in Corollary 3 displays the features i)-iii) described at the beginning of this section. Indeed, the left hand side of (3) manifestly measures the distortion (or non-orthogonality) of the normal spaces. The numerator in the right hand side involves the riemannian sizes of the surfaces whereas the denominator, which is translation-invariant, changes if only one of the surfaces is translated. Fixing any two of these three geometric quantities while letting the other degenerate violates (3), thus allowing for the intersection to be empty.

For instance, the inequality fails if one of the two surfaces is kept fixed while the other is translated by a large vector, as the denominator in (3) tends to infinity, the other two quantities remaining the same. Similarly, start with compact embedded hypersurfaces with boundary for which $M_1 \cap M_2 \neq \varnothing$ and $\overline{\mathcal{G}_1(\operatorname{int} M_1)} \cap \overline{\mathcal{G}_2(\operatorname{int} M_2)} = \varnothing$ (e.g., two flat discs in $\mathbf{R}^3$, centered at distinct points q_1 and q_2, whose interiors intersect). Keeping M_1 constant, while replacing M_2 by smaller subsets converging to $q_2 \in M_2 - M_1$, will eventually make the intersection empty. In this process, the numerator in (3) goes to zero, while the other quantities remain constant. The third alternative can be illustrated by taking two orthogonal circles, centered at different points q_1 and q_2, whose interiors intersect. Rotate one of the circles about its center until both circles become parallel. The distortion of the normal spaces varies from one to infinity, whereas the right hand side of (3) remains constant. The estimate in Corollary 3 is also sharp. To see this, take M_1 and M_2 to be the closed discs of radii $1+\epsilon$ in the xy and yz planes, with centers at the points $q_1 = (1,0,0)$ and $q_2 = (0,0,0)$, respectively, and then let $\epsilon \to 0$.

3. Proof of Theorem 2

We begin with the dynamical interpretation behind the proof of Theorem 2. For simplicity, suppose that the submanifolds are embedded in $\mathbf{R}^n$ and that their interiors do not have a common intersection. Assume also that any configuration of normal spaces, one from (the interior of) each submanifold, gives rise to a direct sum.

We fix M_1 and continuously translate $M_2, \ldots, M_k$ along suitable directions until they intersect M_1 at the same point, after one unit of time. The idea is to undo the motion of the submanifolds $M_2, \ldots, M_k$, in the direction of increasing times, starting say at time $t = -1$, while keeping track of the evolution of the (local) intersection set. Since we are assuming $\bigcap_{j=1}^{k} \operatorname{int} M_j = \varnothing$, the backwards motion will cease to have a common intersection sometime before (or when) one unit of time has elapsed. One wants to control the speed at which the intersection set is propagating. Of course, it is technically easier to observe the evolution of a single point (given by the flow of a vector field, say), rather than that of the entire intersection set. In the case when the sum of the codimensions is strictly less than n it is crucial that we choose curves parametrized by time that move *orthogonally* to the intersection set, in order to minimize speed. As explained, the intersection set must cease to exist before (or at) time $t = 0$. But this can only happen if for some submanifold M_j the appropriate integral curve x_j in the interior of M_j reaches ∂M_j before (or at) time 0. The speed of motion of the intersection set is controlled by the local configuration of normal spaces to the moving submanifolds. In fact, the speed increases if the normal spaces of the submanifolds tend to "rest" on each other. A somewhat similar situation occurs if one rotates a line L_1 in the plane about one of its points, with constant angular speed. The intersection $L_1^\theta \cap L_2$ between the line L_1 rotated by θ, and a line L_2 parallel to L_1, moves faster as $\theta \to 0$. In other words, the speed of $L_1^\theta \cap L_2$ increases when the distortion of the configurations of their normal spaces tends to ∞.

As it was indicated above, one must control the distortion of the direct sum of normal spaces at the intersection sets during the evolution. But since we are simply translating the submanifolds, the supremum of the distortions of the configurations of all normal spaces remains constant throughout the motion. The estimate (2) in the statement of Theorem 2 comes in precisely to guarantee that the intersection set in the backward motion exists in the interiors of all M_j's for more than one unit of time, a contradiction to the original assumption that the interiors of the manifolds are disjoint.

Let us now proceed to give a formal proof of Theorem 2. For $1 \leq j \leq k$ define a_j by

$$f_1(q_1) - a_1 = f_1(q_1) = f_j(q_j) - a_j\,. \tag{4}$$

Hence $a_1 = 0$ and, from (4),

$$(f_1(\operatorname{int} M_1) + ta_1) \cap \cdots \cap (f_k(\operatorname{int} M_k) + ta_k) \neq \varnothing \tag{5}$$

for the value $t = -1$. The theorem will be proved if we can show that (5) holds for $t = 0$. In fact, we will show the validity of (5) for all $t \in [-1, T)$, for some $T > 0$. We begin with t sufficiently close to -1:

LEMMA 2. *There exist $\delta > -1$ and unique smooth curves $x_j\colon [-1, \delta] \to M_j$ such that, for all $t \in [-1, \delta]$,*

$$\begin{aligned} &x_j(t) \in \operatorname{int} M_j,\ x_j(-1) = q_j\,. \\ &f_1(x_1(t)) = f_j(x_j(t)) + ta_j,\quad 1 \leq j \leq k\,. \\ &df_1(x_1(t))x_1'(t) \in \mathcal{G}_1(x_1(t)) \oplus \cdots \oplus \mathcal{G}_k(x_k(t)) \subseteq \mathbf{R}^n\,. \end{aligned} \tag{6}$$

Since the hypotheses and the conclusion are local in nature we may assume, for the purpose of proving Lemma 2, that (a small neighborhood of q_j in) M_j is contained in $\mathbf{R}^n$ and f_j is the inclusion. With this identification, (6) becomes

$$x_j(t) \in \operatorname{int} M_j,\ x_j(-1) = q_j\,.$$
$$x_1(t) = x_j(t) + ta_j,\ \ 1 \le j \le k\,. \tag{7}$$
$$x_1'(t) \in \mathcal{G}_1(x_1(t)) \oplus \cdots \oplus \mathcal{G}_k(x_k(t)) \subseteq \mathbf{R}^n\,.$$

For j fixed and $1 \le l \le l(j) := n - m_j = \operatorname{codim} M_j$, let ξ_j^l be an S^{n-1}-valued smooth function defined in a neighborhood of q_j in $\mathbf{R}^n$ which, when taken together, form a (unitary) frame of $\mathcal{G}_j(p_j)$ for all $p_j \in M_j$ near q_j. If the sum of the codimensions of all submanifolds is n the inclusion in (7) is an equality and the third equation is trivially satisfied. On the other hand, if this inclusion is proper, consider an S^{n-1}-valued function η^α defined in a neighborhood of $(q_1, \ldots, q_k)$ in $M_1 \times \cdots \times M_k$, $1 \le \alpha \le \beta := n - \sum_{j=1}^k l(j)$, such that $\{\eta^\alpha(p_1, \ldots, p_k),\ 1 \le \alpha \le \beta\}$ constitutes a basis of $[\mathcal{G}_1(p_1) \oplus \cdots \oplus \mathcal{G}_k(p_k)]^\perp$ for all $(p_1, \ldots, p_k)$ near $(q_1, \ldots, q_k)$.

With these choices, (7) is equivalent to the following system, for $|t+1|$ small:

$$\begin{cases} \langle x_j', \xi_j^l \rangle = 0, \quad x_j(-1) = q_j\,, \\ x_1' - x_j' = a_j\,. \\ \langle x_1', \eta^\alpha \rangle = 0\,, \end{cases} \tag{8}$$

where $1 \le j \le k$, $1 \le l \le l(j)$, $1 \le \alpha \le \beta$, $\xi_j^l = \xi_j^l\big(x_j(t)\big)$, $\eta^\alpha = \eta^\alpha\big(x_1(t), \ldots, x_k(t)\big)$.

Suppose first that $\beta = 0$, i.e., $\sum_{j=1}^k l(j) = n$. In this case the last equations in (8) should be disregarded. This situation corresponds to a local complete intersection evolving in time, and it is possible to give a simple proof of the lemma from general transversality arguments. Nevertheless, for completeness and for comparison purposes with the case $\beta > 0$, we provide full details.

For $(p_1, \ldots, p_k)$ sufficiently close to $(q_1, \ldots, q_k)$, define

$$\begin{aligned} L_{p_1,\ldots,p_k} \colon (\mathbf{R}^n)^k &\to \mathbf{R}^{l(1)} \times \cdots \times \mathbf{R}^{l(k)} \times (\mathbf{R}^n)^{k-1}\,, \\ (v_1, \ldots, v_k) &\to \big(\langle v_1, \xi_1^1 \rangle, \ldots \langle v_1, \xi_1^{l(1)} \rangle, \ldots, \langle v_k, \xi_k^1 \rangle, \ldots, \\ &\qquad \langle v_k, \xi_k^{l(k)} \rangle, v_2 - v_1, \ldots, v_k - v_1\big)\,. \end{aligned}$$

We observe that $L_{p_1,\ldots,p_k}$ is a linear map between spaces of the same dimension. If $(v_1, \ldots, v_k) \in \ker L_{p_1,\ldots,p_k}$, then $v_1 \in \bigcap_{j=1}^k T_{p_j} M_j$. From $\bigcap_{j=1}^k T_{p_j} M_j \subset T_{p_\alpha} M_\alpha$ one has $\mathcal{G}_\alpha(p_\alpha) = [T_{p_\alpha} M_\alpha]^\perp \subseteq \left[\bigcap_{j=1}^k T_{p_j} M_j\right]^\perp$, and so

$$\mathbf{R}^n = \bigoplus_{\alpha=1}^{k} \mathcal{G}_\alpha(p_\alpha) = \bigoplus_{\alpha=1}^{k} [T_{p_\alpha}M_\alpha]^\perp \subseteq \left[\bigcap_{j=1}^{k} T_{p_j}M_j\right]^\perp .$$

Hence $v_1 \in \bigcap_{j=1}^{k} T_{p_j}M_j = \{0\}$, and since $v_1 = v_2 = \cdots = v_k$ the map $L_{p_1,\dots,p_k}$ is an isomorphism.

We write $X = (x_1, \dots, x_k)$, $V = (v_1, \dots, v_k)$, and

$$L_{p_1,\dots,p_k}(v_1, \dots, v_k) = A(p_1, \dots, p_k)V ,$$

where A is an invertible $(nk \times nk)$ matrix varying smoothly with $(p_1, \dots, p_k)$. One can then easily check that any local solution of the system of ordinary differential equations

$$\begin{cases} X'(t) = A^{-1}\big(x_1(t), \dots, x_k(t)\big)b , \\ x_j(-1) = q_j , \end{cases}$$

where b is given by the right hand side of (8), satisfies (7). This proves the lemma under the assumption $\beta = 0$.

It remains to deal with the case $\beta > 0$. All three sets of relations in (8) are now in force. We proceed in an analogous manner by defining the map

$$\hat{L}_{p_1,\dots,p_k} : (\mathbf{R}^n)^k \to \mathbf{R}^{l(1)} \times \cdots \times \mathbf{R}^{l(k)} \times (\mathbf{R}^n)^{k-1} \times \mathbf{R}^\beta$$

that sends $(v_1, \dots, v_k)$ into

$$\big(\langle v_1, \xi_1^1\rangle, \dots, \langle v_1, \xi_1^{l(1)}\rangle, \dots, \langle v_k, \xi_k^1\rangle, \dots, \langle v_k, \xi_k^{l(k)}\rangle, \\ v_2 - v_1, \dots, v_k - v_1, \langle v_1, \eta^1\rangle, \dots, \langle v_1, \eta^\beta\rangle\big) .$$

Once again, $\hat{L}_{p_1,\dots,p_k}$ is a linear map between spaces of the same dimension. If $(v_1, \dots, v_k)$ lies in the kernel of $\hat{L}_{p_1,\dots,p_k}$ then

$$v_1 \in \left[\bigcap_{j=1}^{k} T_{p_j}M_j\right] \cap \left[\bigoplus_{j=1}^{k} \mathcal{G}_j(p_j)\right]^{\perp\perp} = \left[\bigcap_{j=1}^{k} T_{p_j}M_j\right] \cap \left[\bigoplus_{j=1}^{k} \mathcal{G}_j(p_j)\right] .$$

But, as observed before,

$$\bigoplus_{\alpha=1}^{k} \mathcal{G}_\alpha(p_\alpha) \subseteq \left[\bigcap_{j=1}^{k} T_{p_j}M_j\right]^\perp ,$$

so that $v_1 = 0$, $v_2 = \cdots = v_k = 0$ and $\hat{L}_{p_1,\dots,p_k}$ is injective. As in the previous case, $\hat{L}_{p_1,\dots,p_k}$ is represented by an invertible matrix $A(p_1, \dots, p_k)$ and a local solution of the system

$$\begin{cases} X'(t) = A^{-1}\big(x_1(t), \dots, x_k(t)\big)b , \\ x_j(-1) = q_j , \end{cases}$$

where b is given by (8), satisfies (6). This concludes the proof of Lemma 2.

To continue with the proof of Theorem 2 we observe that the local solutions given by Lemma 2 can be extended to a maximal one, by the usual continuation argument. To be precise, there exist $T \in (-1, \infty)$, smooth curves $x_j\colon [-1, T) \to M_j$, $1 \le j \le k$, $j_0 \in \{1, \dots, k\}$ and $t_m \in [-1, T)$ such that, for all $t \in [-1, T)$,

$$
(9)\qquad
\begin{aligned}
&x_j(t) \in \operatorname{int} M_j,\ x_j(-1) = q_j\,.\\
&f_1(x_1(t)) = f_j(x_j(t)) + ta_j, \quad 1 \le j \le k\,.\\
&df_1(x_1(t))x_1'(t) \in \mathcal{G}_1(x_1(t)) \oplus \cdots \oplus \mathcal{G}_k(x_k(t)) \subseteq \mathbf{R}^n\,.\\
&\lim_{m\to\infty} t_m = T,\ \lim_{m\to\infty} x_{j_0}(t_m) \text{ exists and belongs to } \partial M_{j_0}\,.
\end{aligned}
$$

Our next task is to estimate $|df_{j_0}(x_{j_0}(t))x_{j_0}'(t)|$ for $t \in (-1, T)$ fixed. Since this is a local question we may assume, as we did in the proof of Lemma 2, that $x_{j_0}(s) \in \mathbf{R}^n$ for s near t. Since f_{j_0} is an isometric immersion we must estimate $|x_{j_0}'(t)|$.

Suppose first that $\beta = 0$, so that $\mathbf{R}^n = \mathcal{G}_1(p_1) \oplus \cdots \oplus \mathcal{G}_k(p_k)$ for every choice of points $p_j \in \operatorname{int} M_j$, and choose $\{\xi_j^l,\ 1 \le j \le k,\ 1 \le l \le l(j)\}$ to be a frame that, when properly ordered, is an adapted basis realizing DIS $(\mathcal{G}_1(x_1(t)), \dots, \mathcal{G}_k(x_k(t)))$. Next, we write

$$
(10)\qquad x_{j_0}'(t) = \sum_{\alpha=1}^{l(1)} c_1^\alpha \xi_1^\alpha + \cdots + \sum_{\alpha=1}^{l(k)} c_k^\alpha \xi_k^\alpha\,,
$$

for appropriate coefficients c_j^α, $1 \le j \le k$, $1 \le \alpha \le l(j)$. From (8) we have

$$
(11)\qquad x_{j_0}' = x_j' + b_j,\quad b_j = f_j(q_j) - f_{j_0}(q_{j_0})\,.
$$

From (10) and (11) we obtain a linear system of n equations in the n unknowns c_j^α, by taking inner products with $\xi_{j_0}^l$ and ξ_j^l, $j \ne j_0$, respectively:

$$
(12)\qquad
\begin{aligned}
&\sum_{\alpha=1}^{l(1)} c_1^\alpha \langle \xi_1^\alpha, \xi_{j_0}^l\rangle + \cdots + \sum_{\alpha=1}^{l(k)} c_k^\alpha \langle \xi_k^\alpha, \xi_{j_0}^l\rangle = 0, \quad 1 \le l \le l(j_0)\,,\\
&\sum_{\alpha=1}^{l(1)} c_1^\alpha \langle \xi_1^\alpha, \xi_j^l\rangle + \cdots + \sum_{\alpha=1}^{l(k)} c_k^\alpha \langle \xi_k^\alpha, \xi_j^l\rangle = \langle b_j, \xi_j^l\rangle\,,
\end{aligned}
$$

where, in the last set of equations, $j \in \{1, \dots, \hat{j}_0, \dots, k\}$ and $1 \le l \le l(j)$.

Reordering the equations if necessary, the system can be written in the form $M_{\mathcal{B}} C = B$ (notation as in §2). The n-dimensional vector C formed by the coefficients of $x'_{j_0}(t)$, relative to the adapted basis $\mathcal{B}$, satisfies $C = (M_{\mathcal{B}})^{-1}B$. Since the operator norm of $(M_{\mathcal{B}})^{-1}$ is $\lambda_{\mathcal{B}}^{-1}$,

$$\Big[\sum_{\substack{1\le\alpha\le l(j)\\1\le j\le k}} (c_j^{\alpha})^2\Big]^{\frac12} \le (\lambda_{\mathcal{B}})^{-1}\Big[\sum_{\substack{1\le l\le l(j)\\1\le j\le k}} \langle b_j, \xi_j^l\rangle^2\Big]^{\frac12}. \tag{13}$$

(Notice that $b_{j_0} = 0$, so it is legitimate to include the index j_0 in the summation on the right hand side of (13).)

By the Cauchy-Schwarz inequality (13) implies

$$\Big[\sum_{\substack{1\le\alpha\le l(j)\\1\le j\le k}} (c_j^{\alpha})^2\Big]^{\frac12} \le (\lambda_{\mathcal{B}})^{-1}\Big[\sum_{j=1}^{k} l(j)|b_j|^2\Big]^{\frac12}. \tag{14}$$

Expanding the quadratic form given by $|x'_{j_0}|^2$ and (10), and recalling the definition of $\Lambda_{\mathcal{B}}$ from §2, we have

$$|df_{j_0}x'_{j_0}|^2 = |x'_{j_0}|^2 \le \Lambda_{\mathcal{B}} \sum_{\substack{1\le\alpha\le l(j)\\1\le j\le k}} (c_j^{\alpha})^2. \tag{15}$$

Since the adapted basis $\{\xi_j^l(t)\}$ realizes DIS $\big(\mathcal{G}_1(x_1(t)),\ldots,\mathcal{G}_k(x_k(t))\big)$, (14) and (15) imply that $|x'_{j_0}(t)|$ can be estimated from above by

$$\Big[\text{DIS }\big(\mathcal{G}_1(x_1(t)),\ldots,\mathcal{G}_k(x_k(t))\big)\Big]\Big[\sum_{j=1}^{k}(n-m_j)\,|f_j(q_j)-f_{j_0}(q_{j_0})|^2\Big]^{\frac12}. \tag{16}$$

In particular,

$$\text{Length } x_{j_0}|_{[-1,t_n]} = \int_{-1}^{t_n} |x'_{j_0}(t)|dt$$

$$\le (t_n+1)\Big[\sum_{j=1}^{k}(n-m_j)\,|f_j(q_j)-f_{j_0}(q_{j_0})|^2\Big]^{\frac12}\cdot\Big[\sup_{\substack{p_i\in \mathrm{int}M_i\\1\le i\le k}} DIS\,\big(\mathcal{G}_1(p_1),\ldots,\mathcal{G}_k(p_k)\big)\Big].$$

Letting $n\to\infty$ we have, in view of the last relation in (9),

$$d_{j_0}(q_{j_0},\partial M_{j_0}) \le \limsup_{n\to\infty} \text{Length } x_{j_0}|_{[-1,t_n]}$$

$$\le (T+1)\Big[\sum_{j=1}^{k}(n-m_j)\,|f_j(q_j)-f_{j_0}(q_{j_0})|^2\Big]^{\frac12}\cdot\Big[\sup_{\substack{p_i\in \mathrm{int}M_i\\1\le i\le k}} DIS\,\big(\mathcal{G}_1(p_1),\ldots,\mathcal{G}_k(p_k)\big)\Big].$$

It now follows from (2) that $d_{j_0}(q_{j_0},\partial M_{j_0}) < (T+1)d_{j_0}(q_{j_0},\partial M_{j_0})$, so that $T > 0$. Hence the middle equation in (6) is valid for $t = 0$, and so is (5).

This proves the theorem under the assumption that the sum of the codimensions of all submanifolds is equal to n.

Next, we indicate how the above arguments need to be modified in the case $\beta > 0$. Locally, we may regard f_j as the inclusion map. We also have the situation covered by (9). For $t \in (-1, T)$ fixed we write

$$x'_{j_0}(t) = \sum_{\alpha=1}^{l(1)} c_1^\alpha \xi_1^\alpha + \cdots + \sum_{\alpha=1}^{l(k)} c_k^\alpha \xi_k^\alpha + \sum_{m=1}^{\beta} d_m \eta^m, \tag{17}$$

for appropriate coefficients c_j^α, d_m, where $1 \le j \le k$, $1 \le \alpha \le l(j)$, $1 \le m \le \beta$ and $\{\eta^m\}$ is an orthonormal basis of $[\mathcal{G}_1(x_1(t)) \oplus \cdots \oplus \mathcal{G}_k(x_k(t))]^\perp$. Again, ξ_j^l is chosen so that, with an appropriate ordering, $\{\xi_j^l,\ 1 \le j \le k;\ 1 \le l \le l(j)\}$ constitutes an admissible basis of $\mathcal{G}_1(x_1(t)) \oplus \cdots \oplus \mathcal{G}_k(x_k(t))$ that realizes $\mathrm{DIS}(\mathcal{G}_1(x_1(t)), \ldots, \mathcal{G}_k(x_k(t)))$.

From (11), (17), the third relation in (9) and $x'_1 - x'_{j_0} = a_{j_0} = -b_1$, we have:

$$\begin{aligned} &\sum_{\alpha=1}^{l(1)} c_1^\alpha \langle \xi_1^\alpha, \xi_{j_0}^l \rangle + \cdots + \sum_{\alpha=1}^{l(k)} c_k^\alpha \langle \xi_k^\alpha, \xi_{j_0}^l \rangle = 0, 1 \le l \le l(j_0). \\ &\sum_{\alpha=1}^{l(1)} c_1^\alpha \langle \xi_1^\alpha, \xi_j^l \rangle + \cdots + \sum_{\alpha=1}^{l(k)} c_k^\alpha \langle \xi_k^\alpha, \xi_j^l \rangle = \langle b_j, \xi_j^l \rangle. \\ &d_m = \langle b_1, \eta^m \rangle,\ 1 \le m \le \beta. \end{aligned} \tag{18}$$

(In the second set of equations, $j \in \{1, \ldots, \hat{j_0}, \ldots, k\}$, $1 \le l \le l(j)$.)

We write the above $n \times n$ system in the unknowns c_j^α, d_m in matrix form as $AC = B$. We claim that, as in the case $\beta = 0$,

$$|B| \le \Big[\sum_{j=1}^{k} l(j) |b_j|^2\Big]^{\frac{1}{2}}. \tag{19}$$

To see this we first take $j > 1$ and estimate from the second set of equations in (18) using the Cauchy-Schwarz inequality:

$$\sum_{1 \le l \le l(j)} \langle b_j, \xi_j^l \rangle^2 \le l(j) |b_j|^2, \quad 2 \le j \le k. \tag{20}$$

In order to deal with the case $j = 1$, we let Π be the orthogonal projection onto $\mathcal{G}_1(x_1(t)) \oplus \cdots \oplus \mathcal{G}_k(x_k(t))$. From the last equation in (18),

$$\sum_{m=1}^{\beta} \langle b_1, \eta^m \rangle^2 = |(I - \Pi) b_1|^2 \le l(1) |(I - \Pi) b_1|^2. \tag{21}$$

If $j_0 = 1$ one has $b_1 = 0$ and (19) follows from (20). On the other hand, if $j_0 \neq 1$ the second set of equations in (18) implies, with $j = 1$,

$$\sum_{1 \leq l \leq l(1)} \langle b_1, \xi_1^l \rangle^2 = \sum_{1 \leq l \leq l(1)} \langle \Pi b_1, \xi_1^l \rangle^2 \leq l(1) |\Pi b_1|^2 . \tag{22}$$

The estimate in (19) now follows from (20)-(22).

After rearranging the rows if necessary, we may assume that the matrix A above is a 2×2 matrix of blocks. The off-diagonal blocks are 0, and along the diagonal one has the matrix $M_{\mathcal{B}}$ and the identity matrix $I_{\beta \times \beta}$.

It follows that the operator norm $|A^{-1}|$, which is the reciprocal of the smallest eigenvalue of A, is at most $\lambda_{\mathcal{B}}^{-1}$ (recall $\lambda_{\mathcal{B}} \leq 1$ from (1)). From (19) and $C = A^{-1}B$ one then has an estimate on the Euclidean norm of the vector C:

$$|C| \leq (\lambda_{\mathcal{B}})^{-1} \Big[\sum_{j=1}^{k} l(j) |b_j|^2 \Big]^{\frac{1}{2}} . \tag{23}$$

Next we expand the quadratic form given by $|x'_{j_0}(t)|^2$ and (17). Using the fact that the largest eigenvalue of A is $\max\{1, \Lambda_{\mathcal{B}}\} \leq \Lambda_{\mathcal{B}}$ by (1), one has

$$|x'_{j_0}|^2 \leq \Lambda_{\mathcal{B}} |C|^2. \tag{24}$$

From (23) and (24) one derives the same estimate as in (16). From this point on the proof proceeds exactly as in the previous case $\beta = 0$. This concludes the proof of Theorem 2.

4. Proof of Theorem 1

In this short section both the statement and the proof of Theorem 2 are used to prove Theorem 1. In actuality, Theorem 1 is valid in the more general geometric context of complete isometric immersions:

THEOREM 3. *For* $1 \leq j \leq k \leq n$, $2 \leq k \leq n$, *let* $f_j \colon (M_j, g_j) \to \mathbf{R}^n$ *be an isometric immersion where* M_j *is connected,* $1 \leq \dim M_j = m_j \leq n-1$, $\operatorname{codim} M_j = n - m_j$, $\operatorname{codim} M_1 + \cdots + \operatorname{codim} M_k \leq n$, *and* g_j *is complete. Consider the Gauss map* $\mathcal{G}_j \colon M_j \to G(n - m_j, n)$, $\mathcal{G}_j(p) = [df_j(p) T_p M_j]^{\perp}$. *If for all* $E_j \in \overline{\mathcal{G}_j(M_j)}$, $1 \leq j \leq k$, *the subspaces* $E_1, \ldots, E_k$ *of* $\mathbf{R}^n$ *are in direct sum then* $f_1(M_1) \cap \cdots \cap f_k(M_k) \neq \varnothing$. *Moreover,* $f_1(M_1) \cap \cdots \cap f_k(M_k)$ *reduces to a single point if* $\operatorname{codim} M_1 + \cdots + \operatorname{codim} M_k = n$.

To show existence in Theorem 3, fix points $\overline{q_1} \in M_1, \dots, \overline{q_k} \in M_k$ and assume, by contradiction, that $M_1 \cap \dots \cap M_k = \varnothing$. Exhaust each M_j by a sequence of compact submanifolds with boundary M_j^l, $l \to \infty$. We may assume $\overline{q_j} \in M_j^l$ for every l. The claim is that Theorem 2 applies to the immersions $f_j\colon (M_j^l, g_j) \to \mathbf{R}^n$, $1 \le j \le k$, if l is large enough. Indeed, the hypotheses in Theorem 1 certainly imply that if $V_j \in \mathcal{G}_j(\mathrm{int} M_j^l)$ the subspaces $V_1, \dots, V_k$ of $\mathbf{R}^n$ are in direct sum. Note that the denominator in (2) remains the same as $l \to \infty$, but the numerator tends to infinity, since all induced riemannian metrics g_j are complete. It remains to argue that the left hand side in (2) does not tend to infinity with l. According to Lemma 1, if DIS of a sequence of direct sums tends to infinity then, after passing to convergent subsequences in the appropriate Grassmanians, the summands tend to subspaces which, when taken together, are *not* in direct sum. But this would be an outright contradiction to the central hypothesis in Theorem 3 namely, that for all $V_j \in \overline{\mathcal{G}_j(M_j)}$ the subspaces $V_1, \dots, V_k$ of $\mathbf{R}^n$ *are* in direct sum. This settles the issue of existence in Theorem 3.

Finally, we prove that the intersection set in Theorem 3 reduces to a single point if the sum of all codimensions is n. Since the configurations of normal spaces have uniformly bounded distortion, the estimate given by (16), in the proof of Theorem 2, shows that $|x_j'(t)|$ is uniformly bounded for every j (notice that the special property in (9) enjoyed by the index j_0 was used only later in the proof). Since all manifolds M_j are complete relative to the induced metric one has $T = \infty$, regardless of the initial choice of points $(q_1, \dots, q_k) \in M_1 \times \dots \times M_k$. Hence each such choice yields an element in $f_1(M_1) \cap \dots \cap f_k(M_k)$ by considering the maximal extension of the local solutions given by Lemma 2, and then setting $t = 0$.

Since $[df_1(p_1)T_{p_1}M_1]^\perp \oplus \dots \oplus [df_k(p_k)T_{p_k}M_k]^\perp = \mathbf{R}^n$ for all choices of $p_j \in M_j$, the set $f_1(M_1) \cap \dots \cap f_k(M_k)$ is discrete. By the arguments in the first paragraph this intersection is also non-empty. For $p \in f_1(M_1) \cap \dots \cap f_k(M_k)$, the subset $\mathcal{O}_p$ of $M_1 \times \dots \times M_k$ consisting of all $(q_1, \dots, q_k)$ giving rise to the point p according to the procedure indicated at the end of the last paragraph is non-empty and open. Indeed, to show $\mathcal{O}_p \neq \varnothing$ if $p \in f_1(M_1) \cap \dots \cap f_k(M_k)$ one starts with any $(q_1, \dots, q_k)$ such that all $f_j(q_j)$ are sufficiently close (or equal) to p. Since the trajectories have uniformly bounded speed (by the estimate given by (16)) and $f_1(M_1) \cap \dots \cap f_k(M_k)$ is discrete, the images $f_j(x_j(t))$ of the solutions will necessarily pass through p at time $t = 0$. The fact that $\mathcal{O}_p$ is open is a consequence of the discreteness of the above intersection, together with the continuous dependence of solutions of ordinary differential equations upon the coefficients and the initial conditions, over bounded time intervals. Here one should note that the solutions of (8) are independent of the choice of frames, since they uniquely solve (6). Furthermore, the sets $\mathcal{O}_p$ are pairwise disjoint, by the uniqueness of solutions of initial value problems for ordinary differential equations with smooth coefficients. Hence,

$$M_1 \times \cdots \times M_k = \bigcup_{p \in f_1(M_1) \cap \cdots \cap f_k(M_k)} \mathcal{O}_p ,$$

where, as observed, the sets $\mathcal{O}_p$ are non-empty, open and pairwise disjoint. Connectedness of $M_1 \times \cdots \times M_k$ now implies that there is only one such set, thus showing that $f_1(M_1) \cap \cdots \cap f_k(M_k)$ reduces to a single point. This concludes the proof of Theorem 3.

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