This proves that sp($L$) is finite. If $e^{\lambda_1 z}, \dots, e^{\lambda_s z}$ are the only exponential functions contained in $L$, then every exponential polynomial contained in $L$ must be of the form $\sum_{i=1}^{s} p_i(z)\, e^{\lambda_i z}$, where $p_1, \dots, p_s$ are polynomials. Since the set of all polynomials is dense in $C(\mathbf{R})$ and $L \neq C(\mathbf{R})$, it follows that the degrees of $p_1, \dots, p_s$ must be bounded. As the set of exponential polynomials is dense in $L$, we find that each element of $L$ is an exponential polynomial, which completes the proof of Theorem 2 in the case when $h \equiv 0$.

The general case can be reduced to the previous one in the same way as in the proof of Theorem 1. Again, it is enough to show that $f_n$ is an exponential polynomial. Since $\Delta_b f_n$ satisfies the homogeneous version of (1), it follows that $\Delta_b f_n$ is an exponential polynomial for every $b$. Therefore, by Carroll's theorem [2], $f_n$ is also an exponential polynomial. $\square$

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