1. Introduction

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **49 (2003)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **21.09.2024**

L'Enseignement Mathématique, t. 49 (2003), p. 67–84

SOME REMARKS ON NONCONNECTED COMPACT LIE GROUPS

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ABSTRACT. Let G_o be a connected compact Lie group and let Γ be a finite group. Denote by $\mathcal{E}$ the set of equivalence classes of extensions of Γ by G_o. Using the notion of principal subgroup, we show that two nonconnected compact Lie groups are isomorphic if and only if the cohomology classes corresponding to their naturally associated extensions are in the same orbit under an action of $\mathrm{Out}(G_o) \times \mathrm{Aut}(\Gamma)$ on $\mathcal{E}$. Explicit examples of this cohomological classification are given. A revisited "Sandwich" Theorem and some criteria for the splitting of the extension associated to a compact Lie group are also presented.

1. INTRODUCTION

Naturally associated to a compact Lie group G, there is a group extension $G_o \hookrightarrow G \twoheadrightarrow \Gamma$, where G_o denotes the connected component of the identity of G, and $\Gamma = \pi_0(G) = G/G_o$ denotes the finite group of connected components. The problem we want to address here is the following: given a connected compact Lie group G_o and a finite group Γ, can one classify *up to isomorphism* the compact Lie groups with connected component isomorphic to G_o and with group of components isomorphic to Γ ? Of course, the classical theory of group extensions with nonabelian kernel and its relationship to the appropriate cohomology groups of degree 2 gives a partial answer to our question. However, these cohomology groups classify groups only *up to equivalence* of extensions, leaving the isomorphism question unsolved in general. This is illustrated, in the case of finite groups, by the following well-known example: the cohomology group $H^2(\mathbf{Z}/3;\mathbf{Z}/3)$ is isomorphic to $\mathbf{Z}/3$, but there are only two nonisomorphic groups with 9 elements, two

*) The author was supported by the Japan Society for the Promotion of Science (JSPS) and by the Grant-in-Aid for Scientific Research No. 12000751-00 of the JSPS.

nonequivalent extensions corresponding to the group $\mathbf{Z}/9$. Let us restate the above problem in this context: given two compact Lie groups G and G', how can one tell from the cohomology classes associated to their extensions whether they are *isomorphic* or not?

As the Lie algebra associated to a Lie group only gives information on the connected component, Lie groups have often been studied under the hypothesis of connectedness. However, *nonconnected* compact Lie groups arise quite naturally: the orthogonal group $\mathrm{O}(n)$ of rigid motions that fix the origin of the Euclidean n-space, or more generally, the isometry group of a compact Riemannian manifold are examples of such groups. Moreover, some physicists have revived the idea that nonconnected compact Lie groups might be the relevant objects in certain gauge theories (see [18] for instance). The problem of classification addressed here has a long history, starting mainly with the work of de Siebenthal in the '50s [9]. However his paper, as well as McInnes' recent paper [18], restrict attention to particular cases, and do not solve the problem in full generality. We first became interested in this question when we needed a precise answer as a basic ingredient for a generalization, in the nonconnected setting, of the remarkable theorem of Curtis, Wiederhold, and Williams, saying that *two connected compact Lie groups are isomorphic if and only if the normalizers of their maximal tori are isomorphic* (see [7], and the paper by Osse [20] for a general proof, valid in the non-semisimple situation). This generalization then provides a new proof, in the nonconnected case, of the homotopy-theoretic result that a compact Lie group is, up to isomorphism, characterized by its classifying space [13]. Even though the cohomological classification of compact Lie groups described in this work might be "well-known to the experts", we could not find any reference for it in the literature. Besides, even a recent paper, in which this classification is needed, gives a description of it which does not hold in general (see the Introduction of [13] for more details). Therefore our motivation is twofold. Firstly, we intend to give a precise account of a solution to the classification problem described above, together with examples illustrating the fact that it can be explicitly carried out when given some specific groups G_0 and Γ, but that some care has to be taken to avoid pitfalls. Secondly, we want to pay a tribute to the work of de Siebenthal [8, 9] by showing that, as in the restricted cases he considered, the principal subgroups he defined play a key role in the general case.

Let us briefly describe our main result. For G_0 and Γ as above, let $\mathcal{E} = \mathcal{E}(\Gamma, G_0)$ denote the set of equivalence classes of extension of Γ by G_0. To a given class corresponds an "outer action", i.e. a homomorphism

$\varphi\colon \Gamma \to \mathrm{Out}(G_o)$, where $\mathrm{Out}(G_o)$ is the group of outer automorphism of G_o. Let Z_o denote the center of G_o (recall that Z_o is the direct product of a finite abelian group and a torus); by "restriction", φ gives rise to an action on Z_o, i.e. to a homomorphism $\bar{\varphi}\colon \Gamma \to \mathrm{Aut}(Z_o)$. As explained in Section 3, the classical theory of group extensions applied to this particular case shows that the set $\mathcal{E}$ corresponds to the disjoint union

$$\mathcal{E} = \mathcal{E}(\Gamma, G_o) \approx \coprod_{\varphi \in \mathrm{Hom}(\Gamma, \mathrm{Out}(G_o))} H^2_{\bar{\varphi}}(\Gamma; Z_o)\,.$$

For $\alpha \in \mathrm{Out}(G_o)$, let $\bar{\alpha} \in \mathrm{Aut}(Z_o)$ denote the restricted automorphism. A cohomology class $u \in H^2_{\bar{\varphi}}(\Gamma; Z_o)$ will be canonically identified with the corresponding equivalence class of extensions and will be denoted by $u = \left[Z_o \overset{\mu}{\hookrightarrow} Z \overset{\nu}{\twoheadrightarrow} \Gamma\right]$. In Section 4, we show that the map

$$\big((\alpha,\beta), u\big) \longmapsto (\alpha,\beta)\,.\,u = \left[Z_o \overset{\mu\circ\bar{\alpha}}{\hookrightarrow} Z \overset{\beta\circ\nu}{\twoheadrightarrow} \Gamma\right]$$

defines an action of $\mathrm{Out}(G_o)\times\mathrm{Aut}(\Gamma)$ on the set $\mathcal{E}$. Using principal subgroups, we then prove that this action allows us to pass from *up to equivalence of extensions* to *up to isomorphism of Lie groups*, as stated in the following theorem.

MAIN THEOREM. *Two compact Lie groups G_{u_1} and G_{u_2} are isomorphic if and only if the corresponding cohomology classes $u_1 \in H^2_{\bar{\varphi}_1}(\Gamma; Z_o) \subset \mathcal{E}$ and $u_2 \in H^2_{\bar{\varphi}_2}(\Gamma; Z_o) \subset \mathcal{E}$ are in the same orbit under the action of* $\mathrm{Out}(G_o) \times \mathrm{Aut}(\Gamma)$.

REMARK 1.1. It is well-known that there exists only a finite number of non-isomorphic compact Lie groups of given dimension and number of components (see [23], Theorem 5.9.5). In particular, the number of orbits in the Main Theorem is always finite.

NOTE. The Main Theorem is straightforward for the case in which G_o is abelian, i.e. the connected component is a torus. So, for the rest of this work, we will always suppose that G_o is *nonabelian.*

The paper is organized as follows. Section 2 is based on the work of de Siebenthal. It recalls the notion of principal subgroup and gathers results related to nonconnected compact Lie groups. As already mentioned, the theory of group extension is applied to our situation in Section 3. In particular, it is shown that centralizers of principal subgroups are extensions of Γ by the center Z_0 of G_0 that completely control the situation. The Main Theorem is proved in Section 4 and as an illustration, two examples are then given. The final section relates the approach taken in the present work with the natural question of the splitting of the extension associated to a compact Lie group. As an application of principal subgroups, a revisited "Sandwich" Theorem is proved. Particular cases where the extension is always split are also described, and, finally, a "minimal" extension failing to be split is exhibited.

ACKNOWLEDGMENTS. The material in this note is taken from my Ph.D. thesis [13]. It is a pleasure to thank warmly my advisor Professor U. Suter for his guidance and constant encouragement. I am also indebted to M. Matthey for his careful reading of an earlier version of this paper and for his useful comments.

2. Compact Lie groups: a review

In this section, we recall the existence of subgroups of G whose related extensions have close relationships to that corresponding to G. First, we introduce more notation. Let T be a fixed maximal torus in G_0, and let LT denote its Lie algebra. Let B be a basis of the root system $R = R(G_0, T)$ of G_0 associated to T. Let $\mathcal{H}$ denote the maximal semisimple ideal of the Lie algebra LG_0 of G_0; the *principal diagonal* of G_0 with respect to B is the 1-dimensional subspace given by $D(B) = \{X \in LT : \alpha(X) = \beta(X), \text{ for all } \alpha, \beta \in B\} \cap \mathcal{H} \subset LT$. The image $\Delta = \Delta(B) = \exp(D(B))$ of this subspace under the exponential map is easily seen to be a closed subgroup of T, isomorphic to the circle group $\mathbf{S}^1$. With a slight abuse of language, we will also call this subgroup a *principal diagonal*. We are now ready to recall the definition of one of the key notions of the present work.

DEFINITION 2.1 (de Siebenthal). A *principal subgroup* of G_0 (associated to T) is a connected closed subgroup H such that H is not contained in any proper connected closed subgroup of maximal rank, and such that $\Delta(B) \subset H$ for some basis B of R.

The work of de Siebenthal shows that any compact Lie group possesses a principal subgroup of rank 1, thus isomorphic to SU(2) or SO(3), and that two such principal subgroups are conjugate [8].