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SOME REMARKS ON NONCONNECTED COMPACT LIE GROUPS

by Jean-François HÄMMERLI *)

ABSTRACT. Let G_0 be a connected compact Lie group and let Γ be a finite group. Denote by $\mathcal{E}$ the set of equivalence classes of extensions of Γ by G_0. Using the notion of principal subgroup, we show that two nonconnected compact Lie groups are isomorphic if and only if the cohomology classes corresponding to their naturally associated extensions are in the same orbit under an action of $\mathrm{Out}(G_0) \times \mathrm{Aut}(\Gamma)$ on $\mathcal{E}$. Explicit examples of this cohomological classification are given. A revisited "Sandwich" Theorem and some criteria for the splitting of the extension associated to a compact Lie group are also presented.

1. INTRODUCTION

Naturally associated to a compact Lie group G, there is a group extension $G_0 \hookrightarrow G \twoheadrightarrow \Gamma$, where G_0 denotes the connected component of the identity of G, and $\Gamma = \pi_0(G) = G/G_0$ denotes the finite group of connected components. The problem we want to address here is the following: given a connected compact Lie group G_0 and a finite group Γ, can one classify *up to isomorphism* the compact Lie groups with connected component isomorphic to G_0 and with group of components isomorphic to Γ ? Of course, the classical theory of group extensions with nonabelian kernel and its relationship to the appropriate cohomology groups of degree 2 gives a partial answer to our question. However, these cohomology groups classify groups only *up to equivalence* of extensions, leaving the isomorphism question unsolved in general. This is illustrated, in the case of finite groups, by the following well-known example: the cohomology group $H^2(\mathbf{Z}/3; \mathbf{Z}/3)$ is isomorphic to $\mathbf{Z}/3$, but there are only two nonisomorphic groups with 9 elements, two

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nonequivalent extensions corresponding to the group $\mathbf{Z}/9$. Let us restate the above problem in this context: given two compact Lie groups G and G', how can one tell from the cohomology classes associated to their extensions whether they are *isomorphic* or not?

As the Lie algebra associated to a Lie group only gives information on the connected component, Lie groups have often been studied under the hypothesis of connectedness. However, *nonconnected* compact Lie groups arise quite naturally: the orthogonal group $O(n)$ of rigid motions that fix the origin of the Euclidean n-space, or more generally, the isometry group of a compact Riemannian manifold are examples of such groups. Moreover, some physicists have revived the idea that nonconnected compact Lie groups might be the relevant objects in certain gauge theories (see [18] for instance). The problem of classification addressed here has a long history, starting mainly with the work of de Siebenthal in the '50s [9]. However his paper, as well as McInnes' recent paper [18], restrict attention to particular cases, and do not solve the problem in full generality. We first became interested in this question when we needed a precise answer as a basic ingredient for a generalization, in the nonconnected setting, of the remarkable theorem of Curtis, Wiederhold, and Williams, saying that *two connected compact Lie groups are isomorphic if and only if the normalizers of their maximal tori are isomorphic* (see [7], and the paper by Osse [20] for a general proof, valid in the non-semisimple situation). This generalization then provides a new proof, in the nonconnected case, of the homotopy-theoretic result that a compact Lie group is, up to isomorphism, characterized by its classifying space [13]. Even though the cohomological classification of compact Lie groups described in this work might be "well-known to the experts", we could not find any reference for it in the literature. Besides, even a recent paper, in which this classification is needed, gives a description of it which does not hold in general (see the Introduction of [13] for more details). Therefore our motivation is twofold. Firstly, we intend to give a precise account of a solution to the classification problem described above, together with examples illustrating the fact that it can be explicitly carried out when given some specific groups G_0 and Γ, but that some care has to be taken to avoid pitfalls. Secondly, we want to pay a tribute to the work of de Siebenthal [8, 9] by showing that, as in the restricted cases he considered, the principal subgroups he defined play a key role in the general case.

Let us briefly describe our main result. For G_0 and Γ as above, let $\mathcal{E} = \mathcal{E}(\Gamma, G_0)$ denote the set of equivalence classes of extension of Γ by G_0. To a given class corresponds an "outer action", i.e. a homomorphism

$\varphi\colon \Gamma \to \mathrm{Out}(G_o)$, where $\mathrm{Out}(G_o)$ is the group of outer automorphism of G_o. Let Z_o denote the center of G_o (recall that Z_o is the direct product of a finite abelian group and a torus); by "restriction", φ gives rise to an action on Z_o, i.e. to a homomorphism $\bar{\varphi}\colon \Gamma \to \mathrm{Aut}(Z_o)$. As explained in Section 3, the classical theory of group extensions applied to this particular case shows that the set $\mathcal{E}$ corresponds to the disjoint union

$$\mathcal{E} = \mathcal{E}(\Gamma, G_o) \approx \coprod_{\varphi \in \mathrm{Hom}(\Gamma, \mathrm{Out}(G_o))} H^2_{\bar{\varphi}}(\Gamma; Z_o)\,.$$

For $\alpha \in \mathrm{Out}(G_o)$, let $\bar{\alpha} \in \mathrm{Aut}(Z_o)$ denote the restricted automorphism. A cohomology class $u \in H^2_{\bar{\varphi}}(\Gamma; Z_o)$ will be canonically identified with the corresponding equivalence class of extensions and will be denoted by $u = \left[Z_o \overset{\mu}{\hookrightarrow} Z \overset{\nu}{\twoheadrightarrow} \Gamma\right]$. In Section 4, we show that the map

$$\big((\alpha,\beta),u\big) \longmapsto (\alpha,\beta)\,.\,u = \left[Z_o \overset{\mu\circ\bar{\alpha}}{\hookrightarrow} Z \overset{\beta\circ\nu}{\twoheadrightarrow} \Gamma\right]$$

defines an action of $\mathrm{Out}(G_o) \times \mathrm{Aut}(\Gamma)$ on the set $\mathcal{E}$. Using principal subgroups, we then prove that this action allows us to pass from *up to equivalence of extensions* to *up to isomorphism of Lie groups*, as stated in the following theorem.

MAIN THEOREM. *Two compact Lie groups G_{u_1} and G_{u_2} are isomorphic if and only if the corresponding cohomology classes $u_1 \in H^2_{\bar{\varphi}_1}(\Gamma; Z_o) \subset \mathcal{E}$ and $u_2 \in H^2_{\bar{\varphi}_2}(\Gamma; Z_o) \subset \mathcal{E}$ are in the same orbit under the action of* $\mathrm{Out}(G_o) \times \mathrm{Aut}(\Gamma)$.

REMARK 1.1. It is well-known that there exists only a finite number of non-isomorphic compact Lie groups of given dimension and number of components (see [23], Theorem 5.9.5). In particular, the number of orbits in the Main Theorem is always finite.

NOTE. The Main Theorem is straightforward for the case in which G_o is abelian, i.e. the connected component is a torus. So, for the rest of this work, we will always suppose that G_o is *nonabelian.*

The paper is organized as follows. Section 2 is based on the work of de Siebenthal. It recalls the notion of principal subgroup and gathers results related to nonconnected compact Lie groups. As already mentioned, the theory of group extension is applied to our situation in Section 3. In particular, it is shown that centralizers of principal subgroups are extensions of Γ by the center Z_0 of G_0 that completely control the situation. The Main Theorem is proved in Section 4 and as an illustration, two examples are then given. The final section relates the approach taken in the present work with the natural question of the splitting of the extension associated to a compact Lie group. As an application of principal subgroups, a revisited "Sandwich" Theorem is proved. Particular cases where the extension is always split are also described, and, finally, a "minimal" extension failing to be split is exhibited.

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2. COMPACT LIE GROUPS: A REVIEW

In this section, we recall the existence of subgroups of G whose related extensions have close relationships to that corresponding to G. First, we introduce more notation. Let T be a fixed maximal torus in G_0, and let LT denote its Lie algebra. Let B be a basis of the root system $R = R(G_0, T)$ of G_0 associated to T. Let $\mathcal{H}$ denote the maximal semisimple ideal of the Lie algebra LG_0 of G_0; the *principal diagonal* of G_0 with respect to B is the 1-dimensional subspace given by $D(B) = \{X \in LT : \alpha(X) = \beta(X), \text{ for all } \alpha, \beta \in B\} \cap \mathcal{H} \subset LT$. The image $\Delta = \Delta(B) = \exp(D(B))$ of this subspace under the exponential map is easily seen to be a closed subgroup of T, isomorphic to the circle group $\mathbf{S}^1$. With a slight abuse of language, we will also call this subgroup a *principal diagonal*. We are now ready to recall the definition of one of the key notions of the present work.

DEFINITION 2.1 (de Siebenthal). A *principal subgroup* of G_0 (associated to T) is a connected closed subgroup H such that H is not contained in any proper connected closed subgroup of maximal rank, and such that $\Delta(B) \subset H$ for some basis B of R.

The work of de Siebenthal shows that any compact Lie group possesses a principal subgroup of rank 1, thus isomorphic to SU(2) or SO(3), and that two such principal subgroups are conjugate [8].

NOTE. For the rest of this work, "principal subgroup" will always mean principal subgroup *of rank* 1.

Before stating the main result of this section, which is a direct consequence of the results of de Siebenthal, we introduce three subgroups of G. Let H_T be a principal subgroup associated to T, and let $Z = Z_G(H_T)$ denote its centralizer in G. This subgroup will play a crucial role in the paper. Let also $N = N_G(T)$ be the normalizer of T in G. We will use the convenient notation $N_o = N_{G_o}(T)$ for the intersection of N with G_o, but one should not be confused, N_o is *not* connected (its group of components being the Weyl group of G_o). Finally, we will consider the centralizer $Q = Z_G(\Delta)$ in G of a principal diagonal Δ.

THEOREM 2.2. *For any compact Lie group G there exists a commutative diagram*

$$\begin{array}{ccccc}
Z_o & \hookrightarrow & Z & \twoheadrightarrow & \Gamma \\
\downarrow & & \downarrow & & \| \\
T & \hookrightarrow & Q & \twoheadrightarrow & \Gamma \\
\downarrow & & \downarrow & & \| \\
N_o & \hookrightarrow & N & \twoheadrightarrow & \Gamma \\
\downarrow & & \downarrow & & \| \\
G_o & \hookrightarrow & G & \twoheadrightarrow & \Gamma
\end{array}$$

where each row is a group extension.

Proof. The centralizer of H_T in G_o is equal to the center Z_o (by a theorem of Borel and de Siebenthal [3], this property characterizes the closed subgroups of G_o that are not contained in any proper connected closed subgroup of maximal rank [5, Ex. 15, p. 116]). As Z intersects every component of G ([8], Théorème 4, pp. 253–254), we get an extension $Z_o \hookrightarrow Z \twoheadrightarrow \Gamma$. The other statements are deduced from the fact that $\Delta \subset T$ contains a regular element, i.e. an element that is contained in exactly one maximal torus, namely T in the present case (see [12] or [17] for more details). □

REMARK 2.3. We call the subgroup Q an *extended maximal torus* of G. These subgroups share some important properties with maximal tori: they are all conjugate, and fixing one of them, its conjugates by the elements of G_o cover the whole group G. They appear in the literature under various disguises (see for instance Oliver [19], Section 1, and Segal [22], §4), as explained in [12].

In the final section, we will see how the splitting of the extension associated to G is related to the splitting of the extensions associated to N and Q that appear in Theorem 2.2.

We end this section by recalling a very important result relating the inner, "usual", and outer automorphism groups of a connected compact Lie group. This result is one of the main reasons why the case of compact Lie groups is well controlled when applying the theory of group extensions, as we will see in Section 3. For the proof, we refer to de Siebenthal [9, Théorème, pp. 46–47] (for another approach consult Bourbaki [5], §4.10).

THEOREM 2.4 (de Siebenthal). *Let G_0 be a connected compact Lie group and let $H \subset G_0$ be a principal subgroup. Then the extension*

$$\mathrm{Inn}(G_0) \overset{\iota}{\hookrightarrow} \mathrm{Aut}(G_0) \overset{\pi}{\twoheadrightarrow} \mathrm{Out}(G_0)$$

is split, i.e.

$$\mathrm{Aut}(G_0) \cong \mathrm{Inn}(G_0) \rtimes \mathrm{Out}(G_0)\,.$$

A possible splitting is given by $s\colon \mathrm{Out}(G_0) \to \mathrm{Aut}(G_0)$, where $s(\alpha)$ is the unique automorphism in $\pi^{-1}(\alpha)$ fixing H pointwise.

REMARK 2.5. The fact that the extension associated to $\mathrm{Aut}(G_0)$ is split was known before the work of de Siebenthal, at least in the semisimple case, and appeared in a paper of Dynkin [10].

3. COMPACT LIE GROUPS AND EXTENSIONS

We assume knowledge of the classical relationship between group extensions and related cohomology groups of low degree, as first introduced by Eilenberg and Mac Lane [11]. For readers not familiar with it, the textbooks by Mac Lane [16], Robinson [21], or Adem-Milgram [2], provide a thorough treatment; a more concise approach can be found in Kirillov's book [15], and a sketch in Brown's [6]. We now want to apply this relationship to the case of compact Lie groups. We fix a nonabelian connected compact Lie group G_0, a finite group Γ, and a homomorphism $\varphi\colon \Gamma \to \mathrm{Out}(G_0)$. Recall that Z_0 denotes the center of G_0. Choosing a principal subgroup $H \subset G_0$ and fixing s as in Theorem 2.4, we get the commutative diagram

$$\begin{array}{ccccc} \Gamma & \xrightarrow{\varphi} & \mathrm{Out}(G_o) & \xrightarrow{s} & \mathrm{Aut}(G_o) \\ & \searrow^{\bar{\varphi}} & \downarrow{\scriptstyle res} & & \downarrow{\scriptstyle res} \\ & & \mathrm{Aut}(Z_o) & = & \mathrm{Aut}(Z_o) \end{array}$$

In the sequel, we will use the notation $\sigma_\gamma = (s \circ \varphi)(\gamma)$, for $\gamma \in \Gamma$. Let $\mathcal{E}(\Gamma, G_o, \varphi) \subset \mathcal{E}$ denote the subset of equivalence classes giving rise to φ. In the particular case of compact Lie groups, one has the following results.

PROPOSITION 3.1.

(i) *The set of equivalence classes of extensions* $\mathcal{E}(\Gamma, G_o, \varphi)$ *is in bijection with the cohomology group* $H^2_{\bar{\varphi}}(\Gamma; Z_o)$.

(ii) *For all* $u \in H^2_{\bar{\varphi}}(\Gamma; Z_o)$ *the corresponding extension* $G_o \hookrightarrow G \twoheadrightarrow \Gamma$ *carries a natural structure of Lie group.*

Proof. It suffices to check that $\mathcal{E}(\Gamma, G_o, \varphi) \neq \varnothing$ to prove (i). But this follows from Theorem 2.4: the semidirect product $G = G_o \rtimes_{s\circ\varphi} \Gamma$ exists for any φ. The second statement is easily deduced from classical Lie group theory. □

The bijection in the latter proposition is not canonical, as it depends on the choice of a particular element in $\mathcal{E}(\Gamma, G_o, \varphi)$. On the other hand, there is a canonical bijection between $H^2_{\bar{\varphi}}(\Gamma; Z_o)$ and the set $\mathcal{E}(\Gamma, Z_o, \bar{\varphi})$ of equivalence classes of extension of Γ by Z_o with action given by $\bar{\varphi}$. Therefore, there is a bijection $\Lambda\colon \mathcal{E}(\Gamma, Z_o, \bar{\varphi}) \to \mathcal{E}(\Gamma, G_o, \varphi)$ still depending on the previous choice. Let us describe this bijection by first expliciting the cohomology group $H^2_{\bar{\varphi}}(\Gamma; Z_o) = Z^2_{\bar{\varphi}}(\Gamma; Z_o)/B^2_{\bar{\varphi}}(\Gamma; Z_o)$. Keeping the multiplicative notation in Z_o, the cocycles, i.e. the elements of $Z^2_{\bar{\varphi}}(\Gamma; Z_o)$, are functions $h\colon \Gamma \times \Gamma \to Z_o$ satisfying $h(\gamma_1, e) = h(e, \gamma_2) = e$ (normalization), and

$$(\delta h)(\gamma_1, \gamma_2, \gamma_3) = \sigma_{\gamma_1}(h(\gamma_2, \gamma_3)) \cdot h(\gamma_1\gamma_2, \gamma_3)^{-1} \cdot h(\gamma_1, \gamma_2\gamma_3) \cdot h(\gamma_1, \gamma_2)^{-1} = e$$

for all $\gamma_1, \gamma_2, \gamma_3 \in \Gamma$. The coboundaries, i.e. the elements of $B^2_{\bar{\varphi}}(\Gamma; Z_o)$, are functions $h\colon \Gamma \times \Gamma \to Z_o$ such that there exists a function $k\colon \Gamma \to Z_o$, with $k(e) = e$, satisfying

$$h(\gamma_1, \gamma_2) = (\delta k)(\gamma_1, \gamma_2) = \sigma_{\gamma_1}(k(\gamma_2)) \cdot k(\gamma_1\gamma_2)^{-1} \cdot k(\gamma_1)$$

for all $\gamma_1, \gamma_2 \in \Gamma$. Let us choose the semidirect product $G_o \rtimes \Gamma$ associated to the section s as the extension corresponding to the neutral element in $H^2_{\bar{\varphi}}(\Gamma; Z_o)$. Then, for $u = [h] \in H^2_{\bar{\varphi}}(\Gamma; Z_o)$, the corresponding class of extensions is given by $[G_\circ \hookrightarrow G_h \twoheadrightarrow \Gamma]$, where G_h is the *set* $G_\circ \times \Gamma$ equipped with the multiplication

$$(g,\gamma) *_h (g',\gamma') = \big(g \cdot \sigma_\gamma(g') \cdot h(\gamma,\gamma'), \gamma \cdot \gamma'\big)$$

(see [16], Chapter IV, §4 and §8). We will also denote by $G_\circ \hookrightarrow G_u \twoheadrightarrow \Gamma$ any representative of the class of extensions corresponding to $u \in H^2_{\bar\varphi}(\Gamma; Z_\circ)$. We now give a canonical description of the inverse of Λ, i.e. a description that does *not* depend on the choice of a particular element in $\mathcal{E}(\Gamma, G_\circ, \varphi)$.

LEMMA 3.2. *For any principal subgroup H in $G_\circ$, the map*

$$\Theta\colon \mathcal{E}(\Gamma, G_\circ, \varphi) \longrightarrow \mathcal{E}(\Gamma, Z_\circ, \bar\varphi),\ [G_\circ \hookrightarrow G \twoheadrightarrow \Gamma] \longmapsto [Z_\circ \hookrightarrow Z_G(H) \twoheadrightarrow \Gamma]$$

is the inverse of Λ (and does not depend on the choice of H). In particular it is a bijection.

Proof. As centralizers of principal subgroups are preserved by isomorphisms of G, Θ does not depend on the choice of a representative in $[G_\circ \hookrightarrow G \twoheadrightarrow \Gamma]$. Let $u = [Z_\circ \hookrightarrow E_h \twoheadrightarrow \Gamma] = [h] \in H^2_{\bar\varphi}(\Gamma, Z_\circ)$. Then, we have the commutative diagram

$$\begin{array}{ccccc} Z_\circ & \hookrightarrow & E_h & \twoheadrightarrow & \Gamma \\ \downarrow & & \downarrow & & \| \\ G_\circ & \hookrightarrow & G = G_h & \twoheadrightarrow & \Gamma \end{array}$$

where E_h is $Z_\circ \times \Gamma$ as a set. Let us show that $E_h = Z_{G_h}(H)$, H being the principal subgroup of $G_\circ$ corresponding to the fixed section s. By Theorem 2.2, it is enough to check that E_h is contained in $Z_{G_h}(H)$. Let $(z,\gamma) \in E_h$ and $(x,e) \in H \subset G_\circ \subset G_h$. We calculate

$$\begin{aligned}(z,\gamma) *_h (x,e) &= \big(z \cdot \sigma_\gamma(x) \cdot h(\gamma,e), \gamma\big) \\ &= (z \cdot x, \gamma)\,,\end{aligned}$$

and

$$\begin{aligned}(x,e) *_h (z,\gamma) &= \big(x \cdot \sigma_e(z) \cdot h(e,\gamma), \gamma\big) \\ &= (x \cdot z, \gamma) \\ &= (z \cdot x, \gamma)\,,\end{aligned}$$

by normalization, and because the restriction of σ_γ to H is the identity by the choice of the section s.

Now, as the principal subgroups are all conjugate by an element of G_o (see [8], Théorème, pp. 46–47), so are their centralizers. Therefore, the extensions $Z_o \hookrightarrow Z_G(H) \twoheadrightarrow \Gamma$, for H running through the family of principal subgroups, all belong to the same class. This shows that Θ is well defined and satisfies $\Theta \circ \Lambda = id_{\mathcal{E}(\Gamma, Z_o, \bar{\varphi})}$. As Λ is bijective, this shows that $\Theta = \Lambda^{-1}$. $\square$

We summarize the situation exposed in this section.

THEOREM 3.3. *Suppose given* G_o, *a homomorphism* $\varphi\colon \Gamma \to \mathrm{Out}(G_o)$ *and an extension* $Z_o \hookrightarrow Z \twoheadrightarrow \Gamma$, *for which the homomorphism* $\Gamma \to \mathrm{Aut}(Z_o)$ *coincides with* $\bar{\varphi}$. *Then, up to equivalence of extensions, there exists a unique compact Lie group* G *fitting into the commutative diagram*

$$\begin{array}{ccccc}
Z_o & \hookrightarrow & Z & \twoheadrightarrow & \Gamma \\
\downarrow & & \downarrow & & \| \\
G_o & \hookrightarrow & G & \twoheadrightarrow & \Gamma \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \varphi} \\
\mathrm{Inn}(G_o) & \hookrightarrow & \mathrm{Aut}(G_o) & \twoheadrightarrow & \mathrm{Out}(G_o)
\end{array}$$

where the rows are group extensions. Moreover the given data allow the construction of an extension $G_o \hookrightarrow G \twoheadrightarrow \Gamma$, *in which the subgroup* Z *is the centralizer of a principal subgroup.*

Conversely, the class of the extension $Z_o \hookrightarrow Z \twoheadrightarrow \Gamma$ *in* $G_o \hookrightarrow G \twoheadrightarrow \Gamma$ *can be recovered by taking the centralizer of any principal subgroup.*

4. PROOF OF THE MAIN THEOREM AND EXAMPLES

We are almost ready to show that the map described in the Introduction is an action of $\mathrm{Out}(G_o) \times \mathrm{Aut}(\Gamma)$ on the set

$$\mathcal{E} \approx \coprod_{\varphi \in \mathrm{Hom}(\Gamma, \mathrm{Out}(G_o))} H^2_{\bar{\varphi}}(\Gamma; Z_o)\,.$$

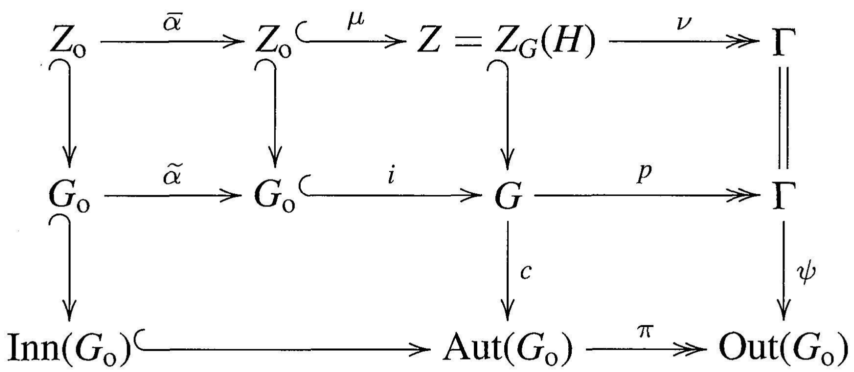
We first introduce some notation. For an element g in a group K, we will write c_g for conjugation by g, i.e. $c_g(x) = gxg^{-1}$, for all x in K. For $\alpha \in \mathrm{Out}(G_o)$, we will choose $\widetilde{\alpha} \in \mathrm{Aut}(G_o)$ such that $\pi(\widetilde{\alpha}) = \alpha$, and we will denote the restricted automorphism by $\bar{\alpha} \in \mathrm{Aut}(Z_o)$. Finally, recall that a cohomology class $u \in H^2_{\overline{\varphi}}(\Gamma; Z_o)$ is canonically identified with the corresponding equivalence class of extensions and is denoted by $u = \left[Z_o \overset{\mu}{\hookrightarrow} Z \overset{\nu}{\twoheadrightarrow} \Gamma\right]$.

LEMMA 4.1.

(i) *The map* $\operatorname{Out}(G_o) \times \mathcal{E} \to \mathcal{E}$, $(\alpha, u) \mapsto u \,.\, \alpha = \alpha^*(u) = \left[Z_o \overset{\mu \circ \bar{\alpha}}{\hookrightarrow} Z \overset{\nu}{\twoheadrightarrow} \Gamma\right]$ *defines a right action. The image* $\alpha^*(u)$ *corresponds to the extension* $G_o \overset{i \circ \tilde{\alpha}}{\hookrightarrow} G \overset{p}{\twoheadrightarrow} \Gamma$, *and belongs to* $H^2_{\overline{\psi}}(\Gamma; Z_o) \subset \mathcal{E}$, *where* $\psi = c_{\alpha^{-1}} \circ \varphi$.

(ii) *The map* $\operatorname{Aut}(\Gamma) \times \mathcal{E} \to \mathcal{E}$, $(\beta, u) \mapsto \beta \,.\, u = \beta_*(u) = \left[Z_o \overset{\mu}{\hookrightarrow} Z \overset{\beta \circ \nu}{\twoheadrightarrow} \Gamma\right]$ *defines a left action. The image* $\beta_*(u)$ *corresponds to the extension* $G_o \overset{i}{\hookrightarrow} G \overset{\beta \circ p}{\twoheadrightarrow} \Gamma$, *and belongs to* $H^2_{\overline{\theta}}(\Gamma; Z_o) \subset \mathcal{E}$, *where* $\theta = \varphi \circ \beta^{-1}$.

Proof. As the proofs of the two parts of the lemma are very similar, we only treat the first one. Consider the following commutative diagram:



The principal subgroups are preserved by isomorphisms. As $\tilde{\alpha}^{-1}(H)$ is clearly centralized by any element in Z, the statement about which extension corresponds to $\alpha^*(u)$ follows from Theorem 3.3. At the same time, this shows that the map is well defined. It is then straightforward to check that it is a right action. For the resulting homomorphism, we choose a set theoretic section $v\colon \Gamma \to G$ of $p\colon G \twoheadrightarrow \Gamma$, and compute for $\gamma \in \Gamma$:

$$\begin{aligned}
\psi(\gamma) &= \pi\left((i \circ \tilde{\alpha})^{-1} \circ c_{v(\gamma)} \circ (i \circ \tilde{\alpha})\right) \\
&= \pi\left(\tilde{\alpha}^{-1} \circ (i^{-1} \circ c_{v(\gamma)} \circ i) \circ \tilde{\alpha}\right) \\
&= \pi(\tilde{\alpha})^{-1} \circ \pi(i^{-1} \circ c_{v(\gamma)} \circ i) \circ \pi(\tilde{\alpha}) \\
&= \alpha^{-1} \circ \varphi(\gamma) \circ \alpha \\
&= (c_{\alpha^{-1}} \circ \varphi)(\gamma)\,.
\end{aligned}$$

□

Clearly for $u \in \mathcal{E}$ and a corresponding representative $G_o \hookrightarrow G_u \twoheadrightarrow \Gamma$, we have $G_u \cong G_{\alpha^*(u)} \cong G_{\beta_*(u)}$ for all $\alpha \in \mathrm{Out}(G_o)$, $\beta \in \mathrm{Aut}(\Gamma)$. Moreover, it is clear that the two actions commute and so we get a left action of $\mathrm{Out}(G_o) \times \mathrm{Aut}(\Gamma)$ on $\mathcal{E}$. Elements in the same orbit represent isomorphic groups; the main result of this paper, stated in the Introduction, tells that the converse is true.

Proof of the Main Theorem: Let $\rho\colon G_{u_1} \to G_{u_2}$ be an isomorphism of compact Lie groups. As the connected component of the identity is preserved by an isomorphism, this gives rise to the commutative diagram

$$\begin{array}{ccccc} G_o & \hookrightarrow & G_{u_1} & \twoheadrightarrow & \Gamma \\ \cong\downarrow \widetilde{\rho} & & \cong\downarrow \rho & & \cong\downarrow \beta \\ G_o & \hookrightarrow & G_{u_2} & \twoheadrightarrow & \Gamma \end{array}$$

Let us define $\alpha = \pi(\widetilde{\rho}) \in \mathrm{Out}(G_o)$ and $\overline{\alpha} = \rho|_{Z_o}$. As the centralizers of principal subgroups are preserved by isomorphisms, and by Theorem 3.3, this induces a new commutative diagram that we write as follows:

$$\begin{array}{ccccc} Z_o & \overset{\mu_1 \circ \overline{\alpha}^{-1}}{\hookrightarrow} & Z_{u_1} & \overset{\beta \circ \nu_1}{\twoheadrightarrow} & \Gamma \\ \| & & \cong\downarrow \overline{\rho}=\rho|_{Z_{u_1}} & & \| \\ Z_o & \underset{\mu_2}{\hookrightarrow} & Z_{u_2} & \underset{\nu_2}{\twoheadrightarrow} & \Gamma \end{array}$$

Thus, by Lemma 4.1, we have $u_2 = (\alpha^{-1})^* \beta_*(u_1)$, and so u_1 and u_2 are in the same orbit. □

REMARK 4.2. The extension $\mathrm{Inn}(G_o) \overset{\iota}{\hookrightarrow} \mathrm{Aut}(G_o) \overset{\pi}{\twoheadrightarrow} \mathrm{Out}(G_o)$ is split; however, other facts are relevant for allowing in the Main Theorem the passage from *up to equivalence* to *up to isomorphism*. The crucial point is that the class of extensions of the center of the connected component of the identity G_o can be represented by subgroups of G, namely centralizers of principal subgroups, that are preserved by isomorphisms and all conjugate by elements in G_o. This also raises two natural questions: are there larger classes of groups for which the Main Theorem holds, and also, can one find explicit examples for which it fails (even when supposing that the extension relating the automorphism groups of the kernel of the extension is split)?

Before proceeding with two examples, we introduce notations for three elements of the group SU(2), which will also appear in the final proposition of the paper. We denote the identity matrix by $\mathbf{1}$ and we set $-\mathbf{1} = \mathrm{diag}(-1,-1)$. We also set

$$j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

EXAMPLE 4.3. We take $G_o = \mathrm{SU}(2)$ and $\Gamma = \mathbf{Z}/2$. As $\mathrm{Out}(G_o)$ is trivial and $Z_o \cong \mathbf{Z}/2$, we have

$$\mathcal{E}(\mathbf{Z}/2, \mathrm{SU}(2)) \approx \coprod_{\varphi \in \mathrm{Hom}(\mathbf{Z}/2, 0)} H^2_{\bar{\varphi}}(\mathbf{Z}/2; \mathbf{Z}/2) = H^2(\mathbf{Z}/2; \mathbf{Z}/2) \cong \mathbf{Z}/2\,.$$

The group $\mathrm{Out}(G_o) \times \mathrm{Aut}(\Gamma)$ being trivial, these two elements correspond to two non-isomorphic compact Lie groups. The first one is clearly $G_{u_0} = \mathrm{SU}(2) \times \mathbf{Z}/2$. Let us give a description of the second one. Conjugating a matrix in SU(2) by j amounts to taking the complex conjugate of each entry in the matrix, i.e. $c_j\colon \mathrm{SU}(2) \to \mathrm{SU}(2)$, $g \mapsto c_j(g) = \bar{g}$. Let us denote by $G_{u_1} = \mathrm{SU}(2) \rtimes_j \mathbf{Z}/2$ the semidirect product where the generator t of $\mathbf{Z}/2$ acts as c_j on SU(2). As the center of G_{u_1} is given by $\langle (j,t) \rangle \cong \mathbf{Z}/4$, G_{u_0} and G_{u_1} are non-isomorphic. Therefore G_{u_1} is the second compact Lie group that we were looking for.

It is clear, from what has been done so far, that the elements in $H^2_{\bar{\varphi}}(\Gamma; Z_o)$ and in $H^2_{\bar{\psi}}(\Gamma; Z_o)$, with $\psi = c_{\alpha^{-1}} \circ \varphi$, will be identified (at least) pairwise under the action of the element $\alpha \in \mathrm{Out}(G_o)$. The second example is intended to show that identifications can even occur inside a given cohomology group (i.e. without changing the "outer" action of Γ on G_o).

EXAMPLE 4.4. We take $G_o = \mathrm{SU}(2) \times \mathrm{SU}(2) \cong \mathrm{Spin}(4)$ and keep $\Gamma = \mathbf{Z}/2$. The outer automorphism group is given by $\mathrm{Out}(G_o) = \langle \tau \rangle$, where $\widetilde{\tau}$ is the automorphism that exchanges the two factors, i.e.

$$\widetilde{\tau}\colon \mathrm{SU}(2) \times \mathrm{SU}(2) \longrightarrow \mathrm{SU}(2) \times \mathrm{SU}(2),\ (g,h) \longmapsto (h,g)\,,$$

and $Z_o = \mathbf{Z}/2 \times \mathbf{Z}/2$. We thus have

$$\begin{aligned} \mathcal{E}(\mathbf{Z}/2, \mathrm{SU}(2) \times \mathrm{SU}(2)) &\approx \coprod_{\varphi \in \mathrm{Hom}(\mathbf{Z}/2, \mathbf{Z}/2)} H^2_{\bar{\varphi}}(\mathbf{Z}/2; \mathbf{Z}/2 \times \mathbf{Z}/2) \\ &= H^2(\mathbf{Z}/2; \mathbf{Z}/2 \times \mathbf{Z}/2) \amalg H^2_{\overline{id}}(\mathbf{Z}/2; \mathbf{Z}/2 \times \mathbf{Z}/2) \\ &\approx (\mathbf{Z}/2 \times \mathbf{Z}/2) \amalg \{0\}\,. \end{aligned}$$

One then verifies that as extensions of the center, i.e. as centralizers of a principal subgroup, these five non-equivalent extensions are in fact represented by only three non-isomorphic groups, namely

$$\begin{aligned} Z_{u_0} &\cong \mathbf{Z}/2 \times \mathbf{Z}/2 \times \mathbf{Z}/2 \\ Z_{u_1} &\cong \mathbf{Z}/2 \times \mathbf{Z}/4 \\ Z_{u_2} &\cong \mathbf{Z}/2 \times \mathbf{Z}/4 \\ Z_{u_3} &\cong \mathbf{Z}/2 \times \mathbf{Z}/4 \end{aligned}$$

for the elements of $H^2(\mathbf{Z}/2; \mathbf{Z}/2 \times \mathbf{Z}/2))$, and

$$Z_{v_0} \cong (\mathbf{Z}/2 \times \mathbf{Z}/2) \rtimes \mathbf{Z}/2 \cong \mathrm{D}_8$$

(where D_8 denotes the group of symmetries of the square) for the element of $H^2_{\overline{id}}(\mathbf{Z}/2; \mathbf{Z}/2 \times \mathbf{Z}/2)$. The group $\mathbf{Z}/2 \times \mathbf{Z}/4$ yields three non-equivalent extensions, because among its three elements of order 2, only one is divisible by 2 (the element $(0, 2)$ in additive notation). Therefore, this element must be characteristic and changing the non-trivial element of $\mathbf{Z}/2 \times \mathbf{Z}/2$ that is mapped to it gives three extensions that must clearly be non-equivalent. At the level of Lie groups, the five non-equivalent extensions are represented by

$$\begin{aligned} G_{u_0} &= \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathbf{Z}/2 \\ G_{u_1} &= (\mathrm{SU}(2) \times \mathrm{SU}(2)) \rtimes_{j \times id} \mathbf{Z}/2 \\ G_{u_2} &= (\mathrm{SU}(2) \times \mathrm{SU}(2)) \rtimes_{id \times j} \mathbf{Z}/2 \\ G_{u_3} &= (\mathrm{SU}(2) \times \mathrm{SU}(2)) \rtimes_{j \times j} \mathbf{Z}/2 \\ G_{v_0} &= (\mathrm{SU}(2) \times \mathrm{SU}(2)) \rtimes_{\tau} \mathbf{Z}/2 \, . \end{aligned}$$

(One checks that $(-\mathbf{1}, \mathbf{1}, e)$ corresponds to the characteristic element of order 2 in Z_{u_1} whereas it is $(\mathbf{1}, -\mathbf{1}, e)$ in Z_{u_2}, and therefore G_{u_1} and G_{u_2} are certainly not equivalent.) Finally, the group $\mathrm{Out}(G_o) \times \mathrm{Aut}(\Gamma) \cong \mathbf{Z}/2$ acts on this set of equivalent extensions, and it is clear that the only non-trivial orbit is $\{G_{u_1}, G_{u_2}\}$. Therefore there are four non-isomorphic extensions of $\mathbf{Z}/2$ by $\mathrm{SU}(2) \times \mathrm{SU}(2) \cong \mathrm{Spin}(4)$.

5. Splitting of the extension associated to a nonconnected compact Lie group

Let G still denote a compact Lie group with a nonabelian connected component; we also assume the other notations introduced previously. In this final section, we make a few observations on the following problem: when is the natural extension associated to G split, i.e. when is G isomorphic to a semidirect product $G_0 \rtimes \Gamma$? Our aim is to relate this problem to the rest of this work. For a deeper analysis one should consult Chapter 6 in the book by Hofmann and Morris [14]. We start with a structure theorem for compact Lie groups based on centralizers of principal subgroups, similar to the "Sandwich Theorem for compact Lie groups" (see [14], Corollary 6.75, p. 272). This theorem shows that any compact Lie group is "sandwiched" in between two semidirect products closely related to it. We then recall a theorem of de Siebenthal and compare, in some particular cases, the question of the splitting of the extension associated to G to that of the extensions associated to the normalizer N of a maximal torus and to the centralizer Q of a principal diagonal, both introduced in Section 2. As an application of the "Sandwich" Theorem, the final proposition presents a "minimal" compact Lie group G such that the associated extension is *not* split.

Let $\overline{G}_0$ denote the adjoint group G_0/Z_0 ; it is well-known that the center of $\overline{G}_0$ is trivial.

THEOREM 5.1. *Let G be a compact Lie group with a nonabelian connected component. Then there exist two surjective homomorphisms*

$$\begin{array}{ccccc} G_s = G_0 \rtimes Z & \xrightarrow{\pi_1} & G & \xrightarrow{\pi_2} & \overline{G} = G/Z_0 \cong \overline{G}_0 \rtimes \Gamma, \\ (g_0, z) & \longmapsto & g_0 \cdot z & & \end{array}$$

where the centralizer Z of a fixed principal subgroup acts on G_0 by conjugation, and where π_2 is the canonical projection corresponding to the normal subgroup Z_0 of G.

For the kernels, we have $\ker \pi_1 \cong Z_0$ *and* $\ker \pi_2 = Z_0$ *; in particular*

$$G \cong (G_0 \rtimes Z)/Z_0 \quad \textit{and} \quad G/Z_0 \cong \overline{G}_0 \rtimes \Gamma .$$

Proof. As $\overline{G}_0$ is centerless, $\overline{G}$ must be isomorphic to $\overline{G}_0 \rtimes \Gamma$ by the proof of Proposition 3.1. The other assertions about π_2 are clear.

The map π_1 is well-defined and surjective (because Z intersects every component of G). Straightforward computations show that it is a homomorphism and that $\ker \pi_1 \cong Z_0$. □

REMARK 5.2. 1) The component of the identity of G_s is equal to G_o if and only if G_o is semisimple.

2) The present version of the "Sandwich" Theorem has the advantage of being more explicit than the one in [14] (the result therein is an existence theorem). Its drawbacks are the fact that Z is not finite if G_o is not semisimple, and that it obviously makes no sense for compact Lie groups with an abelian connected component.

3) Given a homomorphism $\varphi\colon \Gamma \to \mathrm{Out}(G_o)$, and an extension $Z_o \overset{\mu}{\hookrightarrow} E \overset{\nu}{\twoheadrightarrow} \Gamma$ for which the action coincides with the "restriction" $\bar{\varphi}$, there is a more direct way than the cohomological one to recover the corresponding compact Lie group G, i.e. the one that fits into the commutative diagram

$$\begin{array}{ccccc} Z_o & \hookrightarrow & E & \twoheadrightarrow & \Gamma \\ \downarrow & & \downarrow & & \| \\ G_o & \hookrightarrow & G & \twoheadrightarrow & \Gamma \end{array}$$

Let us define the composition $\bar{\sigma}\colon E \overset{\nu}{\twoheadrightarrow} \Gamma \overset{\varphi}{\to} \mathrm{Out}(G_o) \overset{s}{\to} \mathrm{Aut}(G_o)$, where s is as in Theorem 2.4. Then by Bourbaki (see [4], Lemme 7, pp. 210–211), we have

$$G = (G_o \rtimes_{\bar{\sigma}} E)/\Delta Z_o\,,$$

where ΔZ_o is the image of the injection $z_o \mapsto (z_o^{-1}, \mu(z_o))$. Taking $E = Z$, this gives another proof of the assertions concerning π_1 in Theorem 5.1.

Using Cartan subgroups (in the sense of Segal [22], i.e. those Adams called "SS subgroups" in honour of Segal and de Siebenthal [1]), de Siebenthal gave some explicit sufficient conditions for the splitting of the extension associated to G ([9], Théorème p. 74).

THEOREM 5.3 (de Siebenthal). *Let G be a compact Lie group with G_o simply connected, or of adjoint type (i.e. Z_o is trivial). If $\Gamma = \pi_0(G)$ is cyclic then G is a semidirect product, i.e. $G \cong G_o \rtimes \Gamma$.*

A relationship with the splitting of the extension associated to the normalizer of a maximal torus N in G is given in the next proposition.

PROPOSITION 5.4. *If the group of components Γ of G is nilpotent, then the extension $G_o \hookrightarrow G \twoheadrightarrow \Gamma$ is split if and only if the extension $N_o \hookrightarrow N \twoheadrightarrow \Gamma$ is split.*

Proof. The "if" part is clear. Conversely, let $s\colon \Gamma \to G$ be a section. By a result in Bourbaki (see [5], Corollaire 4, p. 49), any nilpotent subgroup of a compact Lie group is contained in the normalizer of some maximal torus. Therefore, if needed after conjugation by an element in G_0, we have $s(\Gamma) \subset N$, and we can conclude that the extension associated to N is split. □

For an extended maximal torus Q, the extensions can be related as follows.

PROPOSITION 5.5. *If the group of components Γ of G is cyclic, then the extension $G_0 \hookrightarrow G \twoheadrightarrow \Gamma$ is split if and only if the extension $T \hookrightarrow Q \twoheadrightarrow \Gamma$ is split.*

Proof. The proposition readily follows from the fact that the conjugates of Q cover G. □

REMARK 5.6. This latter proposition fails in general. An obstruction to the splitting of the extension associated to the extended maximal torus Q can be found in a paper by Oliver; this obstruction involves the representation ring of G and its relation with the family of all p-toral subgroups of G (see [19], Corollary 3.11). In particular, Oliver constructs a compact Lie group $G = \mathrm{SU}(2) \rtimes (\mathbf{Z}/2 \times \mathbf{Z}/2 \times \mathbf{Z}/3)$ such that the extension corresponding to the extended maximal torus Q is *not* split ([19], pp. 376–377).

We conclude with the promised example.

PROPOSITION 5.7. *Let $\mathrm{D}_8 = \langle r, s \mid r^4 = s^2 = e,\ srs^{-1} = r^{-1} \rangle$ be a presentation of the dihedral group. Then the quotient*

$$G = (\mathrm{SU}(2) \times \mathrm{D}_8)/\Delta\mathbf{Z}/2\,,$$

where $\Delta\mathbf{Z}/2$ denotes the central subgroup generated by $(-\mathbf{1}, r^2)$, is a compact Lie group with $G_0 \cong \mathrm{SU}(2)$ and $\Gamma \cong \mathbf{Z}/2 \times \mathbf{Z}/2$, the 4-group of Klein. The associated extension

$$\mathrm{SU}(2) \hookrightarrow G \twoheadrightarrow \mathbf{Z}/2 \times \mathbf{Z}/2$$

is not split. Among the extensions associated to a compact Lie group with a nonabelian connected component, it is a minimal one having the property of being non-split, in the sense that the rank of the connected component and the size of the group of components are minimal. Moreover replacing the *connected component by* SO(3) *(i.e. by "any" group of the same rank), or the group of components by* $\mathbf{Z}/4$ *(i.e. by "any" group of the same size), will force the extension to be split.*

Proof. The assertions about the connected component and the group of components are clear. Let us show that the extension associated to G is not split. Let us denote $[g,\gamma] \in G$ the image of $(g,\gamma) \in \mathrm{SU}(2) \times \mathrm{D}_8$ under the canonical projection. Let $\mathbf{S}^1$ denote the standard maximal torus in SU(2), and let N denote its normalizer in G. We have

$$N = \left\{[t,e] : t \in \mathbf{S}^1\right\} \amalg \left\{[jt,e] : t \in \mathbf{S}^1\right\} \amalg \left\{[t,r] : t \in \mathbf{S}^1\right\} \amalg \left\{[jt,r] : t \in \mathbf{S}^1\right\}$$
$$\amalg \left\{[t,s] : t \in \mathbf{S}^1\right\} \amalg \left\{[jt,s] : t \in \mathbf{S}^1\right\} \amalg \left\{[t,rs] : t \in \mathbf{S}^1\right\} \amalg \left\{[jt,rs] : t \in \mathbf{S}^1\right\} .$$

By contradiction, suppose that the extension associated to G is split, i.e. there exists a section. As $\mathbf{Z}/2 \times \mathbf{Z}/2$ is abelian, thus nilpotent, we deduce, by Proposition 5.4, that the extension associated to N is also split. We want to show that this is not possible by considering the elements of order 2 in N. For $n = 0, 1$, a straightforward calculation shows that in the component corresponding to $r^n s$, an element $[t, r^n s]$ is of order 2 if and only if $t = \pm\mathbf{1}$, and that the sub-component $\left\{[jt, r^n s] : t \in \mathbf{S}^1\right\}$ does not contain any element of order 2. Two of the three non-trivial elements in $\Gamma \cong \mathbf{Z}/2 \times \mathbf{Z}/2$ must thus be mapped by the section to $[\pm\mathbf{1}, s]$ and $[\pm\mathbf{1}, rs]$. Therefore, as the section is a homomorphism, the image of the third non-trivial element is

$$[\pm\mathbf{1}, rs] \cdot [\pm\mathbf{1}, s] = [\pm\mathbf{1}, r] ,$$

which is not of order 2. A contradiction that shows that the extension associated to G is not split.

The property of minimality follows by Theorem 5.3, and by the fact that any extension with SO(3) as normal subgroup is a direct product (because SO(3) is complete, i.e. centerless and with trivial outer automorphism group). □

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