

## 2.4 The algebra of differential-reflection operators

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We want to study the stationary Schrödinger equation:

$$(3) \quad H\psi = \lambda\psi, \quad \lambda \in \mathbf{C}.$$

As in the classical case, it is difficult to say anything explicit about solutions of this equation for a general Schrödinger operator  $H$ , but for the Calogero-Moser operator the situation is much better.

DEFINITION 2.1. A *quantum integral* of  $H$  is a differential operator  $M$  such that

$$[M, H] = 0.$$

We are going to show that there are many quantum integrals of  $H$ , namely that there are  $n$  commuting algebraically independent quantum integrals  $M_1, \dots, M_n$  of  $H$ . By definition, this means that the quantum Calogero-Moser system is completely integrable.

Once we have found  $M_1, \dots, M_n$ , observe that for fixed constants  $\mu_1, \dots, \mu_n$ , the space of solutions of the system

$$\begin{cases} M_1\psi = \mu_1\psi \\ \dots\dots\dots \\ M_n\psi = \mu_n\psi \end{cases}$$

is clearly stable under  $H$ . We will see that this space is in fact finite dimensional. Therefore, the operators  $M_i$  allow one to reduce the problem of solving the partial differential equation  $H\psi = \lambda\psi$  to that of solving a system of ordinary linear differential equations. This phenomenon is called quantum complete integrability.

## 2.4 THE ALGEBRA OF DIFFERENTIAL-REFLECTION OPERATORS .

We are now going to explain how to find quantum integrals for  $H$ , using the Dunkl-Cherednik method.

First let us fix some notation. Given a smooth affine variety  $X$ , we will denote by  $\mathcal{D}(X)$  the ring of differential operators on  $X$ . We are going to consider the case in which  $X$  is the open set  $U$  in  $\mathfrak{h}$  which is the complement of the divisor of the equation  $\delta(x) := \prod_{s \in \Sigma} \alpha_s(x)$ . Clearly  $\mathcal{D}(U) = \mathcal{D}(\mathfrak{h})[1/\delta(x)]$ .

LEMMA 2.2. An element of  $\mathcal{D}(U)$  is completely determined by its action on  $\mathbf{C}[U]^W = \mathbf{C}[U/W]$ .

*Proof.* Recall that the quotient map  $\pi: U \rightarrow U/W$  is finite and unramified. This implies that

$$\mathcal{D}(U) = \mathbf{C}[U] \otimes_{\mathbf{C}[U/W]} \mathcal{D}(U/W).$$

From this we obtain that if  $P \in \mathcal{D}(U)$  is such that  $Pf = 0$  for all  $f \in \mathbf{C}[U/W]$ , then  $P = 0$ .  $\square$

We also have the operators on  $\mathbf{C}[U]$  given by the action of  $W$ . We will denote by  $\mathcal{A}$  the algebra of operators on  $U$  generated by  $\mathcal{D}(U)$  and  $W$ , and call it the algebra of differential-reflection operators. The action of  $W$  on  $U$  induces an action on  $\mathcal{D}(U)$ , so that the subalgebra  $\mathcal{D}(U) \subset \mathcal{A}$  is preserved by conjugation by elements of  $W$ . We have:

PROPOSITION 2.3.  $\mathcal{A} = \mathcal{D}(U) \rtimes W$ , i.e. every element in  $A \in \mathcal{A}$  can be uniquely written as a linear combination

$$A = \sum_{w \in W} P_w w$$

with  $P_w \in \mathcal{D}(U)$ .

*Proof.* The fact that every element in  $\mathcal{A}$  can be expressed as a linear combination  $\sum_{w \in W} P_w w$  is clear. To show that such an expression is unique, assume  $\sum_{w \in W} P_w w = 0$ . Take  $f \in \mathbf{C}[U]$  such that  ${}^w f \neq {}^u f$  for all  $w \neq u$  in  $W$ , and multiply the operator  $\sum P_w w$  on the right by the operator of multiplication by the function  $f^i$ ,  $i \geq 0$ . Then we get

$$\sum_{w \in W} P_w \circ ({}^w f)^i w = \sum_{w \in W} P_w w \circ f^i = 0.$$

Applying both sides of this equation to a function  $g \in \mathbf{C}[U/W]$  we have

$$\sum_{w \in W} (P_w \circ {}^w f^i) g = 0.$$

Thus by Lemma 2.2,  $\sum_{w \in W} P_w \circ {}^w f^i = 0$  for all  $i$ . Therefore, by Vandermonde's determinant formula,  $P_w \circ \prod_{w \neq u} ({}^w f - {}^u f) = 0$  and hence  $P_w = 0$ , for all  $w \in W$ , as desired.  $\square$

Take  $A \in \mathcal{A}$  and write

$$A = \sum_{w \in W} P_w w.$$

We set  $m(A) = \sum_{w \in W} P_w \in \mathcal{D}(U)$ . Notice that if  $f$  is a  $W$ -invariant function, then clearly  $A(f) = m(A)(f)$  and that, by what we have seen in Lemma 2.2,  $m(A)$  is completely determined by its action on invariant functions.

In general,  $m$  is not a homomorphism. However:

**PROPOSITION 2.4.** *Let  $\mathcal{A}^W \subset \mathcal{A}$  denote the subalgebra of elements invariant under conjugation by  $W$ . Then the restriction of  $m$  to  $\mathcal{A}^W$  is an algebra homomorphism.*

*Proof.* If  $A \in \mathcal{A}^W$ , then clearly  $m(A)$  is  $W$ -invariant. Now if we take  $A, B \in \mathcal{A}^W$  and  $f$  a  $W$ -invariant function we have that  $B(f)$  is also  $W$ -invariant. So

$$m(AB)(f) = (AB)(f) = A(B(f)) = A(m(B)(f)) = m(A)(m(B)(f)).$$

Thus  $m(AB)$  and  $m(A)m(B)$  coincide on  $W$ -invariant functions and hence coincide.  $\square$

## 2.5 DUNKL OPERATORS AND SYMMETRIC QUANTUM INTEGRALS

In this subsection we will construct quantum integrals of the Calogero-Moser operator. This construction is due to Heckman [He] and is based on the Dunkl operators, introduced in [Du].

Fix a  $W$ -invariant function  $c: \Sigma \rightarrow \mathbf{C}$  such that  $\beta_s = c_s(c_s + 1)$  for each  $s \in \Sigma$ . Set  $\delta_c := \prod_{s \in \Sigma} \alpha_s(x)^{c_s}$  and define

$$L = \delta_c(x) H \delta_c(x)^{-1}.$$

Then an easy computation shows that

$$L = \Delta - \sum_{s \in \Sigma} \frac{2c_s}{\alpha_s(x)} \partial_{\alpha_s},$$

where, for a vector  $y \in \mathfrak{h}$ , the symbol  $\partial_y$  denotes, as usual, the partial derivative in the  $y$  direction (notice that using the scalar product we are viewing  $\alpha_s$  as a vector in  $\mathfrak{h}$  orthogonal to the hyperplane fixed by  $s$ ).

From now on we will work with  $L$  instead of  $H$  and study the eigenvalue problem

$$(4) \quad L\psi = \lambda\psi.$$

It is clear that  $\psi$  is a solution of this equation if and only if  $\delta_c(x)^{-1}\psi$  is a solution of (3).

Since for any  $s \in \Sigma$  and  $f \in \mathbf{C}[\mathfrak{h}]$  we have that  $f(sx) - f(x)$  is divisible by  $\alpha_s(x)$ , the operator

$$\frac{1}{\alpha_s(x)}(s - 1) \in \mathcal{A}$$

maps  $\mathbf{C}[\mathfrak{h}]$  to itself.