

2.2 The classical Calogero-Moser System

Objektyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **49 (2003)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **20.09.2024**

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern. Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden. Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

called states of the system. The dynamics of the system $x = x(t)$, $p = p(t)$ depends on the Hamiltonian, or energy function, $E(x, p)$ on T^*X . Given E and the initial state $x(0)$, $p(0)$, one can recover the dynamics $x = x(t)$, $p = p(t)$ from Hamilton's differential equations $\frac{df(x, p)}{dt} = \{f, E\}$. If X is locally identified with \mathbf{R}^n by choosing coordinates x_1, \dots, x_n , then T^*X is locally identified with \mathbf{R}^{2n} with coordinates $x_1, \dots, x_n, p_1, \dots, p_n$. In these coordinates, Hamilton's equations may be written in their standard form

$$\dot{x}_i = \frac{\partial E}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial E}{\partial x_i}.$$

A function $I(x, p)$ is called an integral of motion for our system if $\{I, E\} = 0$. Integrals of motion are useful, since for any such integral I the function $I(x(t), p(t))$ is constant, which allows one to reduce the number of variables by 2. Thus, if we are given n functionally independent integrals of motion I_1, \dots, I_n with $\{I_l, I_k\} = 0$ for all $1 \leq l, k \leq n$, then all $2n$ variables x_i, p_i can be excluded, and the system can be completely solved by quadratures. Such a situation is called complete (or Liouville) integrability.

2.2 THE CLASSICAL CALOGERO-MOSER SYSTEM

Quasi-invariants are related to many-particle systems. Consider a system of n particles on the real line \mathbf{R} . A potential is an even function

$$U(x) = U(-x), \quad x \in \mathbf{R}.$$

Two particles at points a, b have energy of interaction $U(a - b)$. The total energy of our system of particles is

$$E = \sum_{i=1}^n \frac{p_i^2}{2} + \sum_{i < j} U(x_i - x_j).$$

Here, x_i are the coordinates of the particles, p_i their momenta. The dynamics of the particles $x_i = x_i(t)$, $p_i = p_i(t)$ is governed by the Hamilton equations with energy function E .

This is a system of nonlinear differential equations, which in general can be difficult to solve explicitly. However, for special potentials this system might be completely integrable. For instance, we will see that this is the case for the Calogero-Moser potential,

$$U(x) = \frac{\gamma}{x^2},$$

γ being a constant.

The Calogero-Moser system has a generalization to arbitrary Coxeter groups. Namely, consider a finite group W generated by reflections acting on the space \mathfrak{h} , and keep the notation of the previous section. Fix a W -invariant nondegenerate scalar product $(-, -)$ on \mathfrak{h} . It determines a scalar product on \mathfrak{h}^* . Define the “energy function”

$$E(x, p) = \frac{(p, p)}{2} + \frac{1}{2} \sum_{s \in \Sigma} \frac{\gamma_s(\alpha_s, \alpha_s)}{\alpha_s(x)^2}, \quad x \in \mathfrak{h}, \quad p \in \mathfrak{h}^*$$

on $T^*\mathfrak{h} = \mathfrak{h} \times \mathfrak{h}^*$, where $\gamma: \Sigma \rightarrow \mathbf{C}$ is a W -invariant function. Notice that although α_s is defined up to a non zero constant, by homogeneity, E is independent of the choice of α_s . We will call the system defined by E the Calogero-Moser system for W .

If W is the symmetric group S_n , $\mathfrak{h} = \mathbf{C}^n$, then Σ is the set of transpositions $s_{i,j}$, $i < j$, and we can take $\alpha_s = e_i - e_j$. Then we clearly obtain the usual Calogero-Moser system.

Below we will see that the Calogero-Moser system for W is completely integrable.

2.3 THE QUANTUM CALOGERO-MOSER SYSTEM

Let us now discuss quantization of the Calogero-Moser system. We start by quantizing the energy E by formally making the substitution

$$p_j \Rightarrow -i\hbar \frac{\partial}{\partial x_j},$$

where \hbar is a parameter (Planck's constant). This yields the Schrödinger operator

$$\widehat{E} := -\frac{\hbar^2}{2} \Delta + \frac{1}{2} \sum_{s \in \Sigma} \frac{\gamma_s(\alpha_s, \alpha_s)}{\alpha_s^2},$$

where Δ denotes the Laplacian.

In particular, in the case of $W = S_n$ we have

$$\widehat{E} = -\frac{\hbar^2}{2} \Delta + \sum_{i < j} \frac{c}{(x_i - x_j)^2},$$

where $\Delta = \sum_i \frac{\partial^2}{\partial x_i^2}$. Setting $\beta_s = \frac{\gamma_s}{2\hbar^2}$, we will from now on consider the operator

$$H := -\frac{2}{\hbar^2} \widehat{E} = \Delta - \sum_{s \in \Sigma} \frac{\beta_s(\alpha_s, \alpha_s)}{\alpha_s^2(x)},$$

called the Calogero-Moser operator.