3.2 The flypes

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **49 (2003)**

Heft 3-4: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **23.09.2024**

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.
Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.
Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

Ein Dienst der *ETH-Bibliothek*
ETH Zürich, Rämistrasse 101, 8092 Zürich, Schweiz, www.library.ethz.ch

http://www.e-periodica.ch

In Figure 21 we illustrate one example of a cut that is not rational. This is a possible cut made in the middle of the representative diagram $N(T)$. Here we see that if T' is the tangle obtained from this cut, so that $N(T') = K$, then $D(T')$ is a connected sum of two non-trivial knots. Hence the denominator $K' = D(T')$ is not prime. Since we know that both the numerator and the denominator of a rational tangle are prime (see [5], p. 91 or [19], Chapter 4, pp. 32–40), it follows that T' is not a rational tangle. Clearly the above argument is generic. It is not hard to see by enumeration that all possible cuts with the exception of the ones we have described will not give rise to rational tangles. We omit the enumeration of these cases.

This completes the proof that all of the rational tangles that close to a given standard rational knot diagram are arithmetically equivalent.

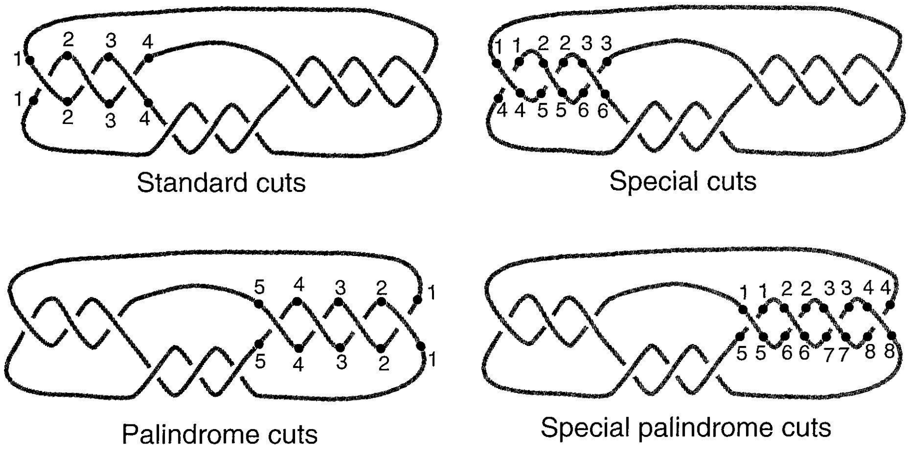

FIGURE 22
Standard, special, palindrome and special palindrome cuts

In Figure 22 we illustrate on a representative rational knot in 3-strand-braid form all the cuts that exhibit that knot as a closure of a rational tangle. Each pair of points is marked with the same number. •

REMARK 4. It follows from the above analysis that if T is a rational tangle in twist form, which is isotopic to the standard form $[[a_1], [a_2], \ldots, [a_n]]$, then all arithmetically equivalent rational tangles can arise by any cut of the above types either on the crossings that add up to the subtangle $[a_1]$ or on the crossings of the subtangle $[a_n]$.

3.2 THE FLYPES

Diagrams for knots and links are represented on the surface of the two-sphere, S^2, and then notationally on a plane for purposes of illustration.

Let $K = N(T)$ be a rational link diagram with T a rational tangle in twist form. By an appropriate sequence of flypes (recall Definition 1) we may assume, without loss of generality, that T is alternating and in continued fraction form, i.e. T is of the form $T = [[a_1], [a_2], \ldots, [a_n]]$ with the a_i's all positive or all negative. From the ambiguity of the first crossing of a rational tangle we may assume that n is odd. Moreover, from the analysis of the bottom twists in the previous subsection we may assume that T is in reduced form. Then the numerator $K = N(T)$ will be a reduced alternating knot diagram. This follows from the primality of K.

Let K and K' be two isotopic, reduced, alternating rational knot diagrams. By the Tait Conjecture they will differ by a finite sequence of flypes. In considering how rational knots can be cut open to produce rational tangles, we will examine how the cuts are affected by flyping. We analyze all possible flypes to prove that it is sufficient to consider the cuts on a single alternating reduced diagram for a given rational knot K. Hence the proof will be complete at that point. We need first two definitions and an observation about flypes.

DEFINITION 3. We shall call *region of a flype* the part of the knot diagram that contains precisely the subtangle and the crossing that participate in the flype. The region of a flype can be enclosed by a simple closed curve on the plane that intersects the tangle in four points.

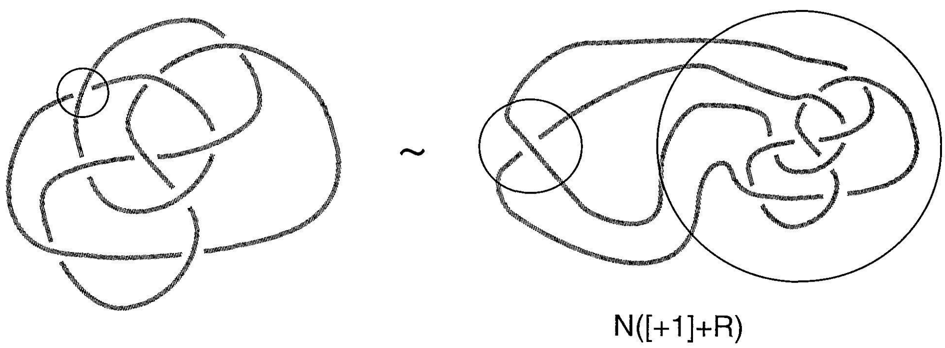

FIGURE 23
Decomposing into $N([\pm 1] + R)$

DEFINITION 4. A *pancake flip* of a knot diagram in the plane is an isotopy move that rotates the diagram by 180° in space around a horizontal or vertical axis on its plane and then it replaces it on the plane. Note that any knot diagram in S^2 can be regarded as a knot diagram in a plane.

In fact, the pancake flip is actually obtained by flypes so long as we allow as background moves isotopies of the diagram in S^2. To see this, note as in Figure 23 that we can assume without loss of generality that the diagram in question is of the form $N([\pm 1] + R)$ for some tangle R not necessarily rational. (Isolate one crossing at the 'outer edge' of the diagram in the plane and decompose the diagram into this crossing and a complementary tangle, as shown in Figure 23.) In order to place the diagram in this form we only need to use isotopies of the diagram in the plane.

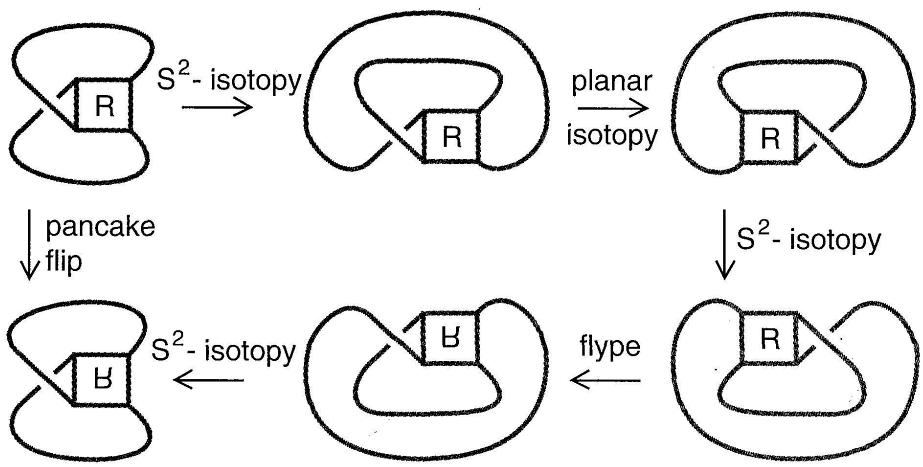

FIGURE 24
Pancake flip

Note now, as in Figure 24, that the pancake flip applied to $N([\pm 1] + R)$ yields a diagram that can be obtained by a combination of a planar isotopy, S^2-isotopies and a flype. (By an S^2*-isotopy* we mean the sliding of an arc around the back of the sphere.) This is valid for R any 2-tangle. We will use this remark in our study of rational knots and links.

We continue with a general remark about the form of a flype in any knot or link in S^2. View Figure 25. First look at parts A and B of this figure. Diagram A is shown as a composition of a crossing and two tangles P and Q. Part B is obtained from a flype of part A, where the flype is applied to the crossing in conjunction with the tangle P. This is the general pattern of the application of a flype. The flype is applied to a composition of a crossing with a tangle, while the rest of the diagram can be regarded as contained within a second tangle that is left fixed under the flyping.

Now look at diagrams C and D. Diagram D is obtained by a flype using Q and a crossing on diagram C. But diagram C is isotopic by a planar isotopy

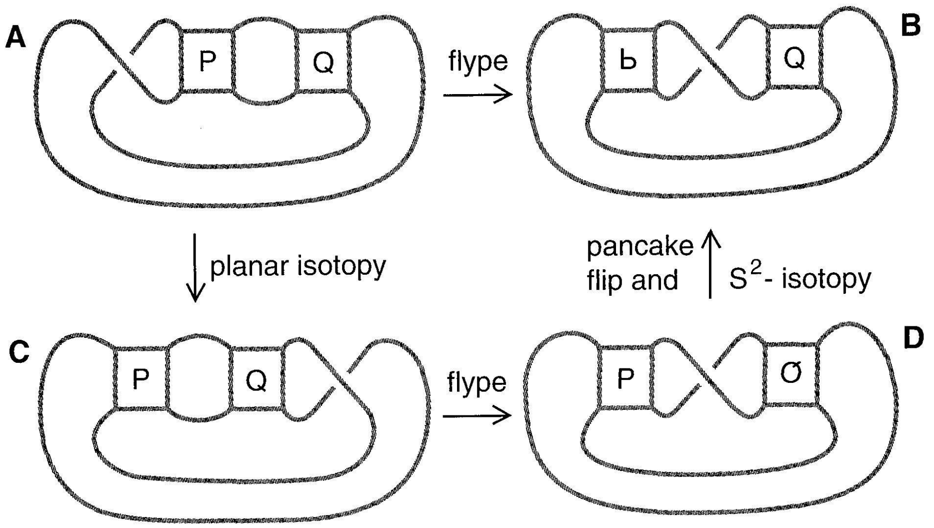

FIGURE 25
The complementary flype

to diagram A, and diagrams B and D are related by a pancake flip (combined with an isotopy that swings two arcs around S^2). Thus we see that:

Up to a pancake flip one can choose to keep either of the tangles P or Q fixed in performing a flype.

Let now $K = N(T)$ and $K' = N(T')$ be two reduced alternating rational knot diagrams that differ by a flype. The rational tangles T and T' are in reduced alternating twist form and without loss of generality T may be assumed to be in continued fraction form. Then, recall from Section 2 that the region of the flype on K can either include a rational truncation of T or some crossings of a subtangle $[a_i]$, see Figure 26. In the first case the two subtangles into which K decomposes are both rational and each will be called the *complementary tangle* of the other. In the second case the flype has really trivial effect and the complementary tangle is not rational, unless $i = 1$ or n.

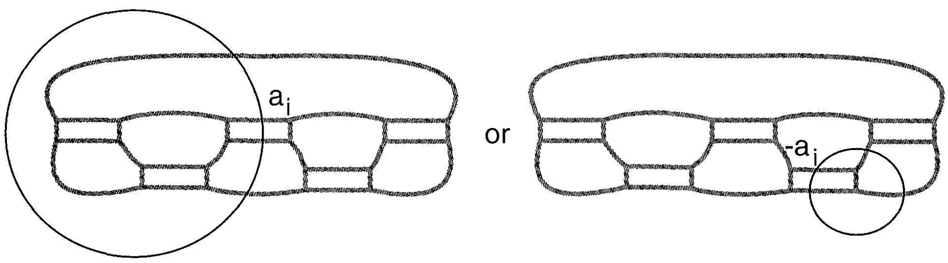

FIGURE 26
Flypes of rational knots

For the cutpoints of T on $K = N(T)$ there are three possibilities:

1. they are outside the region of the flype,
2. they are inside the flyped subtangle,
3. they are inside the region of the flype and outside the flyped subtangle.

If the cutpoints are outside the region of the flype, then the flype is taking place inside the tangle T and so there is nothing to check, since the new tangle is isotopic and thus arithmetically equivalent to T.

We concentrate now on the first case of the region of a flype. If the cutpoints are inside the flyped subtangle then, by Figure 25, this flype can be seen as a flype of the complementary tangle followed by a pancake flip. The region of the flype of the complementary tangle does not contain the cut points, so it is a rational flype that isotopes the tangle to itself. The pancake flip also does not affect the arithmetic, because its effect on the level of the tangle T is simply a horizontal or a vertical flip, and we know that a flipped rational tangle is isotopic to itself.

If the region of the flype encircles a number of crossings of some $[a_i]$ then the cutpoints cannot lie in the region, unless $i = 1$ or n. If the cutpoints do not lie in the region of the flype, there is nothing to check. If they do, then the complementary tangle is isotopic to T, and the pancake flip produces an isotopic tangle.

Finally, if the cutpoints are inside the region of the flype and outside the flyped subtangle, i.e. they are near the crossing of the flype, then there are three cases to check. These are illustrated in Figure 27.

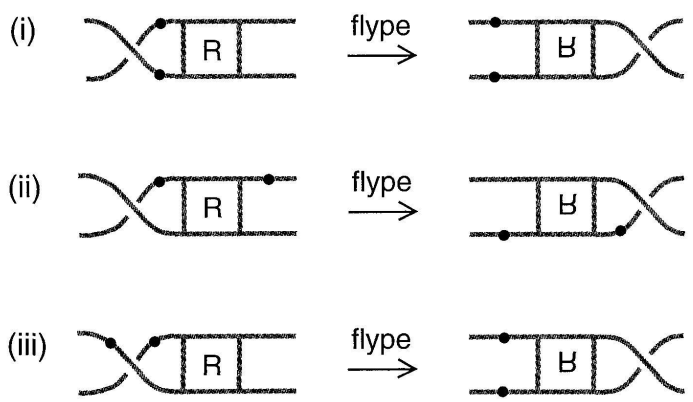

FIGURE 27

Flype and cut interaction

In each of these cases the flype is illustrated with respect to a crossing and a tangle R that is a subtangle of the link $K = N(T)$. Cases (i) and

(ii) are taken care of by the trick of the complementary flype. Namely, as in Figure 25, we transfer the crossing of the flype around S^2. Using this crossing we do a tangle flype of the complementary tangle, then we do a horizontal pancake flip and finally an S^2-isotopy, to end up with the right-hand sides of Figure 27.

In case (iii) we note that after the flype the position of the cut points is outside the region of a flyping move that can be performed on the diagram K' to return to the original diagram K, see Figure 28. This means that after performing the return flype the tangle T' is isotopic to the tangle T''. One can now observe that if the original cut produces a rational tangle, then the cut after the returned flype also produces a rational tangle, and this is arithmetically equivalent to the tangle T. More precisely, the tangle T'' is the result of a special cut on $N(T)$.

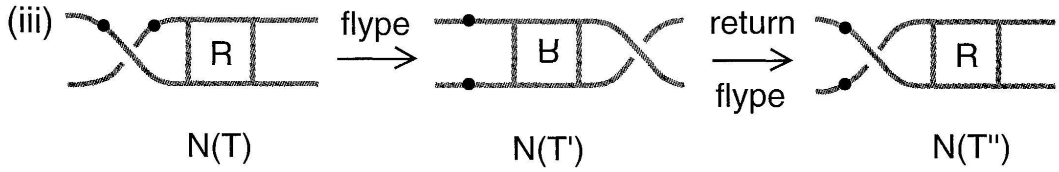

FIGURE 28
Flype and special cut

With the above argument we conclude the proof of the main direction of Theorem 2. From our analysis it follows that:

If $K = N(T)$ is a rational knot diagram with T a rational tangle then, up to bottom twists, any other rational tangle that closes to this knot is available as a cut on the given diagram.

We will now show the converse. We wish to show that if two rational tangles are arithmetically equivalent, then their numerators are isotopic knots. Let T_1, T_2 be rational tangles with $F(T_1) = \frac{p}{q}$ and $F(T_2) = \frac{p}{q'}$, with $|p| > |q|$ and $|p| > |q'|$, and assume first $qq' \equiv 1 \mod p$. If $\frac{p}{q} = [a_1, a_2, \ldots, a_n]$, with n odd, and $\frac{p}{q''} = [a_n, a_{n-1}, \ldots, a_1]$ is the corresponding palindrome continued fraction, then it follows from the Palindrome Theorem that $qq'' \equiv 1 \mod p$. Furthermore, it follows by induction that in a product of the form

$$M(a_1)M(a_2)\cdots M(a_n) = \begin{pmatrix} p & q'' \\ q & u \end{pmatrix}$$

we have that $p > q$ and $p > q''$, $q \geq u$ and $q'' \geq u$ whenever $a_1, a_2, \ldots, a_n$ are positive integers. (With the exception in the case of

$M(1)$ where the first two inequalities are replaced by equalities.) The induction step involves multiplying a matrix in the above form by one more matrix $M(a)$, and observing that the inequalities persist in the product matrix.

Hence, in our discussion we can conclude that $|p| > |q''|$. Since $|p| > |q'|$ and $|p| > |q''|$, it follows that $q' = q''$, since they are both reduced residue solutions of a $\bmod p$ equation with a unique solution. Hence $[a_n, a_{n-1}, \ldots, a_1] = \frac{p}{q'}$, and, by the uniqueness of the canonical form for rational tangles, T_2 has to be:

$$T_2 = [[a_n], [a_{n-1}], \ldots, [a_1]].$$

For these tangles we know that $N(T_1) = N(T_2)$. Let now T_3 be another rational tangle with fraction

$$\frac{p}{q' + kp} = \frac{1}{\frac{q'}{p} + k}.$$

By the Conway Theorem, this is the fraction of the rational tangle

$$\frac{1}{\frac{1}{T_2} + [k]} = T_2 * \frac{1}{[k]}.$$

Hence we have (recall the analysis of the bottom twists):

$$N(\frac{1}{\frac{1}{T_2} + [k]}) \sim N(T_2).$$

Finally, let $F(S_1) = \frac{p}{q}$ and $F(S_2) = \frac{p}{q+kp}$. Then

$$\frac{p}{q + kp} = \frac{1}{\frac{q}{p} + k},$$

which is the fraction of the rational tangle

$$\frac{1}{\frac{1}{S_1} + [k]} = S_1 * \frac{1}{[k]}.$$

Thus

$$N(S_1) \sim N(S_2).$$

The proof of Theorem 2 is now complete. □

We close the section with two remarks.

REMARK 5. In the above discussion about flypes we used the fact that the tangles and flyping tangles involved were rational. One can consider the question of *arbitrary alternating tangles* T *that close to form links isotopic to a given alternating diagram* K. Our analysis of cuts occurring before and after a flype goes through to show that *for every alternating tangle* T*, that closes to a diagram isotopic to a given alternating diagram* K*, there is a cut on the diagram* K *that produces a tangle that is arithmetically equivalent to* T. Thus it makes sense to consider the collection of tangles that close to an arbitrary alternating link up to this arithmetic equivalence. In the general case of alternating links this shows that on a given diagram of the alternating link we can consider all cuts that produce alternating tangles and thereby obtain all such tangles, up to a certain arithmetical equivalence, that close to links isotopic to K.

Even for rational links there can be more than one equivalence class of such tangles. For example, $N(1/[3] + 1/[3]) = N([-6])$ and $F(1/[3] + 1/[3]) = 2/3$ while $F([-6]) = -6$. Since these fractions have different numerators their tangles (one of which is not rational) lie in different equivalence classes. These remarks lead us to consider the set of arithmetical equivalence classes of altenating tangles that close to a given alternating link and to search for an analogue of Schubert's Theorem in this general setting.

REMARK 6. DNA supercoils, replicates and recombines with the help of certain enzymes. *Site-specific recombination* is one of the ways nature alters the genetic code of an organism, either by moving a block of DNA to another position on the molecule or by integrating a block of alien DNA into a host genome. In [7] it was made possible for the first time to see knotted DNA in an electron micrograph with sufficient resolution to actually identify the topological type of these knots and links. It was possible to design an experiment involving successive DNA recombinations and to examine the topology of the products. In [7] the knotted DNA produced by such successive recombinations was consistent with the hypothesis that all recombinations were of the type of a positive half twist as in $[+1]$. Then D.W. Sumners and C. Ernst [9] proposed a *tangle model for successive DNA recombinations* and showed, in the case of the experiments in question, that there was no other topological possibility for the recombination mechanism than the positive half twist $[+1]$. Their work depends essentially on the classification theorem for rational knots. This constitutes a unique use of topological mathematics as a theoretical underpinning for a problem in molecular biology.

4. Rational knots and their mirror images

In this section we give an application of Theorem 2. An unoriented knot or link K is said to be *achiral* if it is topologically equivalent to its mirror image $-K$. If a link is not equivalent to its mirror image then it is said be *chiral*. One then can speak of the *chirality* of a given knot or link, meaning whether it is chiral or achiral. Chirality plays an important role in the applications of knot theory to chemistry and molecular biology. In [8] the authors find an explicit formula for the number of achiral rational knots among all rational knots with n crossings. It is interesting to use the classification of rational knots and links to determine their chirality. Indeed, we have the following well-known result (for example see [35] and [16], p. 24, Exercise 2.1.4; compare also with [31]):

THEOREM 5. *Let $K = N(T)$ be an unoriented rational knot or link, presented as the numerator of a rational tangle T. Suppose that $F(T) = p/q$ with p and q relatively prime. Then K is achiral if and only if $q^2 \equiv -1 \mod p$. It follows that the tangle T has to be of the form $[[a_1],[a_2],\ldots,[a_k],[a_k],\ldots,[a_2],[a_1]]$ for any integers $a_1,\ldots,a_k$.*

Note that in this description we are using a representation of the tangle with an even number of terms. The leftmost twists $[a_1]$ are horizontal, thus $|p| > |q|$. The rightmost starting twists are then vertical.

Proof. With $-T$ the mirror image of the tangle T, we have that $-K = N(-T)$ and $F(-T) = p/(-q)$. If K is isotopic to $-K$, it follows from the classification theorem for rational knots that either $q(-q) \equiv 1 \mod p$ or $q \equiv -q \mod p$. Without loss of generality we can assume that $0 < q < p$. Hence $2q$ is not divisible by p and therefore it is not the case that $q \equiv -q \mod p$. Hence $q^2 \equiv -1 \mod p$.

Conversely, if $q^2 \equiv -1 \mod p$, then it follows from the Palindrome Theorem that *the continued fraction expansion of p/q has to be palindromic with an even number of terms.* To see this, let $p/q = [c_1,\ldots,c_n]$ with n even, and let $p'/q' = [c_n,\ldots,c_1]$. The Palindrome theorem tells us that $p' = p$ and that $qq' \equiv -1 \mod p$. Thus we have that both q and q' satisfy