2. Rational tangles and their invariant fractions

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various aspects of rational knots and rational tangles can be found in many places in the literature, see for example [6], [35], [29], [5], [2], [22], [16], [19].

2. Rational tangles and their invariant fractions

In this section we recall from [15] the facts that we need about rational tangles, continued fractions and the classification of rational tangles. We intend the paper to be as self-contained as possible.

A 2*-tangle* is a proper embedding of two unoriented arcs and a finite number of circles in a 3-ball B^3, so that the four endpoints lie in the boundary of B^3. A *rational tangle* is a proper embedding of two unoriented arcs α_1, α_2 in a 3-ball B^3, so that the four endpoints lie in the boundary of B^3, and such that there exists a homeomorphism of pairs:

$$\overline{h}\colon (B^3, \alpha_1, \alpha_2) \to (D^2 \times I,\ \{x, y\} \times I) \quad \text{(a trivial tangle)}.$$

This is equivalent to saying that rational tangles have specific representatives obtained by applying a finite number of consecutive twists of neighboring endpoints starting from two unknotted and unlinked arcs. Such a pair of arcs comprise the [0] or [∞] tangles, depending on their position in the plane, see illustrations in Figure 2.

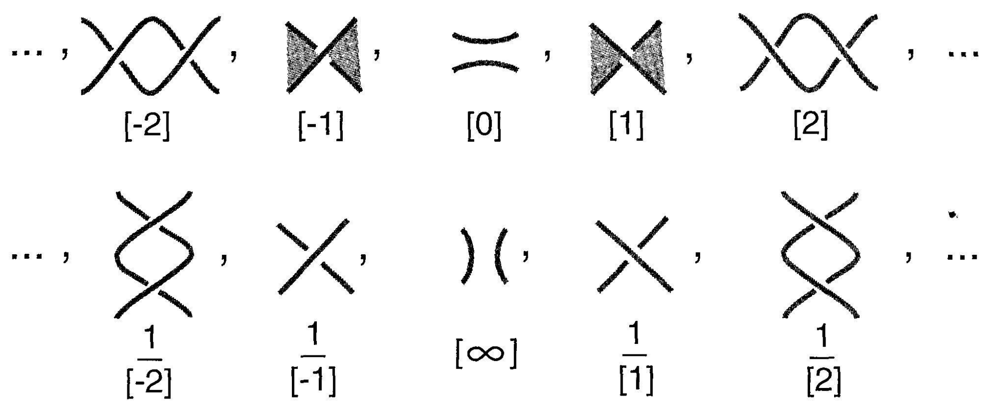

FIGURE 2
The elementary rational tangles and the types of crossings

We shall use this characterizing property of a rational tangle as our definition, and we shall then say that the rational tangle is in *twist form*. See Figure 3 for an example.

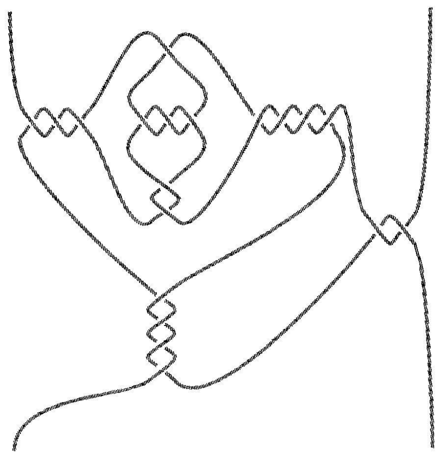

FIGURE 3
A rational tangle in twist form

To see the equivalence of the above definitions, let S^2 denote the two-dimensional sphere, which is the boundary of the 3-ball B^3 and let p denote four specified points in S^2. Let further $h\colon (S^2,p) \to (S^2,p)$ be a self-homeomorphism of S^2 with the four points. This extends to a self-homeomorphism $\overline{h}$ of the 3-ball B^3 (see [29], page 10). Further, let a denote the two straight arcs $\{x,y\} \times I$ joining pairs of the four points in the boundary of B^3. Consider now $\overline{h}(a)$. We call this the tangle induced by h. We note that up to isotopy (see definition below) h is a composition of braidings of pairs of points in S^2 (see [24], pages 61 to 65). Each such braiding induces a twist in the corresponding tangle. So, if h is a composition of braidings of pairs of points, then the extension $\overline{h}$ is a composition of twists of neighboring end arcs. Thus $\overline{h}(a)$ is a rational tangle and every rational tangle can be obtained this way.

A *tangle diagram* is a regular projection of the tangle on a meridian disc. Throughout the paper by 'tangle' we will mean 'regular tangle diagram'. The type of crossings of knots and 2-tangles follow the checkerboard rule: shade the regions of the tangle (knot) in two colors, starting from the left (outside) to the right (inside) with grey, and so that adjacent regions have different colors. Crossings in the tangle are said to be of *positive type* if they are arranged with respect to the shading as exemplified in Figure 2 by the tangle $[+1]$, i.e. they have the region on the right shaded as one walks towards the crossing along the over-arc. Crossings of the reverse type are said to be of *negative type* and they are exemplified in Figure 2 by the tangle $[-1]$. The reader should note that our crossing type conventions are the opposite of those of Conway in [6] and of those of Kawauchi in [16]. Our conventions agree with those of Ernst and Sumners [10], [40] which in turn follow the standard conventions of biologists.

We are interested in tangles up to isotopy. Two rational tangles, T, S, in B^3 are *isotopic*, denoted by $T \sim S$, if and only if any two diagrams of them have identical configurations of their four endpoints on the boundary of the projection disc, and they differ by a finite sequence of the well-known Reidemeister moves [27], which take place in the interior of the disc. Of course, each twisting operation used in the definition of a rational tangle changes the isotopy class of the tangle to which it is applied.

2-TANGLE OPERATIONS. The symmetry of the four endpoints of 2-tangles allows for the following well-defined (up to isotopy) operations in the class of 2-tangles, as described in Figure 4. We have *the sum* of two 2-tangles, denoted by '$+$' and *the product* of two 2-tangles, denoted by '$*$'. This product '$*$' is not to be confused with Conway's product '$\cdot$' in [6].

In view of these operations we can say that a rational tangle is created inductively by consecutive additions of the tangles $[\pm 1]$ on the right or on the left and multiplications by the tangles $[\pm 1]$ at the bottom or at the top, starting from the tangles $[0]$ or $[\infty]$. And since, when we start creating a rational tangle, the very first crossing can be equally seen as a horizontal or as a vertical one, we may always assume that we start twisting from the tangle $[0]$. Addition and multiplication of tangles are not commutative. Also, they do not preserve the class of rational tangles. The sum (product) of two rational tangles is rational if and only if one of the two consists in a number of horizontal (vertical) twists.

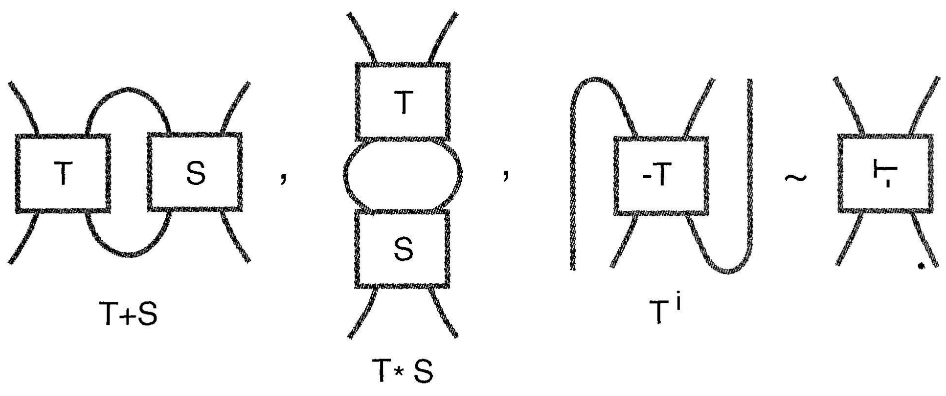

FIGURE 4
Addition, multiplication and inversion of 2-tangles

The *mirror image* of a tangle T, denoted $-T$, is obtained from T by switching all the crossings. So we have $-[n] = [-n]$ and $-\frac{1}{[n]} = \frac{1}{[-n]}$. Finally, *the rotation* of T, denoted T^r, is obtained by rotating T on its plane counterclockwise by 90°, whilst *the inverse* of T, denoted T^i, is defined to be $-T^r$. Thus inversion is accomplished by rotation and mirror image. For example, $[n]^i = \frac{1}{[n]}$ and $\frac{1}{[n]}^i = [n]$. Note that T^r and T^i are in general not isotopic to T.

Moreover, by joining with simple arcs the two upper and the two lower endpoints of a 2-tangle T, we obtain a knot called the *Numerator* of T, denoted by $N(T)$. Joining with simple arcs each pair of the corresponding top and bottom endpoints of T we obtain the *Denominator* of T, denoted by $D(T)$. We have $N(T) = D(T^r)$ and $D(T) = N(T^r)$. We point out that the numerator closure of the sum of two rational tangles is still a rational knot or link. But the denominator closure of the sum of two rational tangles is not necessarily a rational knot or link, think for example of the sum $\frac{1}{[3]} + \frac{1}{[3]}$.

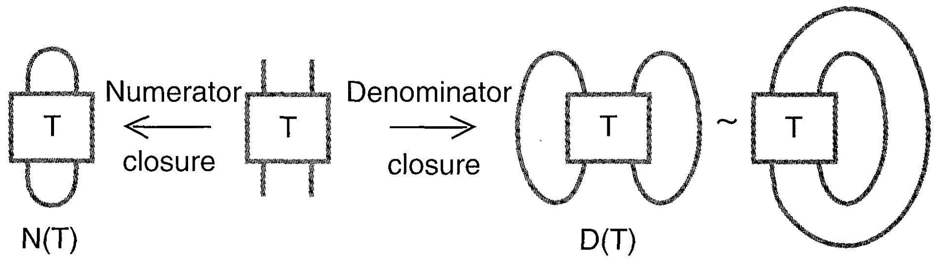

FIGURE 5
The numerator and denominator of a 2-tangle

RATIONAL TANGLE ISOTOPIES. We define now two isotopy moves for rational tangles that play a crucial role in the theory of rational knots and rational tangles.

DEFINITION 1. A *flype* is an isotopy of a 2-tangle T (or a knot or link) applied on a 2-subtangle of the form $[\pm 1] + t$ or $[\pm 1] * t$ as illustrated in Figure 6. A flype fixes the endpoints of the subtangle on which it is applied. A flype shall be called *rational* if the 2-subtangle on which it applies is rational.

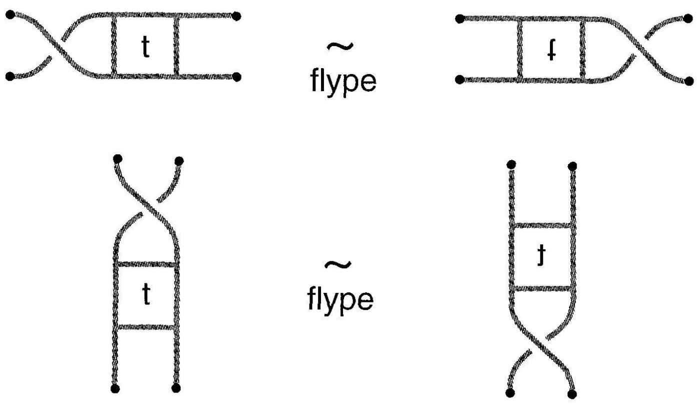

FIGURE 6
The flype moves

We define the *truncation* of a rational tangle to be the result of partially untwisting the tangle. For rational tangles, flypes are of very specific types. Indeed, we have:

Let T be a rational tangle in twist form. Then

(i) *T does not contain any non-rational 2-subtangles.*

(ii) *Every 2-subtangle of T is a truncation of T.*

For a proof of these statements we refer the reader to our paper [15]. As a corollary we have that all flypes of a rational tangle T are rational.

DEFINITION 2. A *flip* is a rotation in space of a 2-tangle by $180°$. We say that T^{hflip} is the *horizontal flip* of the 2-tangle T if T^{hflip} is obtained from T by a $180°$ rotation around a horizontal axis on the plane of T, and T^{vflip} is the *vertical flip* of the tangle T if T^{vflip} is obtained from T by a $180°$ rotation around a vertical axis on the plane of T. See Figure 7 for illustrations.

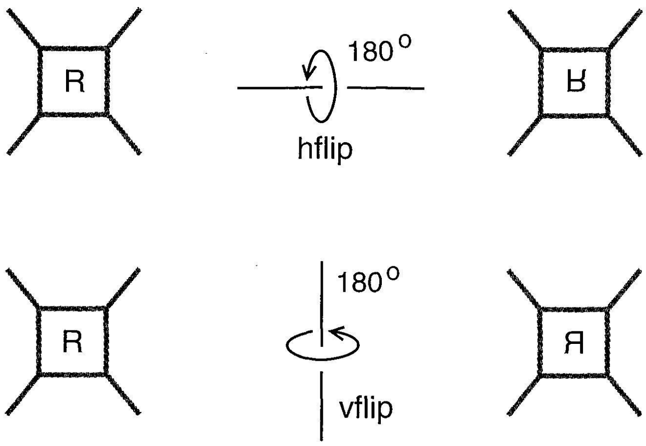

FIGURE 7
The horizontal and the vertical flip

Note that a flip switches the endpoints of the tangle and, in general, a flipped tangle is not isotopic to the original one; the following is a remarkable property of rational tangles:

THE FLIPPING LEMMA. *If T is rational, then:*

(i) $T \sim T^{hflip}$, (ii) $T \sim T^{vflip}$ *and* (iii) $T \sim (T^i)^i = (T^r)^r$.

To see (i) and (ii) we apply induction and a sequence of flypes, see [15] for details. $(T^i)^i = (T^r)^r$ is the tangle obtained from T by rotating it on its plane by $180°$, so statement (iii) follows by applying a vertical flip after a horizontal flip. Note that the above statements are obvious for the tangles $[0], [\infty]$, $[n]$ and $\frac{1}{[n]}$. Statement (iii) says that *for rational tangles the inversion is an operation of order* 2. For this reason we shall denote the inverse of a rational tangle T by $1/T$, and hence the rotation of the tangle T will be denoted by $-1/T$. This explains the notation for the tangles $\frac{1}{[n]}$. For arbitrary 2-tangles the inversion is an order 4 operation. Another consequence of the above property is that addition and multiplication by $[\pm 1]$ are commutative.

STANDARD FORM, CONTINUED FRACTION FORM AND CANONICAL FORM FOR RATIONAL TANGLES. Recall that the twists generating the rational tangles could take place between the right, left, top or bottom endpoints of a previously created rational tangle. Using obvious flypes on appropriate subtangles one can always bring the twists all to the right (or all to the left) and to the bottom (or to the top) of the tangle. We shall then say that the rational tangle is in *standard form*. For example Figure 1 illustrates the tangle $(([3]*\frac{1}{[-2]})+[2])$ in standard form. In order to read out the standard form of a rational tangle in twist form we transcribe it as an algebraic sum using horizontal and vertical twists. For example, Figure 3 illustrates the tangle $((([3]*\frac{1}{[3]})+[-1])*\frac{1}{[-4]})+[2]$ in non-standard form.

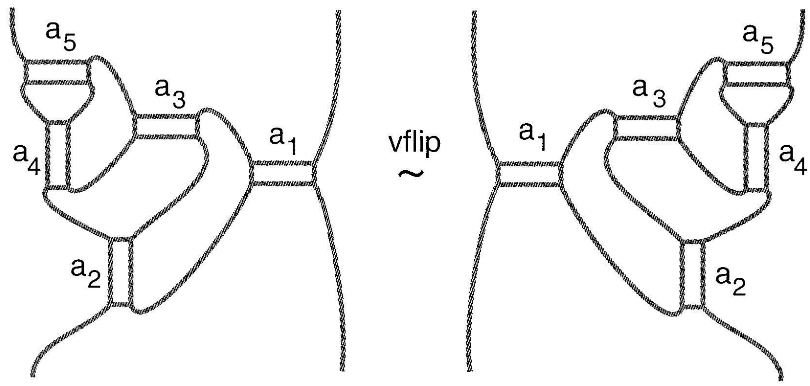

FIGURE 8
The standard representations

Figure 8 illustrates two equivalent (by the Flipping Lemma) ways of representing an abstract rational tangle in standard form: the *standard representation* of a rational tangle. In either illustration the rational tangle begins to twist from the tangle $[a_n]$ ($[a_5]$ in Figure 8), and it untwists from the tangle $[a_1]$. Note that the tangle in Figure 8 has an odd number of sets of twists ($n = 5$) and this causes $[a_1]$ to be horizontal. If n is even and $[a_n]$ is horizontal then $[a_1]$ has to be vertical.

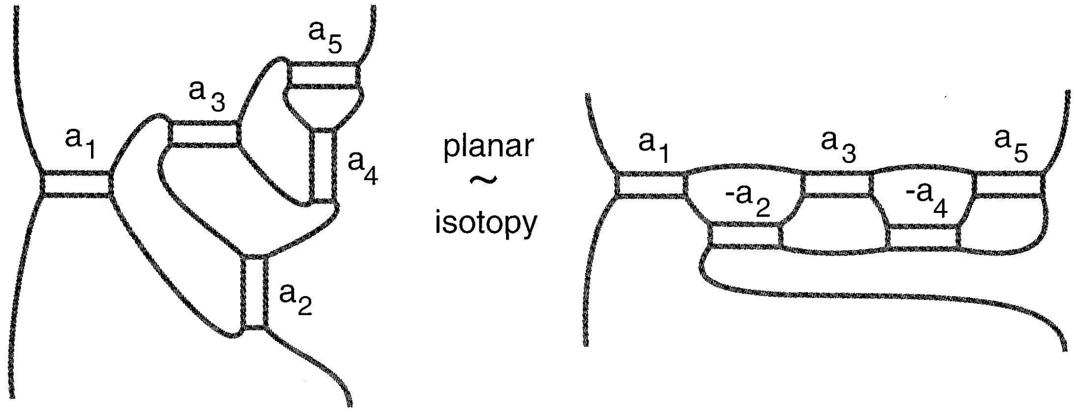

FIGURE 9
The standard and the 3-strand-braid representation

Another way of representing an abstract rational tangle in standard form is illustrated in Figure 9. This is the 3*-strand-braid representation.* For an example see Figure 10. As Figure 9 shows, the 3-strand-braid representation is actually a compressed version of the standard representation, so the two representations are equivalent by a planar rotation. The upper row of crossings of the 3-strand-braid representation corresponds to the horizontal crossings of the standard representation and the lower row to the vertical ones. Note that, even though the type of crossings does not change by this planar rotation, we need to draw the mirror images of the even terms, since when we rotate them to the vertical position we obtain crossings of the opposite type in the local tangles. In order to bear in mind this change of the local signs we put on the geometric picture the minuses on the even terms. We shall use both ways of representation for extracting the properties of rational knots and tangles.

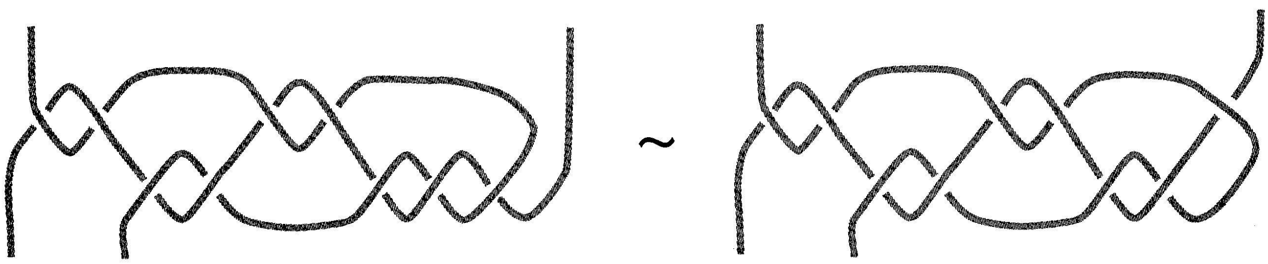

FIGURE 10
The ambiguity of the first crossing

From the above one may associate to a rational tangle diagram in standard form a vector of integers $(a_1, a_2, \ldots, a_n)$, where the first entry denotes the place where the tangle starts unravelling and the last entry where it begins to twist. For example the tangle of Figure 1 is associated to the vector $(2, -2, 3)$, while the tangle of Figure 3 corresponds after a sequence of flypes to the vector $(2, -4, -1, 3, 3)$. The vector associated to a rational tangle diagram is *unique* up to breaking the entry a_n by a unit, i.e. $(a_1, a_2, \ldots, a_n) = (a_1, a_2, \ldots, a_n - 1, 1)$, if $a_n > 0$, and $(a_1, a_2, \ldots, a_n) = (a_1, a_2, \ldots, a_n + 1, -1)$, if $a_n < 0$. This follows from the ambiguity of the very first crossing, see Figure 10. If a rational tangle changes by an isotopy, the associated vector might also change.

REMARK 1. The same ambiguity implies that the number n in the above notation may be assumed to be *odd*. We shall make this assumption for proving Theorems 2 and 3.

The next thing to observe is that a rational tangle in standard form can be described algebraically by a continued fraction built from the integer tangles $[a_1], [a_2], \ldots, [a_n]$ with all numerators equal to 1, namely by an expression of the type:

$$[[a_1], [a_2], \ldots, [a_n]] := [a_1] + \cfrac{1}{[a_2] + \cdots + \cfrac{1}{[a_{n-1}] + \cfrac{1}{[a_n]}}}$$

for $a_2, \ldots, a_n \in \mathbf{Z} - \{0\}$ and n even or odd. We allow $[a_1]$ to be the tangle $[0]$. This expression follows inductively from the equation

$$T * \frac{1}{[n]} = \cfrac{1}{[n] + \cfrac{1}{T}}.$$

Then a rational tangle is said to be in *continued fraction form*. For example, Figure 1 illustrates the rational tangle $[[2], [-2], [3]]$, while the tangles of Figure 8 and 9 all depict the abstract rational tangle $[[a_1], [a_2], [a_3], [a_4], [a_5]]$.

The tangle equation $T * \frac{1}{[n]} = \frac{1}{[n] + \frac{1}{T}}$ implies also that the two simple algebraic operations: *addition of* $[+1]$ *or* $[-1]$ and *inversion* between rational tangles generate the whole class of rational tangles. For $T = [[a_1], [a_2], \ldots, [a_n]]$ the following statements are now straightforward.

1. $T + [\pm 1] = [[a_1 \pm 1], [a_2], \ldots, [a_n]]$,

2. $\frac{1}{T} = [[0], [a_1], [a_2], \ldots, [a_n]]$,

3. $-T = [[-a_1], [-a_2], \ldots, [-a_n]]$,

4. $T = [[a_1], [a_2], \ldots, [a_n - 1], [1]]$, if $a_n > 0$,

and $T = [[a_1], [a_2], \ldots, [a_n + 1], [-1]]$, if $a_n < 0$.

A tangle is said to be *alternating* if the crossings alternate from under to over as we go along any component or arc of the weave. Similarly, a knot is *alternating* if it possesses an alternating diagram. We shall see that rational tangles and rational knots are alternating. Notice that, according to the checkerboard shading (see Figure 2 and the corresponding discussion), the only way the weave alternates is if any two adjacent crossings are of the same type, and this propagates to the whole diagram. Thus, *a tangle or a knot diagram with all crossings of the same type is alternating*, and this characterizes alternating tangle and knot diagrams. It is important to note that flypes preserve the alternating structure. Moreover, flypes are the only isotopy moves needed in the statement of the Tait Conjecture for alternating knots. An important property of rational tangles is now the following:

A rational tangle diagram in standard form can be always isotoped to an alternating one.

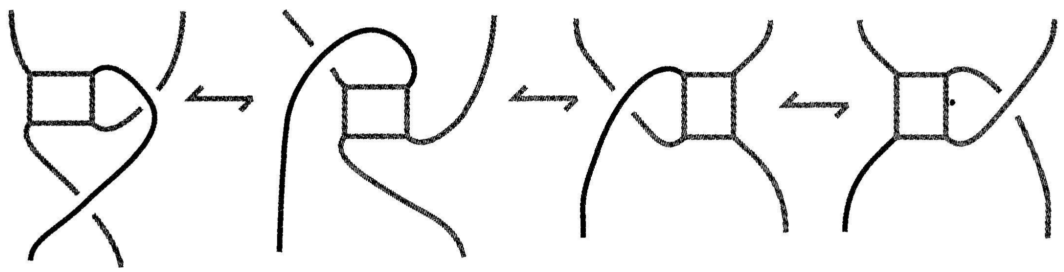

FIGURE 11
Reducing to the alternating form

The process is inductive on the number of crossings and the basic isotopy move is illustrated in Figure 11, see [15] for details. We point out that this isotopy applies to rational tangles in standard form where all the crossings are on the right and on the bottom. We shall say that a rational tangle $T = [[a_1], [a_2], \dots, [a_n]]$ is in *canonical form* if T is alternating and n is odd. From Remark 1 we can always assume n to be odd, so in order to bring a rational tangle to the canonical form we just have to apply the isotopy moves described in Figure 11. Note that T alternating implies that the a_i's are all of the same sign.

The alternating nature of the rational tangles will be very useful to us in classifying rational knots and links. It turns out from the classification of alternating knots that *two alternating tangles are isotopic if and only if they differ by a sequence of flypes.* (See [41], [20]. See also [34].) It is easy to see that the closure of an alternating rational tangle is an alternating knot. Thus we have:

Rational knots are alternating, since they possess a diagram that is the closure of an alternating rational tangle.

CONTINUED FRACTIONS AND THE CLASSIFICATION OF RATIONAL TANGLES. From the above discussion it makes sense to assign to a rational tangle in standard form, $T = [[a_1], [a_2], \dots, [a_n]]$, for $a_1 \in \mathbf{Z}$, $a_2, \dots, a_n \in \mathbf{Z} - \{0\}$ and n even or odd, the continued fraction

$$F(T) = [a_1, a_2, \dots, a_n] := a_1 + \cfrac{1}{a_2 + \cdots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}},$$

if $T \neq [\infty]$, and $F([\infty]) := \infty = \frac{1}{0}$, as a formal expression. This rational number or infinity shall be called *the fraction of T*. The fraction is a topological invariant of the tangle T. We explain briefly below how to see this.

The subject of continued fractions is of perennial interest to mathematicians. See for example [17], [23], [18], [47]. In this paper we shall only consider continued fractions of the above type, i.e. with all numerators equal to 1. As in the case of rational tangles we allow the term a_1 to be zero. Clearly, the two simple algebraic operations *addition of* $+1$ *or* -1 and *inversion* generate inductively the whole class of continued fractions starting from zero. For any rational number $\frac{p}{q}$ the following statements are really straightforward.

1. There are $a_1 \in \mathbf{Z}$, $a_2, \ldots, a_n \in \mathbf{Z} - \{0\}$ such that $\frac{p}{q} = [a_1, a_2, \ldots, a_n]$,

2. $\frac{p}{q} \pm 1 = [a_1 \pm 1, a_2, \ldots, a_n]$,

3. $\frac{q}{p} = [0, a_1, a_2, \ldots, a_n]$,

4. $-\frac{p}{q} = [-a_1, -a_2, \ldots, -a_n]$,

5. $\frac{p}{q} = [a_1, a_2, \ldots, a_n - 1, 1]$, if $a_n > 0$,

   and $\frac{p}{q} = [a_1, a_2, \ldots, a_n + 1, -1]$, if $a_n < 0$.

Property 1 is a consequence of Euclid's algorithm, see for example [17]. Combining the above we obtain the following properties for the tangle fraction.

1. $F(T + [\pm 1]) = F(T) \pm 1$,

2. $F(\frac{1}{T}) = \frac{1}{F(T)}$,

3. $F(-T) = -F(T)$.

The last ingredient for the classification of rational tangles is the following fact about continued fractions: *Every continued fraction* $[a_1, a_2, \ldots, a_n]$ *can be transformed to a unique canonical form* $[\beta_1, \beta_2, \ldots, \beta_m]$*, where all* β_i*'s are positive or all negative integers and* m *is odd.*

One way to see this is to evaluate the continued fraction and then apply Euclid's algorithm, keeping all remainders of the same sign. There is also an algorithm that can be applied directly to the initial continued fraction to obtain its canonical form. This algorithm works in parallel with the algorithm for the canonical form of rational tangles, see [15] for details.

From the Tait conjecture for alternating rational tangles, from the uniqueness of the canonical form of continued fractions and from the above properties of the fraction we derive that the fraction not only is an isotopy invariant of rational tangles but it also classifies rational tangles. This is the Conway Theorem. See [15] for details of the proof. For the isotopy type of a rational tangle with fraction $\frac{p}{q}$ we shall use the notation $[\frac{p}{q}]$. Finally, it is easy to see the following useful result about rational tangles:

Suppose that $T + [n]$ *is a rational tangle, then* T *is a rational tangle.*