12.1 Statement of the theorem

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **49 (2003)**

Heft 3-4: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **25.09.2024**

Proposition 10.1 provides a $\kappa(r,s) = Bi(A(r,s), A(r,s))$ such that this bigon is $\kappa(r,s)$-thin. Thus the given (r,s)-chain bigon is $\delta(r,s)$-thin, with $\delta(r,s) = \kappa(r,s) + 2A(r,s)$. By Lemma 11.1, the given forest-stack, which is a $(1,2)$-quasi geodesic metric space, is $2\delta(1,6)$-hyperbolic. □

12. Back to mapping-telescopes

In this section we elucidate the relationships between forest-stacks and mapping-telescopes.

12.1 Statement of the theorem

An $\mathbf{R}$-*tree* (see [9], [2] among many others) is a metric space such that any two points are joined by a unique arc and this arc is a geodesic for the metric. In particular an $\mathbf{R}$-tree is a topological tree. An $\mathbf{R}$-*forest* is a union of disjoint $\mathbf{R}$-trees.

LEMMA 12.1. *Let (Γ, d_Γ) be an $\mathbf{R}$-forest and let $\psi \colon \Gamma \to \Gamma$ be a forest-map of Γ. Let (K_ψ, f, σ_t) be the mapping-telescope of (ψ, Γ) equipped with a structure of forest-stack as defined in Section 2. Then there is a horizontal metric $\mathcal{H} = (m_r)_{r \in \mathbf{R}}$ on K_ψ such that*

1. *The $\mathbf{R}$-forests $(f^{-1}(r), m_r)$ and $(f^{-1}(r+1), m_{r+1})$ are isometric. Each stratum $(f^{-1}(n), m_n)$, $n \in \mathbf{Z}$, is isometric to (Γ, d_Γ).*
2. *For any real r and any horizontal geodesic $g \in f^{-1}(r)$, the map*

$$l_{r,g} \colon \begin{cases} +1 - r] \to \mathbf{R}^+ \\ \quad t \mapsto |\sigma_t(g)|_{r+t} \end{cases} .$$

is monotone.

Such a horizontal metric is called a horizontal d_Γ-metric. *The telescopic metric associated to a horizontal d_Γ-metric is called a* mapping-telescope d_Γ-metric.

Proof. We make each $\Gamma \times \{n\}$, $n \in \mathbf{Z}$, an $\mathbf{R}$-forest isometric to Γ. We consider a cover of Γ by geodesics of length 1 which intersect only at their endpoints. Each $\Gamma \times \{n\}$ inherits the same cover. There is a disc $D_{e,n}$ in K_ψ for each such horizontal geodesic e in $\Gamma \times \{n\}$. This disc is bounded by e, $\psi(e)$ and the orbit-segments between the endpoints of e and those of $\psi(e)$.

We foliate this disc by segments with endpoints in, and transverse to, the orbit-segments in its boundary. Then we assign a length to each such segment so that the collection of lengths varies continuously and monotonically, from the length of e to that of $\psi(e)$. We thus obtain a horizontal metric on the mapping-telescope. Furthermore each stratum $f^{-1}(n)$, $n \in \mathbf{Z}$, is isometric to (Γ, d_Γ). And the maps denoted by $l_{r,g}$ in Lemma 12.1 are monotone by construction. By definition of a mapping-telescope, the discs $D_{e,n}$ between $\Gamma \times \{n\}$ and $\Gamma \times \{n+1\}$ are copies of the discs $D_{e,n'}$ between $\Gamma \times \{n'\}$ and $\Gamma \times \{n'+1\}$, for any n, n' in $\mathbf{Z}$. This allows us to choose the horizontal metric to satisfy the further condition that $(f^{-1}(r), m_r)$ be isometric with $(f^{-1}(r+1), m_{r+1})$ for any real number r. □

We now define dynamical properties for $\mathbf{R}$-forest maps.

DEFINITION 12.2. Let (Γ, d_Γ) be an $\mathbf{R}$-forest. A forest-map ψ of (Γ, d_Γ) is *weakly bi-Lipschitz* if there exist $\mu \geq 1$ and $K \geq 0$ such that $\mu d_\Gamma(x,y) \geq d_\Gamma(\psi(x), \psi(y)) \geq \frac{1}{\mu} d_\Gamma(x,y) - K$.

DEFINITION 12.3. Let (Γ, d_Γ) be an $\mathbf{R}$-forest. A forest-map ψ of (Γ, d_Γ) is *hyperbolic* if it is weakly bi-Lipschitz and there exist $\lambda > 1$, $N \geq 1$, $M \geq 0$ such that for any pair of points x, y in Γ with $d_\Gamma(x,y) \geq M$, either $d_\Gamma(\psi^N(x), \psi^N(y)) \geq \lambda d_\Gamma(x,y)$ or $d_\Gamma(x_N, y_N) \geq \lambda d_\Gamma(x,y)$ for some x_N, y_N with $\psi^N(x_N) = x$, $\psi^N(y_N) = y$.

A hyperbolic forest-map ψ of (Γ, d_Γ) is *strongly hyperbolic* if, for any pair of points x, y with $d_\Gamma(x,y) \geq M$ and each connected component containing both a preimage of x and a preimage of y under ψ^N, there is at least one pair of such preimages x_N, y_N for which $d_\Gamma(x_N, y_N) \geq \lambda\, d_\Gamma(x,y)$.

If the forest Γ is a tree then a hyperbolic forest-map is strongly hyperbolic (similarly we saw that a hyperbolic semi-flow on a forest-stack whose strata are connected is strongly hyperbolic).

Our theorem about mapping-telescopes is

THEOREM 12.4. *Let (Γ, d_Γ) be an $\mathbf{R}$-forest. Let ψ be a strongly hyperbolic forest-map of (Γ, d_Γ) whose mapping-telescope K_ψ is connected. Then K_ψ is a Gromov-hyperbolic metric space for any mapping-telescope d_Γ-metric.*