ON THE ENTROPY OF HOLOMORPHIC MAPS

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Our purpose is to calculate the topological entropy of a holomorphic map f of the complex projective space $\mathbf{C}P^m$ into itself. Every such map is given by $(m+1)$ homogeneous polynomials in $\mathbf{C}^{m+1}$ each of the same degree p, and the topological degree $\deg f$ is equal to p^m. When $m = 1$, the space $\mathbf{C}P^m$ is the Riemann sphere S^2 and such maps are given by rational functions of one variable. Geometrically, they are conformal maps $S^2 \to S^2$ of positive degree.

The topological *entropy* $h(f)$, defined in the next section, measures the asymptotic complexity of the iterates f^k, and it is usually hard to calculate.

MAIN RESULT. *If* $f\colon \mathbf{C}P^m \to \mathbf{C}P^m$ *is holomorphic, then*

$$h(f) = \log(\deg f)\,. \tag{0.0}$$

REMARKS AND ACKNOWLEDGEMENTS. We prove here only the inequality

$$h(f) \leq \log(\deg f)\,. \tag{0.1}$$

The opposite statement $h(f) \geq \log(\deg f)$ was established by Misiurewicz and Przytycki [4] for all smooth maps. They proved even more: if f is a C^2-smooth endomorphism of a compact manifold and points having at least d preimages are dense then $h(f) \geq \log(\deg f)$. For example, every smooth map of zero degree of a closed manifold *onto* itself has entropy not less than $\log 2$.

Our paper owes very much to Sheldon Newhouse. Inequality (0.1) is a response to his very first question to me on arriving at IHES. He conjectured (0.1) in the case $m = 1$, suggested a possibility of analogous estimates for real polynomial maps and provided an example for a class of maps $\mathbf{R}^2 \to \mathbf{R}^2$ given by polynomials of degree 2. His further interest in the problem forced me to write this paper. I also appreciate the hospitality of IHES, which made possible my involvement in this story. I am especially thankful to Dennis Sullivan, who took pains to read the paper and to clean it up of multiple errors.

STRUCTURE OF THE PAPER. We start with a geometric outlook on topological entropy and reduce our inequality (0.1) to standard facts about minimal varieties. We discuss next the real algebraic analogue of (0.1) and a generalization to maps. We conclude with an estimate of the entropy involving the mean curvature.

§1. NOTATION AND DEFINITIONS

For a space X we denote by X^k the product $X \times X \times \ldots \times X$ (k factors). A *graph* Γ over X is by definition an arbitrary set $\Gamma \subset X^2$. When X is finite this is the usual definition of an oriented graph (with loops). The graph of a map $X \to X$ gives another example.

For a graph Γ we denote by $\Gamma_k \subset X^k$ the set of strings $(x_1, \ldots, x_i, \ldots, x_k)$, $x_i \in X$, where each pair $(x_{i-1}, x_i) \in X^2$ is contained in Γ.

When X is endowed with a metric, we call ϵ*-cubes* products in X^k of balls from X of radius ϵ. For a set $Y \subset X^k$ we denote by $\operatorname{Cap}_\epsilon Y$ the minimal number of ϵ-cubes needed to cover Y.

ENTROPY

Set $h_\epsilon(\Gamma) = \limsup_{k\to\infty} \frac{1}{k} \log \operatorname{Cap}_\epsilon \Gamma_k$, and $h(\Gamma) = \lim_{\epsilon\to 0} h_\epsilon(\Gamma)$, for $\Gamma \subset X^2$.

When f is an endomorphism $X \to X$, we define its *entropy* $h(f)$ as the entropy of its graph Γ_f. If the space X is compact, the definition does not depend on the choice of the metric [2]. Observe that the entropy of a general graph Γ is equal to the entropy of the shift in $\Gamma_\infty \subset X^\infty$: Γ_∞ is the space of doubly infinite strings $(x_i)_{i=\ldots,-1,0,1,\ldots}$ with the product topology, and the shift maps (x_i) to (x_{i+1}). For finite X, we come to the usual definition of the Markov shift.

VOLUME

From now on, X is a Riemannian manifold and $n = \dim \Gamma$, $\Gamma \subset X^2$. We denote by $\operatorname{Vol} \Gamma_k$ the n-dimensional volume of $\Gamma_k \subset X^k$, i.e. the

n-dimensional Hausdorff measure with respect to the Riemann product metric in X^k. Set

$$\operatorname{lov} \Gamma = \limsup_{k\to\infty} \frac{1}{k} \log \operatorname{Vol} \Gamma_k .$$

For an f we set $\operatorname{lov} f = \operatorname{lov} \Gamma_f$. This is a smooth invariant of f (it does not depend on the choice of the Riemann metric).

Our invariant "lov" is sometimes more accessible than entropy and for a holomorphic f we are going to prove that

$$h(f) \leq \operatorname{lov} f . \tag{1.0}$$

DENSITY

Denote by $\operatorname{Dens}_\epsilon(\Gamma_k, \gamma)$, for $\gamma \in \Gamma_k \subset X^k$, the n-dimensional measure of the intersection of Γ_k with the ball (in the Riemannian product metric) of radius ϵ centered at γ. Set $\operatorname{Dens}_\epsilon(\Gamma_k) = \inf_{k\in\Gamma_k} \operatorname{Dens}_\epsilon(\Gamma_k, \gamma)$, and then $\operatorname{lodn}_\epsilon \Gamma = \liminf_{k\to\infty} \frac{1}{k} \log \operatorname{Dens}_\epsilon \Gamma_k$, and finally

$$\operatorname{lodn} \Gamma = \lim_{\epsilon\to 0} \operatorname{lodn}_\epsilon \Gamma .$$

Observe that $\operatorname{Vol} \geq \operatorname{Cap}_{2\epsilon} \operatorname{Dens}_\epsilon$ and hence that

$$h(V) \leq \operatorname{lov} \Gamma - \operatorname{lodn} \Gamma . \tag{1.1}$$

§2. ESTIMATES OF DENSITY

Consider a Riemannian manifold W (it will be X^k in the sequel) and an n-dimensional subvariety $V \subset W$. We suppose that the boundary of V (if there is such) does not intersect the ball $B_\epsilon \subset W$ of radius $\epsilon > 0$ centered at a point $v_0 \in V$. We suppose also that the injectivity radius of W at v_0 is at least ϵ, i.e. the exponential map $T_{v_0}(W) \to W$ is injective in the ϵ-ball in $T_{v_0}(W)$.

DENSITY OF A MINIMAL VARIETY

If the sectional curvature in B_ϵ is not greater than K and V is minimal, then

$$\operatorname{Vol}(V \cap B_\epsilon) \geq C > 0 , \tag{2.0}$$

where the constant C depends on n, K, and ϵ, but does not depend on $\dim W$.

The proof is given below. This fact is well known and C is equal to the volume of the ϵ-ball in the n-dimensional space of constant curvature K. Our application of (2.0) to complex geometry is based on

FEDERER'S THEOREM. *Analytic subvarieties of a Kähler manifold are minimal.*

Thus we can apply (2.0), conclude that $\operatorname{lodn}\Gamma = 0$ and obtain (1.0) in the Kähler case by using (1.1).

Proof of (2.0). We restrict ourselves to the case when W is the Euclidean space and V is nonsingular. Denote by A_ϵ the $(n-1)$-dimensional volume of the boundary $V \cap \partial B_\epsilon$.

Minimality of V implies

$$V_\epsilon \geq \operatorname{Vol} \operatorname{Co}_\epsilon = \frac{\epsilon}{n} A_\epsilon \,, \tag{2.1}$$

where $\operatorname{Co}_\epsilon$ is the cone over A_ϵ centered at v_0.

On the other hand

$$V_\epsilon \geq \int_0^\epsilon A_\tau \, d\tau \,. \tag{2.2}$$

Regularity of V implies that

$$\lim_{\epsilon \to 0} \frac{V_\epsilon}{\epsilon^n} = \lim_{\epsilon \to 0} \frac{A_\epsilon}{n\epsilon^{n-1}} = C_n \,, \tag{2.3}$$

where C_n is the volume of the unit ball in $\mathbf{R}^n$.

Combining (2.1), (2.2) and (2.3), we get

$$V_\epsilon \geq C_n \epsilon^n \,, \tag{2.4}$$

which implies (2.0) in the Euclidean case.

Proof of (1.0). As we mentioned above, (2.4) implies (1.0), but only in the Euclidean case where (2.4) is proven. But the local nature of the density enables us to reduce the general case to the Euclidean one: near each point $x \in X$ we equip the complex manifold X (we suppose that X is compact without boundary) with a flat (i.e. Euclidean) Kähler structure and use the product structure near each point from X^k. Independence of "lodn" upon the choice of the metric allows one to apply (2.4) to derive the vanishing of "lodn" and thus the desired inequality $h \leq \operatorname{lov}$.

§ 3. KÄHLER MANIFOLDS

We view a Kähler manifold as a Riemannian manifold W with a closed 2-form ω. Every submanifold V of dimension $n = 2m$ satisfies the Wirtinger inequality

$$\mathrm{Vol}\,(V) \geq \int_V (\omega)^m\,, \tag{3.0}$$

and equality

$$\mathrm{Vol}\,(V) = \int_V (\omega)^m \tag{3.1}$$

holds if and only if V is complex analytic (of complex dimension m). Observe that (3.0) and (3.1) imply the Federer theorem.

Start now with a Kähler manifold X of real dimension $n = 2m$ and apply (3.1) to the iterated graph $(\Gamma_f)_k \subset X^k$ of an endomorphism $f\colon X \to X$. We get

$$\mathrm{Vol}\,(\Gamma_f)_k = \left\langle \Big(\sum\nolimits_\alpha^k\Big)^m, [X] \right\rangle, \tag{3.2}$$

where $\alpha \in H^2(X, \mathbf{R})$ is the cohomology class represented by the structural 2-form, $\Big(\sum\nolimits_\alpha^k\Big) = \sum_{i=1}^k (f^*)^i(\alpha)$, and $[X]$ is the fundamental class of X.

When $X = \mathbf{C}P^m$ and $\deg f = d = p^m$ we have

$$\Big(\sum\nolimits_\alpha^k\Big) = \frac{p^{k+1}-1}{p-1}\,\alpha$$

and conclude that

$$\mathrm{lov}\ \Gamma_f = \log \deg f\,. \tag{3.3}$$

Together with (1.0) this implies our main inequality (0.1).

REMARKS. (3.3) holds whenever α is an eigenvector of the operator $f^*\colon H^2(X, \mathbf{R}) \to H^2(X, \mathbf{R})$ but not generally, as shown by linear endomorphisms of tori.

When X is complex but not Kähler, neither "lov" nor entropy can be estimated in homological terms. Moreover, the entropy of a holomorphic vector field can be non-zero. (In the Kähler case, maps homotopic to the identity have "lov" $= 0$.)

Take a complex semi-simple Lie group and factor it by a discrete uniform subgroup. The group translations in this factor can have non-zero entropy. To be specific, we take $SL_2(\mathbf{C})$ acting by isometries in three-dimensional hyperbolic space. Thus geodesic flows on compact 3-dimensional hyperbolic manifolds are factors of translations of the above type and their entropy must be positive.

HOPF MANIFOLDS

The Hopf manifold H^m is diffeomorphic to $S^1 \times S^{2m-1}$. As a complex manifold it is the factor of $\mathbf{C}^m \setminus 0$ by the following action of $\mathbf{Z}$:

$$x \longmapsto z_0^r x\ , \qquad x \in \mathbf{C}^m \setminus 0\,,\ z_0 \in \mathbf{C}\,,\ |z_0| \neq 0,1\,,\ r \in \mathbf{Z}\,.$$

There is a natural fibration $H^m \to \mathbf{C}P^{m-1}$ and each endomorphism of $\mathbf{C}P^{m-1}$ extends to H^m. When $m > 1$ Hopf manifolds are not Kähler; nevertheless, for any endomorphism $f\colon H^m \to H^m$ we have

$$h(f) = \operatorname{lov} f = \log \deg f\,. \tag{3.4}$$

Proof. Each endomorphism f preserves the fibers of the fibration $H^m \to \mathbf{C}P^{m-1}$ and "lov" is additive in the following sense.

Given a holomorphic fibration $H \to V$ with fibers T_v, $v \in V$, equiped with Kähler structures. Suppose that structural cohomology classes $\alpha_v \in H^2(T_v;\mathbf{R})$ are parallel under the holonomy action of $\pi_1(V)$. If $f\colon H \to H$ is a fiber-preserving endomorphism, one can define

$$\Big(\overset{k}{\textstyle\sum_\alpha}\Big) = \sum_{i=1}^{k} (f^*)^i(\alpha_{v_0}) \in H^2(T_{v_0};\mathbf{R})\,,$$

$v_0 \in V$, and an endomorphism $g\colon V \to V$ induced by f.

THE ADDITION FORMULA

$$\operatorname{lov} f = \operatorname{lov}\, g + \lim_{k\to\infty} \frac{1}{k} \log\Big\langle \Big(\overset{k}{\textstyle\sum_\alpha}\Big)^m, [T_{v_0}]\Big\rangle\,,$$

where $m = \dim_{\mathbf{C}} T_{v_0}$, and $[T_{v_0}]$ is the fundamental class of T_{v_0}.

This formula is almost as obvious as (3.2) and, together with (3.3), it yields (3.4).

GENERALIZATIONS AND PROBLEMS

In the previous discussion, we avoided mentioning the fact of the scarcity of complex endomorphisms. I am not able to add any interesting examples to those considered above[1]). Generic manifolds have no endomorphisms. Every surjective endomorphism f is finite-to-one and when $\deg f = 1$ it is injective. More generally, if V and V' are complex (not necessarily compact or Kähler) manifolds of equal dimensions and their even Betti numbers are finite and equal (i.e. $b_{2i} = b'_{2i}$) then every proper surjective holomorphic map $f\colon V \to V'$ is finite-to-one and when $\deg f = 1$ the map is injective. The finitness condition cannot be omitted; take $\mathbf{C}^2$, blow it up at all points from a lattice. The endomorphism of the resulting manifold induced by the transformation $\mathbf{C}^2 \to \mathbf{C}^2$, $x \longmapsto \frac{x}{2}$, has infinitely many blow-downs.

The lack of endomorphisms can be offset by abundance of general holomorphic graphs. The most regular asymptotic behavior is displayed by graphs $\Gamma \subset X \times X$ of finite type when both projections $\Gamma \to X$ are finite-to-one. In the finite type case the infinitely iterated graph Γ_∞ can be viewed as a $2m$-dimensional ($m = \dim_{\mathbf{C}} \Gamma = \dim_{\mathbf{C}} X$) compact set 'foliated' by complex m-dimensional leaves and having Cantor sets as transversal sections. The holomorphic finite type graphs probably have finite "lov" and entropy and at least in the Kähler case this can be proved as follows: Denote by $\gamma \in H^n(X \times X; \mathbf{R})$, $n = 2m$, $m = \dim_{\mathbf{C}} \Gamma = \dim_{\mathbf{C}} X$, the class dual to $[\Gamma]$ and by $\gamma_{i,i+1} \in H^n(X^k; \mathbf{R})$ the class induced from γ by projecting X^k onto the product of its i-th and $(i+1)$-th factors. Denote by $\beta \in H^2(X^k; \mathbf{R})$ the class represented by the structural 2-form. One can easily see that

$$\operatorname{Vol} \Gamma_k = \left\langle \beta^m \prod_{i=1}^{k-1} \gamma_{i,i+1}, [X^k] \right\rangle ;$$

thus "lov" is finite.

In the end, I must admit my inability to prove (or disprove) the inequality $h(V) \geq \operatorname{lov} \Gamma$ even when Γ is a graph of an endomorphism. (Of course, I mean here only the holomorphic case. For smooth endomorphisms, the situation $h(f) = 0$, $\operatorname{lov} f > 0$ occurs already for maps $\mathbf{S}^1 \to \mathbf{S}^1$ and, probably, for higher dimensions the opposite: $h(f) > 0$, $\operatorname{lov} f = 0$ can also happen.) The inequality $h(\Gamma) \geq \operatorname{lov} \Gamma$ reminds one of the Shub entropy conjecture [5] proposing a lower estimate for the entropy in homological terms. In the complex-analytic context, one has more homology theories to provide further speculations.

[1]) There are some in the Appendix.

§4. REAL ALGEBRAIC MAPS

We are going to show that the entropy of a real algebraic map of "algebraic degree p" cannot exceed $n \log p$, where n is the dimension. One way of approach is to complexify the whole situation, i.e. to take the Zariski closure of the graph of the map, and to apply reasoning from the previous section. This enables us to solve both problems: to introduce the notion of degree and to prove the inequality $h \leq n \log p$.

In order to avoid passing to complex numbers and to make the proof applicable to piecewise algebraic (say, piecewise linear) maps we shall now present a different argument based on Bézout's theorem.

Let us start for the sake of simplicity with a map f given by two polynomials in $\mathbf{R}^2$ of degree p. Suppose that f sends a square $S \subset \mathbf{R}^2$ into itself and try to estimate the entropy of $f\colon S \to S$. Divide S into pieces S_j of size $\leq \epsilon$ by straight lines l_i, $i = 1, 2, \ldots, r$; see Figure 1.

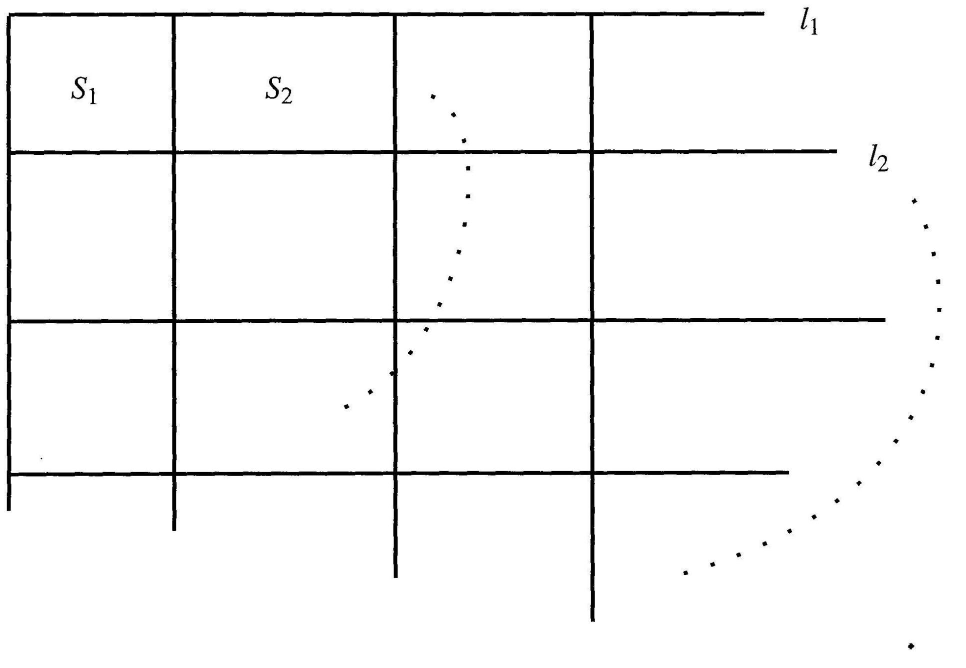

FIGURE 1

Try now to cover the iterated graph $(\Gamma_f)_k$ by products of these pieces. Observe that a product $S_{j_1} \times S_{j_2} \times \cdots \times S_{j_k} \subset S^k$ intersects $(\Gamma_f)_k$ if and only if the intersection $S_{j_1} \cap f^{-1}(S_{j_2}) \cap \cdots \cap f^{-(k-1)}(S_{j_k})$ is not empty. Now let us estimate the number N_k of all such non-empty intersections. This number is not greater than the number of components in

$$S \setminus \bigcup_{\substack{1 \leq \mu \leq k \\ 1 \leq i \leq r}} f^{-\mu}(l_i)$$

and thus N_k cannot exceed 4 times the number of all pairwise intersections of all lines $f^{-\mu}(l_i)$, provided the map $f\colon S \to S$ was injective. The system of lines $\{l_i\}$ can be represented as the zero-set of a polynomial of degree r and thus the system $\{f^{-\mu}(l_i)\}$ given by a polynomial of degree $k \cdot r \cdot p^k$; by Bézout's theorem the number of intersections is not greater than $k^2 r^2 p^{2k}$. So $N_k \leq 4k^2 r^2 p^{2k}$ and $\lim_{k\to\infty} \frac{1}{k} \log N_k \leq 2 \log p$.

In the general case (when $n > 2$, or $n = 2$ and f is not injective), there appears a complication pointed out by J. Milnor (and communicated to me by Newhouse): the components in the complement $S \setminus \bigcup f^{-\mu}(l_i)$ can contain no points of intersections of lines (or surfaces when $n > 2$). A typical 'bad' picture for $n = 3$ is shown in Figure 2.

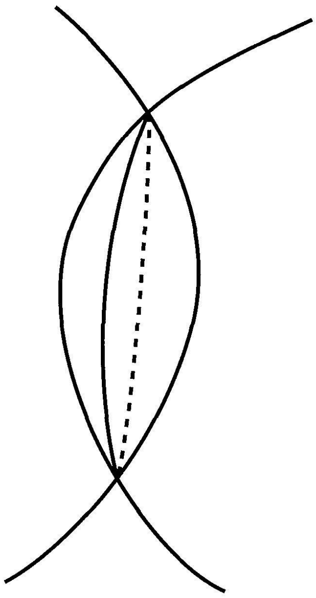

FIGURE 2

But in any case, each component must contain in its boundary a component of an algebraic variety determined by certain intersections of $f^{-\mu}(l_i)$; to estimate the number of those, Milnor suggested using his (and Thom's) theorem (see [3]):

The number of all components of the zero-set of a system of polynomials of degree D in $\mathbf{R}^n$ is not greater than $(2D)^n$.

(The actual Milnor inequality is more precise and also takes into account the Betti numbers in positive dimensions.)

It seems more natural to apply the Milnor theorem not in the space X itself, but in the product X^k, in particular when we deal with an algebraic graph. Unfortunately, the direct application leads to the weaker estimate $h \leq n \log(2p)$, due to the coefficient 2 in the Milnor theorem. But in our rather special situation, this 2 can easily be removed and we always have $h \leq n \log p$.

PERIODIC POINTS

The argument above is the same as in the Artin-Mazur theorem on periodic points [1]: the number of *isolated* periodic points of a map of degree p cannot exceed $(np)^k$, where k is the period. The number n is the dimension of the Euclidean (or projective) space in which the manifold X is realized, and p is the degree of polynomials defining the graph $\Gamma \subset X \times X \subset \mathbf{R}^{2n}$ of the map. The points of period k correspond to the intersection of $\Gamma_k \subset \mathbf{R}^{kn}$ with the preimage of the diagonal in $X \times X$ by the projection of X^k to the product of its first and last factors. Artin and Mazur make use of Bézout's theorem, which immediately yields the needed estimate. Notice that the Milnor-Thom inequality implies an analogous estimate for all Betti numbers of the sets of periodic points.

Artin and Mazur combine their estimate with the Nash theorem on approximation of a smooth map by algebraic ones (i.e. with algebraic graphs) and conclude: For a dense set of smooth maps, the number of *isolated* points k is not greater than $(\text{const})^k$. Omitting 'isolated' seems not completely trivial (though geometrically obvious) in the pure algebraic situation. (One even expects a 'generic algebraic' map to have no invariant algebraic manifolds of positive dimension, unlike the smooth case where invariant manifolds can be persistent.)

The following argument, communicated to me by Newhouse, allows one to get rid of the 'isolated'.

A map $X \to X$, $X \subset \mathbf{R}^n$, can be extended to a neighbourhood of X by a map F strongly expanding in directions normal to X. The invariant manifold X of this extension is stable under small perturbation of F and thus the general situation is reduced to the simple case of polynomial maps in $\mathbf{R}^n$.

GEOMETRIC APPROACH

The last remark undermines the role of algebraic maps in differential dynamics (but, of course, algebraic dynamics is in many respects more interesting than differential anyway) and we can go even further replacing degree by a kind of geometric complexity (in spirit of Thom) of a smooth map, making use of a quantitative form of the Thom transversality theorem instead of Bézout's theorem. The quantitative transversality can be used also for counting periodic orbits of a vector field and periodic points of a transformation preserving an additional structure (volume, symplectic form, etc.) and it yields the Artin-Mazur estimate for dense sets of such maps. Unfortunately, the detailed proof (at least the one I know of) is more lengthy than the algebraic one, and I shall treat the subject somewhere else.

REMARK. The quantitative transversality theory has been developed by Y. Yomdin (see p. 124 in [5′] for a brief introduction) but does not suffice, as it stands, for the diff-version of the Artin-Mazur theorem. In fact, one needs here an adequate notion of genericity (compare remark on p. 31 in [5′]) as is shown in [1′] for smooth maps. I have never returned to this issue and can only conjecture the extension (and sharpening) of the corresponding results in [1′] to structure preserving maps and/or vector fields.

§5. QUASICONFORMAL MAPS

For a smooth map $f\colon X \to Y$ from one oriented n-dimensional Riemannian manifold into another, we denote by $D_x f$ its differential at x, by $\|D_x f\|$ the norm of the differential, by $\det D_x f$ its Jacobian, and by $\lambda_x f$ the ratio $\|D_x f\|^n / \det D_x f$ called the *conformal dilation* at $x \in X$. A map is called λ-quasiconformal if, for almost all x, the differential $D_x f$ exists, $\det D_x f$ does not vanish, and $\lambda_x f \leq \lambda$. A quasiconformal map must have locally positive degree. If $n = 1$, each locally diffeomorphic map is conformal (i.e. 1-quasiconformal).

When $n = 2$, conformal maps are complex analytic and for $n > 2$ all conformal maps are locally diffeomorphic. In particular, when $n > 2$, every non-injective conformal endomorphism is conjugate to a homothety of a flat Riemannian manifold. When $\lambda > 1$, there are (not locally injective) λ-quasiconformal maps in all dimensions $n > 1$. They are locally homeomorphic outside a codimension 2 branching set. At that set, they are never $(n > 2)$ smooth.

GENERALISATION OF (0.1)

If f is a λ-quasiconformal endomorphism of a closed n-dimensional manifold, then

(5.0)
$$h(f) \leq \log \deg f + n \log \lambda .$$

(Compare with the standard estimate $h(f) \leq \sup_{x \in X} n \log \|D_x f\|$.)

Proof. As before, we estimate the density and volume of the iterated graph and we need an analogue of (2.0) only for the Euclidean space. The only new ingredient here is the following obvious fact:

Consider an n-dimensional $V \subset (\mathbf{R}^n)^k$ with all projections $V \to \mathbf{R}^n$ $\tilde{\lambda}$-quasiconformal and having volumes not greater than $\mu > 0$. (The volume of a map is the integral of its Jacobian.) Then

(5.1)
$$\operatorname{Vol} V \leq k^{n+1} \tilde{\lambda} \mu .$$

Combining (5.1) with the isoperimetric inequality applied to the projection $V \to \mathbf{R}^n$ realizing μ we conclude that

(5.2)
$$\operatorname{Vol} V \leq C k^{n+1} \tilde{\lambda} A^{\frac{n}{n-1}} , \qquad C = C(n) ,$$

where A denotes the $(n-1)$-dimensional volume of the boundary ∂V. (In other words, graphs of quasiconformal maps are quasiminimal.)

Now, using the same notation as in Section 2, we conclude that

(5.3)
$$V_\epsilon \leq C k^{n+1} \tilde{\lambda} A_\epsilon^{\frac{n}{n+1}}$$

and combining (5.3) with (2.2) and (2.3), we have:

(5.4)
$$V_\epsilon \geq \operatorname{const}_n K^{\text{constant}_n} \epsilon^n \tilde{\lambda}^{1-n} .$$

Combining (5.4) and (1.1) and observing that projections of the iterated graph $(\Gamma_f)_k$ of a λ-quasiconformal f are λ^k-quasiconformal we obtain:

$$h(f) \leq \operatorname{lov} \Gamma_f + (n-1) \log \lambda .$$

To complete the proof, we apply (5.1) again and get

$$\operatorname{lov} \Gamma_f \leq \log \deg f + \log \lambda .$$

§6. Mean curvature

Let X be a closed n-dimensional manifold with a Riemannian metric g. Suppose that iterated graphs $\Gamma_k \subset X^k$ are smooth of dimension n. Denote by $\mathrm{Cu}(\gamma)$, $\gamma \in \Gamma_k$, the absolute value of the mean curvature of Γ_k at γ. Set

$$\mathrm{lome}_g \Gamma = \limsup_{k\to\infty} \frac{1}{k} \log\Big(1 + \int_{\Gamma_k} [\mathrm{Cu}(\gamma)]^n d\gamma\Big) .$$

When Γ_k are minimal and $\mathrm{lome}_g = 0$ we know that $h \leq$ "lov".

More generally,

$$h(\gamma) \leq \mathrm{lov}\, \Gamma + \mathrm{lome}_g \Gamma . \tag{6.0}$$

Proof. Despite the possible dependence of "lome" upon the choice of g, we can proceed as before and reduce (6.0) to the following local estimate:

Take V in the Euclidean space $\mathbf{R}^{\ell=kn}$ and suppose its boundary does not intersect the ball $B_{2\epsilon}$ centered at $v_0 \in V$. Then

$$\epsilon^{-n} \mathrm{Vol}\, V + \int_V Cu^n(v) dv \geq C_1 \ell^{C_2} , \tag{6.1}$$

where C_1 and C_2 are constants depending only on n.

To prove (6.1) we consider the normal bundle N of V and its canonical map F into $\mathbf{R}^\ell$. The Jacobian J of F at a point $v + \nu t$ (where $v \in V$, and ν is the unit vector at v normal to V) is equal to $\prod_{i=1}^n (1 + k_i t)$, where k_i are the principal curvatures in the direction ν.

If the distance from $v+\nu t$ to V is equal to t, then $1+k_i t \geq 0$, $i = 1, \ldots, h$, and so

$$J \leq A_n\big(1 + t^n Cu^n(v)\big) . \tag{6.2}$$

Now we observe that every point of the ball B_ϵ can be joined by a shortest normal with V and so

$$C_\ell \epsilon^\ell = \mathrm{Vol}\, B_\epsilon \leq A_n C_{\ell-n} \epsilon^{\ell-n} \int_V \big(1 + \epsilon^n Cu^n(v)\big)\, dv ,$$

where C_ℓ and $C_{\ell-n}$ are volumes of unit balls in $\mathbf{R}^\ell$ and $\mathbf{R}^{\ell-n}$. The last inequality implies (6.1) and so (6.0) is proved.

The inequality (6.2) was extended by Karcher and Heinze to general Riemannian manifolds. Discussions with Karcher about such inequalities influenced my reasoning in this section.

APPENDIX: EXAMPLES OF HOLOMORPHIC ENDOMORPHISMS

The following examples appeared after the discussion I had with Spencer Bloch and David Mumford.

(1) TWISTED HOPF MANIFOLDS

Divide $\mathbf{C}^n \setminus 0$ by the action of a linear operator A without eigenvalues in the unit disk. All endomorphisms of the quotient $(n > 1)$ come from polynomial maps $\widetilde{f}\colon \mathbf{C}^n \to \mathbf{C}^n$. For such an endomorphism, its entropy and "lov" are probably equal to "log deg".

EXAMPLE. $A\colon (z_1, z_2) \longmapsto (\lambda_1 z_1, \lambda_2 z_2)$ and $\widetilde{f}\colon (z_1, z_2) \longmapsto (z_1^p, z_2^p)$.

(2) GENERALIZED HOPF MANIFOLDS

Let $f_0\colon X_0 \to X_0$ be an endomorphism. Take a line bundle L over X_0 such that $f_0^*(L) = \bigotimes^p L$ $(:= L \otimes \cdots \otimes L,\ \ p$ times$)$. Locating such an L is usually quite easy by looking at $\mathrm{Pic}(X_0)$. Denote by Y the total space of L. There is a fiberwise map $\widetilde{f}\colon Y \to Y$ lifting f_0 and acting on fibers as $z \longmapsto z^p$. If we divide Y by a fiberwise action of $\mathbf{Z}$ (it is $z \longmapsto z_0 z$, $z_0 \neq 0$, in each fiber) we get $f\colon Y/\mathbf{Z} \to Y/\mathbf{Z}$.

There is another way to compactify Y by taking the total space of the one-dimensional projective bundle associated to L. The endomorphism $\widetilde{f}$ canonically extends to this compactification.

(3) THE CALABI-ECKMANN MANIFOLDS

Let us take $\left(\mathbf{C}^k \times \mathbf{C}^\ell\right) \setminus \left((\mathbf{C}^k \times 0) \cup (0 \times \mathbf{C}^\ell)\right)$ and divide by the following action of $\mathbf{C}$:

$$(z_1, z_2) \longmapsto \left(A_1^\lambda z_1, A_2^\lambda z_2\right), \qquad \lambda \in \mathbf{C}.$$

A_1 and A_2 are appropriate linear operators in $\mathbf{C}^k$ and $\mathbf{C}^\ell$. For example, $A_1^\lambda = \exp \lambda$, $A_2^\lambda = \exp i\lambda$, where λ is a scalar. In the last case, the factor manifold possesses an endomorphism f which lifts to the following polynomial map

$$\mathbf{C}^k \times \mathbf{C}^\ell \to \mathbf{C}^k \times \mathbf{C}^\ell \ :\ (z_1, \ldots, z_{k+\ell}) \longmapsto (z_1^p, \ldots, z_{k+\ell}^p).$$

Recall that the Calabi-Eckmann manifolds are diffeomorphic to $\mathbf{S}^{2k-1} \times \mathbf{S}^{2\ell-1}$. The above map f has degree $d = (2(k + \ell - 1))^p$ and $h(f) = \mathrm{lov}\, f = \log d$.

(4) BLOWING UP

Let us take $W \subset V_0$ and an endomorphism $f\colon V_0 \to V_0$ such that $f^{-1}(W) = W$. The endomorphism f can be sometimes lifted to the manifold V obtained by blowing up W.

EXAMPLE. $V_0 = \mathbf{C}P^1 \times \mathbf{C}P^1$, W is the single point $(0,0)$, and $f\colon (z_1, z_2) \mapsto (z_1^p, z_2^p)$.

(5) CONCLUDING REMARKS

A typical compact complex manifold has very few endomorphisms. For example, manifolds with nontrivial Kobayashi volume have no endomorphisms of degree ≥ 2. Do Grassmann manifolds have such endomorphisms? (No, see [3′].)

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NOTE DES ÉDITEURS

L'article qui précède a été rédigé en 1976 et a circulé sous la forme d'une prépublication de SUNY, dès 1977. Nous l'avons imprimé ici en l'état, à l'exception de la dernière remarque du §4, de l'adjonction de trois références et de la correction d'un petit nombre de fautes de frappe. Serge Cantat, que nous remercions, a bien voulu rédiger à cette occasion un court texte pour orienter le lecteur vers quelques-uns des nombreux travaux influencés par l'article de Mikhaïl Gromov.

NOTES SUR L'ARTICLE DE M. GROMOV

par Serge CANTAT

0.1. L'article de M. Gromov a considérablement influencé les travaux sur la conjecture de Shub (reliant entropie topologique et action sur l'homologie) ainsi que l'étude des systèmes dynamiques holomorphes, notamment en ce qui concerne la dynamique à plusieurs variables.

Le texte [14] propose une preuve alternative des résultats principaux obtenus par M. Gromov. Ceux de Shmuel Friedland ([6], [7] et [8]) proposent diverses extensions de ces résultats au cadre des transformations méromorphes des variétés kählériennes (voir [5]).

Les deux pages qui suivent ne concernent que la dynamique holomorphe. Il convient toutefois de noter les articles suivants, qui sont liés à d'autres aspects de l'article de Gromov et contiennent de nombreuses références: [15] et plus généralement l'ensemble des travaux de S. Newhouse sur le sujet, ainsi que [1], qui s'inscrit dans la lignée des travaux d'Artin et Mazur, de Gromov et de Yomdin.

0.2. Bien souvent, on couple les résultats obtenus par M. Gromov à ceux de Y. Yomdin (voir [10], [19]) et de S. Newhouse (voir [15]). Le théorème suivant est un exemple typique.