# 3. Metrics of specified curvature

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THEOREM 2.5. *Let $\varphi\colon I \to \mathbf{R}$ be a momentum profile. There exists an abstract surface of revolution $(D, g_\varphi)$, unique up to isometry, such that*

- *The image of the moment map $\tau\colon D \to \mathbf{R}$ is $I$.*
- *The orbit $\{\tau = \tau_0\}$ has length $2\pi\sqrt{\varphi(\tau_0)}$ for all $\tau_0 \in I$.*

*The Gaussian curvature of $g_\varphi$ is $K = -\frac{1}{2}\varphi''(\tau)$ wherever the metric is smooth, and the angular defect at a fixed point is $(2 - |\varphi'|)\pi$. The metric is complete at an end $\{\tau = \beta\}$ if and only if one of the following holds:*

(INFINITE-AREA END) $|\beta| = \infty$ *and* $\displaystyle\int_{\tau_0}^{\beta} \frac{dx}{\sqrt{\varphi(x)}}$ *diverges.*

(SMOOTH EXTENSION) $\beta$ *is finite,* $\varphi(\beta) = 0$, *and* $|\varphi'(\beta)| = 2$.

(FINITE-AREA END) $\beta$ *is finite,* $\varphi(\beta) = 0$, *and* $\varphi'(\beta) = 0$.

## 3. METRICS OF SPECIFIED CURVATURE

In momentum coordinates, specifying the Gaussian curvature of a metric in terms of zonal area is a matter of integrating twice. The construction is therefore well-adapted to exhibiting a variety of interesting metrics.

### CONSTANT CURVATURE

Theorem 2.5 and Proposition 2.3 give a simple classification of surfaces of revolution that have constant Gaussian curvature, together with an easy characterization of when the abstract surface embeds in $\mathbf{R}^3$ as a surface of revolution. Many surfaces of constant negative curvature (such as the pseudosphere) are seen to be portions of *complete* abstract surfaces of revolution.

SMOOTH, COMPLETE METRICS. A metric of constant Gaussian curvature corresponds to a quadratic profile $\varphi$, and the metric is smooth and complete if and only if

- $\varphi \geq 0$ on $\mathbf{R}$, or
- $|\varphi'(\beta)| = 2$ at some (hence each) root of $\varphi$.

Table 3.1 lists smooth, complete surfaces of revolution that have constant Gaussian curvature. Most of these metrics embed only partially in $\mathbf{R}^3$ as surfaces of revolution, and *no* zone of the Poincaré metric (on the disk $\Delta$) embeds as a surface of revolution. The pseudosphere is the zone in the punctured disk $\Delta^\times$ corresponding to the momentum interval $(0, 1/c^2)$. In the last column, the annulus is determined up to conformal equivalence by the ratio of the inner and outer radii. Each metric is scaled to have curvature $\pm c^2$ or $0$, metrics are grouped by the sign of their curvature, and the momentum profiles are translated to have $\alpha = 0$ when possible. For each integrand in Table 3.1, the integrals in equation (2.6) are elementary, and the $t$ integrals can be inverted explicitly.

TABLE 3.1
Smooth, complete, constant-curvature surfaces of revolution

| | $\mathbf{P}^1$ | $\mathbf{C}$ | $\mathbf{C}^\times$ | $\Delta$ | $\Delta^\times$ | Annulus |
|---|---|---|---|---|---|---|
| $\varphi(\tau)$ | $2\tau - c^2\tau^2$ | $2\tau$ | $\lambda^2$ | $2\tau + c^2\tau^2$ | $c^2\tau^2$ | $\lambda^2 + c^2\tau^2$ |
| $\tau \in$ | $[0, 2c^{-2}]$ | $[0, \infty)$ | $\mathbf{R}$ | $[0, \infty)$ | $(0, \infty)$ | $(-\infty, \infty)$ |
| $t \in$ | $[-\infty, \infty]$ | $[-\infty, \infty)$ | $\mathbf{R}$ | $[-\infty, 0)$ | $(-\infty, 0)$ | $\left(-\frac{\pi}{c\lambda}, \frac{\pi}{c\lambda}\right)$ |

OTHER EXAMPLES. Every quadratic polynomial that is positive somewhere gives rise to a surface of constant Gaussian curvature via the momentum construction, though with exactly one exception (the round sphere) the resulting metric is singular and/or not embedable in $\mathbf{R}^3$. There is a pleasant correspondence between quadratic profiles and the "standard zoo" presented in elementary differential geometry. Figures 3.1 and 3.2 depict the negative curvature case; positive curvature is similar.

A remarkable family arises from quadratic profiles $\varphi(\tau) = \lambda^2 - \tau^2$ with $\lambda > 1$. At their smooth points these metrics have unit curvature, yet they admit closed geodesics of length $2\pi\lambda > 2\pi$, in seeming contradiction with Myers' theorem. The discrepancy is resolved by (2.11): The conical singularities at the fixed points carry negative curvature. Viewing these examples as surfaces in $\mathbf{R}^3$, the explanation is different: The portion that embeds is not complete.

AREA, DISTANCE, AND ORBIT LENGTH. An abstract surface of revolution is said to have *bounded orbits* if the profile is a bounded function. If a surface has bounded orbits, and if an end of the surface has finite length, then the end also has finite area. Inversely, if the orbits are not bounded at an end of

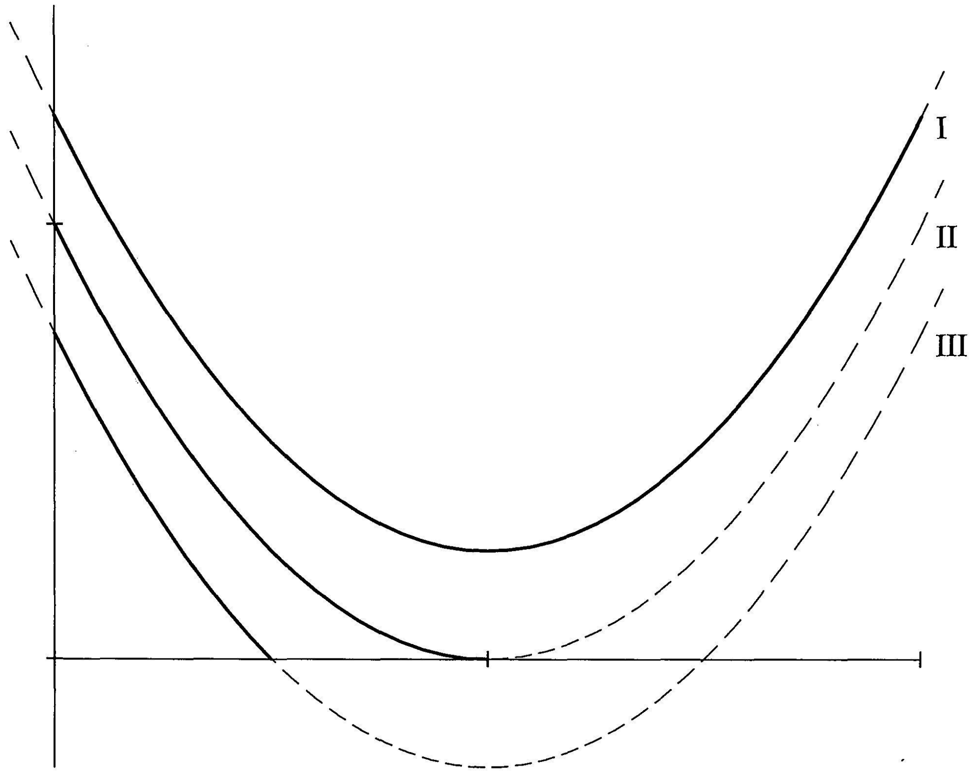


FIGURE 3.1
Quadratic momentum profiles; the heavy portion of each defines a metric that embeds in $\mathbf{R}^3$ as a surface of revolution, see Figure 3.2

infinite length, then the end necessarily has infinite area. Remarkably, these are the only general conclusions that can be drawn about abstract surfaces of revolution; the intuition furnished by surfaces of revolution in $\mathbf{R}^3$ can be misleading! Two examples illustrate what can happen:

- The data $I = [0, \infty)$, $\varphi(\tau) = 2\tau + \tau^3$ define a metric of infinite area on the disk, in which the distance to the edge of the disk is finite.
- The data $I = [0, 1)$, $\varphi(\tau) = 2\tau/(1 - \tau)$ give a metric on the disk with unbounded orbits but having finite area.

Neither metric is complete, and neither can be extended non-trivially.

## EXTREMAL METRICS

The *Calabi energy* of a metric $g$ is the integral of the square of the Gaussian curvature,

$$E(g) = \int_\Sigma K^2 \, dA\,.$$

A metric is *extremal* in the sense of Calabi if the metric is critical for the energy among all smooth metrics of fixed area.

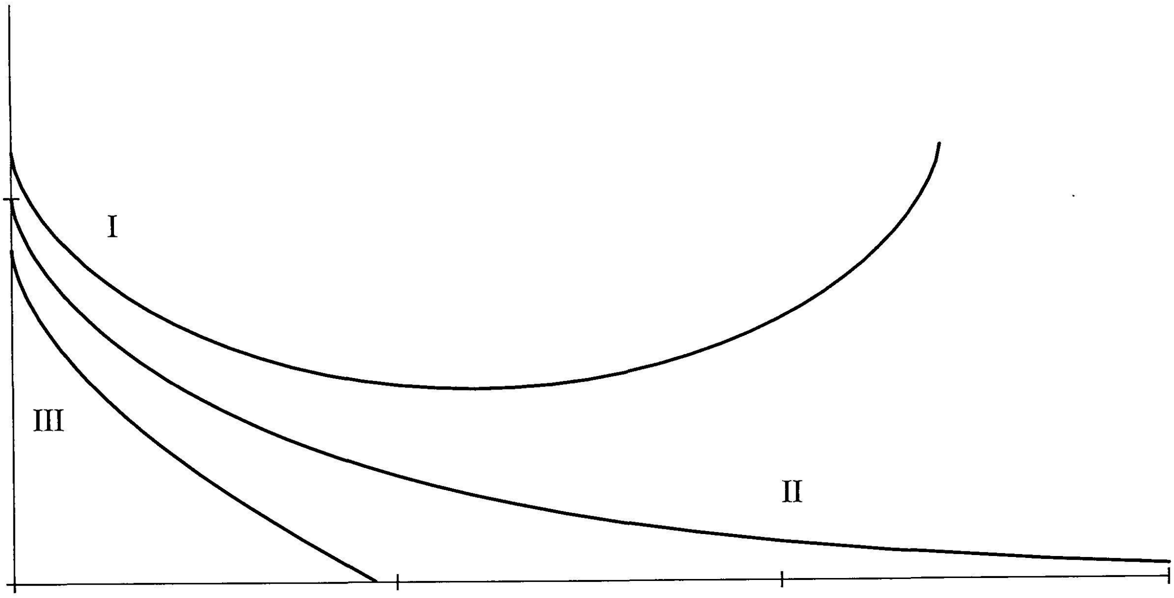


FIGURE 3.2
The geometric profiles that correspond to the momentum profiles in Figure 3.1. Each generates a surface of constant curvature $-1$

THE CALABI ENERGY FOR METRICS WITH CONICAL SINGULARITIES. The space of surfaces of revolution of area $4\pi$ is identified with the space of momentum profiles $\varphi\colon [-1,1] \to \mathbf{R}$. Assume from now on that profiles are of class $C^4$ and vanish at $\pm 1$. The Gaussian curvature is $K = -\frac{1}{2}\varphi''(\tau)$ for $\tau \in (-1,1)$, while the curvature form is the distribution

$$K\,dA = \pi\Big[\big(2+\varphi'(1)\big)\delta_{(1)} + \big(2-\varphi'(-1)\big)\delta_{(-1)}\Big] - \tfrac{1}{2}\varphi''(\tau)\,dA\,.$$

The Calabi energy is the integral of $K^2\,dA$ over $\Sigma$:

$$(3.1)\quad E(g_\varphi) = \frac{\pi}{2}\left[-\varphi''(1)\big(2+\varphi'(1)\big) - \varphi''(-1)\big(2-\varphi'(-1)\big) + \int_{-1}^{1}(\varphi'')^2\right].$$

Differentiating with respect to $\varphi$ and integrating by parts twice yields

$$(3.2)\quad \frac{2}{\pi}\dot{E}(g_\varphi) = -2\big(\dot{\varphi}''(1)+\dot{\varphi}''(-1)\big) + (\varphi''\dot{\varphi}' - \varphi'\dot{\varphi}'')\Big|_{-1}^{1} + 2\int_{-1}^{1}\varphi^{(4)}\dot{\varphi}\,.$$

For variations supported in $(-1,1)$, the boundary term contributes nothing. Consequently, $g_\varphi$ is extremal *only if* $\varphi$ is a cubic polynomial.

REMARK 3.1. The Euler-Lagrange equation (due to Calabi [2]) for a smooth extremal metric on a compact holomorphic manifold is simple and striking: The scalar curvature of the metric is a *holomorphy potential* — a function whose gradient is a holomorphic vector field. To see how this condition is related to $\varphi$ being cubic, observe first that $\varphi$ is cubic if and only if the Gaussian curvature (a.k.a. the scalar curvature, since $\Sigma$ is a complex curve) is an affine function of $\tau$. But the complex gradient of $\tau$ is the holomorphic vector field $w\frac{\partial}{\partial w}$. Conversely, affine functions of $\tau$ are the *only* $S^1$-invariant functions with holomorphic gradient, for if $f(\tau)$ is a holomorphy potential on $\mathbf{P}^1$, then $f'(\tau)$ is a global holomorphic function, hence constant.

Calabi [2] showed that the round metric is the only *smooth* extremal metric of area $4\pi$ on the sphere. This fact is easily recovered for surfaces of revolution. Smoothness at the ends of the momentum interval means $\varphi'(\pm 1) = \mp 2$. If the profile is not quadratic, there exists $\beta > 1$ such that $\varphi(\tau) = c(1-\tau^2)(\beta - \tau)$. This implies $|\varphi'(-1)| \neq |\varphi'(1)|$, so the metric is not smooth.

Equation (3.2) contains an additional condition for extremality,

$$-2\big(\dot{\varphi}''(1) + \dot{\varphi}''(-1)\big) + (\varphi''\dot{\varphi}' - \varphi'\dot{\varphi}'')\Big|_{-1}^{1} = 0 \quad \text{for all } \dot{\varphi}, \tag{3.3}$$

which says that the energy supported at the fixed points is constant infinitesimally, a condition on the domain of the energy functional as much as a restriction on $\varphi$. Without some constraint on the space of metrics, (3.3) is not satisfied, *even for the round metric*. Indeed, the family of profiles $\varphi(\tau) = c(1-\tau^2)$, with $c > 0$, determines a family of metrics for which the curvature concentrates at the fixed points as $c \to 0$, and the energy does not achieve its infimum.

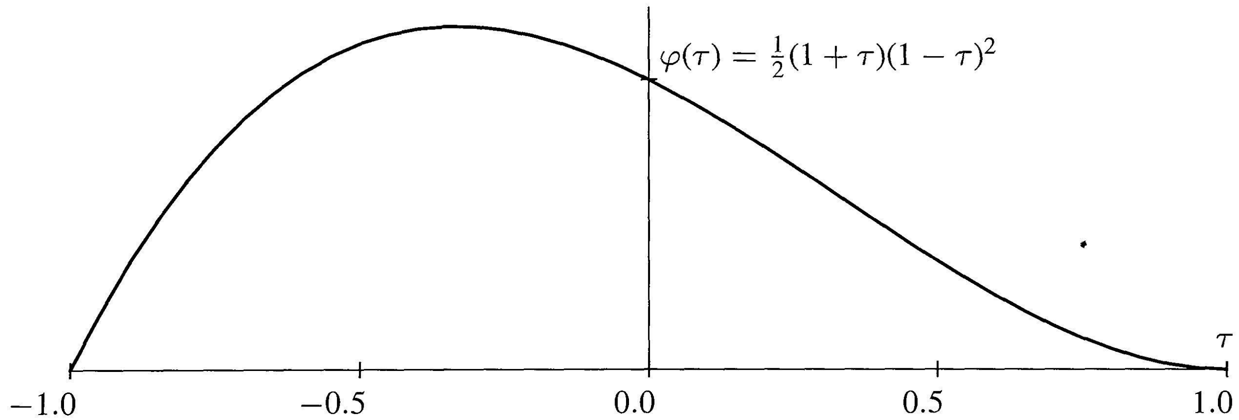


FIGURE 3.3

A cubic momentum profile defining a smooth metric

Two natural constraints on the variation are:

- The energy carried by each end is fixed, so (3.3) holds by fiat.
- The cone angles are fixed, i.e., $\dot{\varphi}'$ vanishes at the endpoints.

When the energy carried by each end is fixed, every metric of constant curvature is critical. In addition, every cubic polynomial

$$\varphi(\tau) = c(1+\tau)(1-\tau)(\beta-\tau)$$

with $\beta \geq 1$ and $c > 0$ gives rise to a critical metric. Among these is a unique *smooth* metric, corresponding to the profile $\varphi(\tau) = \frac{1}{2}(1+\tau)(1-\tau)^2$. The metric $g_\varphi$ is defined on $\mathbf{C}$ and has area $4\pi$. Because $|\varphi'(\tau)| \leq 2$ for $-1 \leq \tau < 1$ (with equality if and only if $\tau = -1$), the surface $(\mathbf{C}, g_\varphi)$ embeds isometrically in $\mathbf{R}^3$, as a "teardrop" of radius $\frac{4}{3\sqrt{3}}$ and with an infinitely long tail, Figure 3.4.

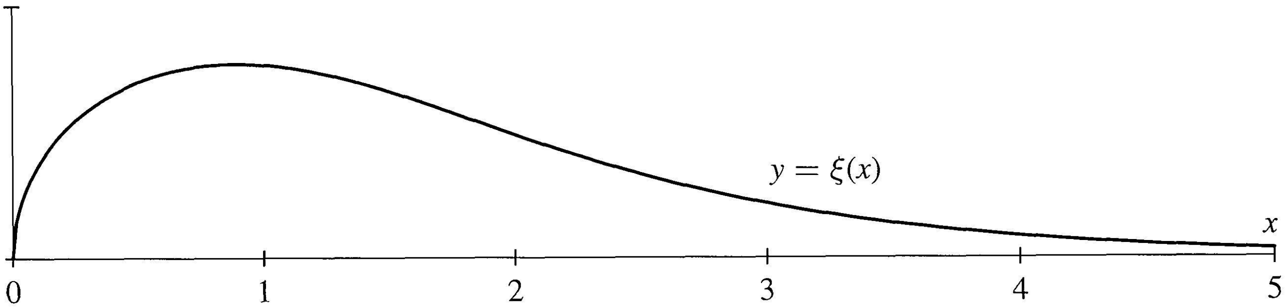


FIGURE 3.4
The embedded geometric profile

Under the weaker restriction that the variation fixes cone angles, the round metric is critical, but is the *only* such metric. Indeed, (3.3) becomes

$$-2\big(\dot\varphi''(1) + \dot\varphi''(-1)\big) - (\varphi'\dot\varphi'')\Big|_{-1}^{1} = 0 \quad \text{for all } \dot\varphi\,.$$

Since $\varphi$ is cubic, there exist constants $a_i$ such that $\varphi'(\tau) = a_1 + 2a_2\tau + 3a_3\tau^2$. The metric closes up at both ends, so $\int_{-1}^{1}\varphi' = 0$, or $a_1 + a_3 = 0$. A short calculation shows that

$$(a_3 + a_2 + 1)\dot\varphi''(1) + (-a_3 + a_2 + 1)\dot\varphi''(-1) = 0 \quad \text{for all } \dot\varphi\,.$$

Consequently, $a_3 = a_2 + 1 = 0$; this means $\varphi$ is a quadratic polynomial with leading coefficient $-1$ that vanishes at $\pm 1$, so $g_\varphi$ is the round metric.

## HISTORY AND ACKNOWLEDGEMENTS

Though the momentum construction arises naturally in symplectic geometry, the author first encountered versions of it in the differential geometry literature. An instance of the integral transform (2.6) appears in a remark of Calabi [1]. The construction as treated in this note perhaps owes its biggest debt to a paper of Koiso and Sakane [6], in which momentum coordinates are used to construct positive Einstein-Kähler metrics. The paper [4] is in part an attempt to frame various differential-geometric constructions in "momentum" language, while simultaneously unifying and generalizing existing results. The momentum construction for surfaces of revolution is elementary, but seems not to be widely appreciated. It is hoped that the present note will help popularize this little gem of differential geometry.

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