

2. ISOFOLDS AND ISOFANS

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of even lattices in $\mathbf{R}^n$ with root system R and $\Gamma(R)$ -orbits of subgroups H in $G(R)$ with $\mathbf{n}(h) \in 2\mathbf{Z} \setminus \{2\}$ for all $h \in H$. Unimodular lattices correspond to isotropic subgroups.

2. ISOFOLDS AND ISOFANS

Given any root system R , we want to determine whether or not a complete even unimodular lattice Λ exists such that $\Lambda_{\text{rt}} = R$. This is equivalent to determining whether or not $(G(R), b_R)$ has an admissible isotropic subgroup. Suppose R' is another root system such that the bilinear form modules $(G(R'), b_{R'})$, $(G(R), b_R)$ are isomorphic. Let φ denote such an isomorphism. As φ is a bilinear form module isomorphism, $b_{R'}(g'_1, g'_2) = b_R(\varphi(g'_1), \varphi(g'_2))$ for all $g'_1, g'_2 \in G(R')$. Recall that the bilinear forms have values in $\mathbf{Q}/\mathbf{Z}$, so that

$$\mathbf{n}(g') \equiv \mathbf{n}(\varphi(g')) \pmod{\mathbf{Z}} \text{ for all } g' \in G(R').$$

If $(G(R'), b_{R'})$ has an isotropic subgroup H' , it may be possible to use H' to construct an admissible isotropic subgroup H for $(G(R), b_R)$.

DEFINITION. In the notation above, let

$$\varphi: (G(R'), b_{R'}) \rightarrow (G(R), b_R)$$

be an isomorphism of bilinear form modules, where $\text{rk } R' < \text{rk } R$. The isomorphism φ is called an *isofan* if

$$\begin{aligned} \mathbf{n}(g') &\equiv \mathbf{n}(\varphi(g')) \pmod{2\mathbf{Z}}, \\ \mathbf{n}(g') &\leq \mathbf{n}(\varphi(g')) \end{aligned}$$

for all $g' \in G(R')$. The inverse φ^{-1} of the isofan φ is called an *isofold*.

EXAMPLE 1. The simplest example of an isofan was given by Venkov [V]. Consider the root system D_k , $k \geq 2$, where D_2 is identified with $2A_1$. Recall that an admissible representative system for $(G(D_k), b_{D_k})$ can be given by $d_{k,0}, d_{k,1}, d_{k,2}, d_{k,3}$, the norms of the representatives being $0, \frac{k}{4}, 1, \frac{k}{4}$, respectively. Thus, for any integer k_1 satisfying $k_1 \equiv k \pmod{8}$, the norms of $d_{k_1,i}$ and $d_{k,i}$ differ by an integral multiple of 2 for $0 \leq i \leq 3$.

Let φ_{D_k} be the group isomorphism given by

$$\varphi_{D_k}: G(D_k) \rightarrow G(D_{k+8}); \quad d_{k,i} \mapsto d_{k+8,i} \quad (0 \leq i \leq 3).$$

This isomorphism preserves the bilinear form in the prescribed manner

$$b_{D_k}(d_{k,i}, d_{k,j}) = b_{D_{k+8}}(\varphi_{D_k}(d_{k,i}), \varphi_{D_k}(d_{k,j})) \quad (0 \leq i, j \leq 3),$$

so in fact it is an isomorphism of the bilinear form modules. It also preserves norms modulo $2\mathbf{Z}$, as noted above. Moreover, $\mathbf{n}(d_{k,i}) \leq \mathbf{n}(\varphi_{D_k}(d_{k,i}))$. Thus φ_{D_k} is an isofan and $\varphi_{D_k}^{-1}$ an isofold.

It is well known that $R' := D_{16}$ is the root system of a complete even unimodular lattice [W2]. An admissible isotropic subgroup for $G(D_{16})$ is given by $H' = \{d_{16,0}, d_{16,1}\}$. Form the subgroup $H := \varphi_{D_{16}}(H') = \{d_{24,0}, d_{24,1}\}$. The map φ preserves the orthogonality relations and the norms modulo $2\mathbf{Z}$, whereby the norms may not decrease under the mapping. Since the group structures are also isomorphic, H is an admissible isotropic subgroup of $G(D_{24})$. Consequently, D_{24} is the root system of a complete even unimodular lattice. By induction, we get a family of complete even unimodular lattices; namely, D_{16+8i} is the root system of the complete even unimodular lattice generated over $\mathbf{Z}$ by D_{16+8i} and the vector $d_{16+8i,1} = \frac{1}{2} \sum_{j=1}^{16+8i} e_j \in \mathbf{R}^{16+8i}$ for $i \in \mathbf{Z}$, $i \geq 0$.

EXAMPLE 2. To find all isometry classes of even unimodular lattices for the root system $E_7 + D_4 + 21A_1$, we will use an application of the fanning method. This root system appears in work of Conway and Pless [CP]; however, they provide no indication as to how an admissible isotropic subgroup, or self-dual doubly-even code, was found for $G(E_7 + D_4 + 21A_1)$.

Begin with the isofold

$$\eta: G(E_7 + D_4) \rightarrow 3G(A_1)$$

$$e_{7,1} \mapsto a^1 + a^2 + a^3; \quad d_{4,1} \mapsto a^1 + a^2; \quad d_{4,3} \mapsto a^2 + a^3,$$

where a^i refers to $a_{1,1}$ in the i th copy of $G(A_1)$ in $3G(A_1)$. Next, extend η to all of $G(E_7 + D_4 + 21A_1)$ by letting it act on $21G(A_1)$ as $\eta(a^i) = a^{i+3}$, $0 \leq i \leq 21$. Then $\eta: G(E_7 + D_4 + 21A_1) \rightarrow 24G(A_1)$ is an isofold. In order to construct an admissible isotropic subgroup for $G(E_7 + D_4 + 21A_1)$, we will apply isofans to isotropic subgroups of $24G(A_1)$.

It is well known that $24A_1$ is the root system of an even unimodular lattice [N]. The only admissible isotropic subgroup, up to equivalence, for its discriminant group can be identified with the self-dual doubly-even binary code of length 24 known as the Golay code. Letting $a^i = a_{1,1}^i$, this isotropic subgroup H' is generated (up to equivalence) by

$$\begin{aligned}
h'_1 &= a^1 + a^2 + a^3 + a^4 \\
&\quad + a^5 + a^6 + a^7 + a^8 \\
h'_2 &= a^1 + a^2 + a^3 + a^4 \\
&\quad + a^9 + a^{10} + a^{11} + a^{12} \\
h'_3 &= a^1 + a^2 + a^3 + a^4 \\
&\quad + a^{13} + a^{14} + a^{15} + a^{16} \\
h'_4 &= a^1 + a^2 + a^3 + a^4 \\
&\quad + a^{17} + a^{18} + a^{19} + a^{20} \\
h'_5 &= a^1 + a^2 + a^3 + a^4 \\
&\quad + a^{21} + a^{22} + a^{23} + a^{24} \\
h'_6 &= a^1 + a^2 + a^3 + a^5 \\
&\quad + a^9 + a^{13} + a^{17} + a^{21} \\
h'_7 &= a^1 + a^2 + a^3 + a^6 \\
&\quad + a^9 + a^{14} + a^{18} + a^{22} \\
h'_8 &= a^1 + a^2 + a^3 + a^7 \\
&\quad + a^9 + a^{15} + a^{19} + a^{23} \\
h'_9 &= a^1 + a^2 + a^3 + a^5 \\
&\quad + a^{10} + a^{14} + a^{19} + a^{24} \\
h'_{10} &= a^1 + a^2 + a^3 + a^5 \\
&\quad + a^{11} + a^{15} + a^{20} + a^{22} \\
h'_{11} &= a^2 + a^3 + a^4 + a^5 \\
&\quad + a^9 + a^{14} + a^{20} + a^{23} \\
h'_{12} &= a^1 + a^2 + a^4 + a^5 \\
&\quad + a^9 + a^{15} + a^{18} + a^{24}
\end{aligned}$$

(see, for example, [Ko]). Applying the isofan

$$\varphi = \eta^{-1}: 24G(A_1) \rightarrow G(E_7 + D_4 + 21A_1),$$

obtained from the extended isofold defined above, to the generators of H' yields generators for an admissible isotropic subgroup H :

$$\begin{aligned}
h'_1 &= a^1 + a^2 + a^3 + a^4 \\
&\quad + a^5 + a^6 + a^7 + a^8 \\
h'_2 &= a^1 + a^2 + a^3 + a^4 \\
&\quad + a^9 + a^{10} + a^{11} + a^{12} \\
h'_3 &= a^1 + a^2 + a^3 + a^4 \\
&\quad + a^{13} + a^{14} + a^{15} + a^{16} \\
h'_4 &= a^1 + a^2 + a^3 + a^4 \\
&\quad + a^{17} + a^{18} + a^{19} + a^{20} \\
h'_5 &= a^1 + a^2 + a^3 + a^4 \\
&\quad + a^{21} + e_{7,1} \\
h'_6 &= a^1 + a^2 + a^3 + a^5 \\
&\quad + a^9 + a^{13} + a^{17} + a^{21} \\
h'_7 &= a^1 + a^2 + a^3 + a^6 \\
&\quad + a^9 + a^{14} + a^{18} + e_{7,1} + d_{4,2} \\
h'_8 &= a^1 + a^2 + a^3 + a^7 \\
&\quad + a^9 + a^{15} + a^{19} + e_{7,1} + d_{4,3} \\
h'_9 &= a^1 + a^2 + a^3 + a^5 \\
&\quad + a^{10} + a^{14} + a^{19} + e_{7,1} + d_{4,1} \\
h'_{10} &= a^1 + a^2 + a^3 + a^5 \\
&\quad + a^{11} + a^{15} + a^{20} + e_{7,1} + d_{4,2} \\
h'_{11} &= a^2 + a^3 + a^4 + a^5 \\
&\quad + a^9 + a^{14} + a^{20} + e_{7,1} + d_{4,3} \\
h'_{12} &= a^1 + a^2 + a^4 + a^5 \\
&\quad + a^9 + a^{15} + a^{18} + e_{7,1} + d_{4,1}
\end{aligned}$$

This isotropic subgroup represents the only $\Gamma(21A_1 + E_7 + D_4)$ -orbit of subgroups that correspond to even unimodular lattices. If there were another such orbit, there would be an admissible isotropic subgroup $K \subset G(21A_1 + E_7 + D_4)$ not in the orbit of H . This means that $\eta(K)$ is an isotropic subgroup of $24G(A_1)$ in a different orbit than that of H' . Therefore, $\eta(K)$ is inadmissible, meaning that new roots have been created. The resulting root system, however, must still have at least 12 summands of A_1 , otherwise some roots of $\eta(K)$ must come from roots in K . Also, the rank of the resulting root system must be 24. The only root system of an even unimodular lattice satisfying these two conditions is $24A_1$.

EXAMPLE 3. This example demonstrates that inequivalent even unimodular lattices can share the same root system; in this case, $4D_8$. Consider the isofold

$$\eta := \eta_{G(4D_8)}: 4G(D_8) \rightarrow 2G(D_4) + 2G(D_8)$$

$$d_{8,j}^1 \mapsto d_{4,j}^1 + d_{4,2}^2, \quad d_{8,j}^2 \mapsto d_{4,2}^1 + d_{4,j}^2, \quad d_{8,j}^3 \mapsto d_{8,j}^1, \quad d_{8,j}^4 \mapsto d_{8,j}^2, \quad j \in \{1, 3\}.$$

There are no even unimodular lattices with root system $2D_4 + 2D_8$ [N]. If $4G(D_8)$ has an admissible isotropic subgroup H , $\eta(H)$ must then be an isotropic subgroup of $G(2D_4 + 2D_8)$ containing at least one element r of norm 2. Since $\mathbf{n}(\eta^{-1}(r)) \geq 4$, the possibilities for r are

$$d_{4,j}^i + d_{8,2}^k, \quad d_{4,j}^1 + d_{4,\ell}^2, \quad i, k \in \{1, 2\}, \quad j, \ell \in \{1, 3\}.$$

The root system has now been changed and must be determined. If a root of the first type occurs, then D_4 joins with D_8 to give D_{12} . Since $D_{12} + D_4 + D_8$ is not the root system of a complete even unimodular lattice, we appropriately introduce another root of the first type, resulting in $2D_{12}$, which indeed is the root system of a complete even unimodular lattice. If a root of the second type is introduced, the two D_4 combine to a D_8 , so that the new root system is $3D_8$. Each of these root systems, $2D_{12}$ and $3D_8$, has a unique isometry class of even unimodular lattices.

Assume first that two roots of the first type are present. Without loss of generality, these roots may be taken to be $d_{4,1}^1 + d_{8,2}^1$ and $d_{4,1}^2 + d_{8,2}^2$. There is only one orbit of admissible isotropic subgroups of $2G(D_{12})$. One representative of this orbit is generated by $d_{12,1}^1 + d_{12,2}^2$, $d_{12,2}^1 + d_{12,1}^2$. From this, we will create an inadmissible isotropic subgroup of $G(2D_4 + 2D_8)$. First, rewrite the generators of the isotropic subgroup in terms of $G(D_4 + D_8 + D_4 + D_8)$, making sure that orthogonality relations between all elements are preserved: $d_{12,1}^1 + d_{12,2}^2$ may either be $d_{4,2}^1 + d_{8,1}^1 + d_{8,2}^2$ or $d_{4,3}^2 + d_{8,1}^1 + d_{8,2}^2$, and $d_{12,2}^1 + d_{12,1}^2$ may be either $d_{4,2}^2 + d_{8,2}^1 + d_{8,1}^2$ or $d_{4,3}^1 + d_{8,2}^1 + d_{8,1}^2$. For example,

using the first choices, generators for an inadmissible isotropic subgroup of $G(2D_4 + 2D_8)$ are

$$d_{4,2}^1 + d_{8,1}^1 + d_{8,2}^2, \quad d_{4,2}^2 + d_{8,2}^1 + d_{8,1}^2, \quad d_{4,1}^1 + d_{8,2}^1, \quad d_{4,1}^2 + d_{8,2}^2.$$

Now fan these generators using η^{-1} to get an admissible isotropic subgroup of $G(4D_8)$:

$$d_{8,2}^1 + d_{8,1}^3 + d_{8,2}^4, \quad d_{8,2}^2 + d_{8,2}^3 + d_{8,1}^4, \quad d_{8,1}^1 + d_{8,2}^2 + d_{8,2}^3, \quad d_{8,2}^1 + d_{8,1}^2 + d_{8,2}^4.$$

Had we used any other choices given above, we would have obtained an equivalent isotropic subgroup. Note that this isotropic subgroup has one word of norm 8.

In a similar fashion, take the generators of a representative of the only orbit of admissible isotropic subgroups of $3G(D_8)$:

$$d_{8,2}^1 + d_{8,2}^2 + d_{8,3}^3, \quad d_{8,2}^1 + d_{8,3}^2 + d_{8,2}^3, \quad d_{8,3}^1 + d_{8,2}^2 + d_{8,2}^3.$$

We shall now break apart the third copy of $G(D_8)$ into $2G(D_4)$ by introducing the root $d_{4,1}^1 + d_{4,1}^2$. The next step is to rewrite $d_{8,2}^3$ and $d_{8,3}^3$ in terms of $2G(D_4)$. Since the results will have to be orthogonal to the root, this narrows down the choices considerably. Indeed, $d_{8,2}^3$ will have to be $d_{4,1}^1$ (which is equivalent to $d_{4,1}^2$), whereas, up to equivalence, $d_{8,3}^3$ can be either $d_{4,3}^1 + d_{4,3}^2$ or $d_{4,2}^1 + d_{4,3}^2$. Using the first choice, form the generators for an inadmissible isotropic subgroup

$$d_{4,1}^1 + d_{4,1}^2, \quad d_{4,3}^1 + d_{4,3}^2 + d_{8,2}^1 + d_{8,2}^2, \quad d_{4,1}^1 + d_{8,2}^1 + d_{8,3}^2, \quad d_{4,1}^1 + d_{8,3}^1 + d_{8,2}^2$$

for $2G(D_4) + 2G(D_8)$ and fan using η^{-1} to yield generators for an admissible metabolizer of $4G(D_8)$:

$$d_{8,3}^1 + d_{8,3}^2, \quad d_{4,1}^1 + d_{4,1}^2 + d_{8,2}^3 + d_{8,2}^4, \\ d_{8,1}^1 + d_{8,2}^2 + d_{8,2}^3 + d_{8,3}^4, \quad d_{8,1}^1 + d_{8,2}^2 + d_{8,3}^3 + d_{8,2}^4.$$

This subgroup has two elements of norm 8, and as such is inequivalent to the admissible isotropic subgroup obtained by breaking apart $2D_{12}$.

On the other hand, if we rewrite $d_{8,3}^3$ as $d_{4,2}^1 + d_{4,3}^2$, an inadmissible isotropic subgroup for $2G(D_4) + 2G(D_8)$ is generated by

$$d_{4,1}^1 + d_{4,1}^2, \quad d_{4,2}^1 + d_{4,3}^2 + d_{8,2}^1 + d_{8,2}^2, \quad d_{4,1}^1 + d_{8,2}^1 + d_{8,3}^2, \quad d_{4,1}^1 + d_{8,3}^1 + d_{8,2}^2.$$

Apply η^{-1} to these to obtain generators for an admissible isotropic subgroup for $4G(D_8)$:

$$d_{8,3}^1 + d_{8,3}^2, \quad d_{8,3}^2 + d_{8,2}^3 + d_{8,2}^4, \quad d_{8,1}^1 + d_{8,2}^2 + d_{8,2}^3 + d_{8,3}^4, \quad d_{8,1}^1 + d_{8,2}^2 + d_{8,3}^3 + d_{8,2}^4.$$

Exchanging $d_{8,1}^i$ for $d_{8,3}^i$ and vice versa for $i = 3, 4$, we recover the same isotropic subgroup as the first one obtained from $2G(D_{12})$. Since all possibilities up to equivalence have been exhausted, there are exactly two distinct isometry classes of complete even unimodular lattices with root system $4D_8$.

EXAMPLE 4. This example deals with a root system of nonzero deficiency; i.e., the maximum number of mutually orthogonal roots is less than the rank of the root lattice. Kervaire [Ke] determined that there is exactly one isometry class of complete even unimodular lattices with the root system $10A_2 + 2E_6$. In his proof, he used results on conference matrices, a topic treated in coding theory. Here, we offer a different proof based on the fanning method.

Define the isofold

$$\eta: 10G(A_2) + 2G(E_6) \rightarrow 12G(A_2)$$

$$a_{2,j}^i \mapsto a_{2,j}^i, \quad 1 \leq i \leq 10, j \in \{0, 1, 2\}, \quad e_{6,1}^1 \mapsto a_{2,1}^1 + a_{2,1}^2, \quad e_{6,1}^2 \mapsto a_{2,1}^1 + a_{2,2}^2.$$

Niemeier showed in [N] that there is exactly one isometry class of complete even unimodular lattices with root system $12A_2$. Thus, there is exactly one orbit of admissible isotropic subgroups in $12G(A_2)$. A representative subgroup H' of this orbit is generated by

$$\begin{aligned} & a_{2,1}^1 + a_{2,1}^2 + a_{2,1}^3 + a_{2,1}^4 + a_{2,1}^5 + a_{2,1}^6 \\ & a_{2,1}^1 + a_{2,1}^2 + a_{2,2}^3 + a_{2,2}^4 + a_{2,1}^7 + a_{2,1}^8 \\ & a_{2,1}^1 + a_{2,1}^2 + a_{2,2}^5 + a_{2,2}^6 + a_{2,1}^9 + a_{2,1}^{10} \\ & a_{2,1}^1 + a_{2,1}^2 + a_{2,2}^3 + a_{2,2}^5 + a_{2,1}^{11} + a_{2,1}^{12} \\ & a_{2,1}^7 + a_{2,2}^8 + a_{2,1}^9 + a_{2,2}^{10} + a_{2,1}^{11} + a_{2,2}^{12} \end{aligned}$$

The inverse of η acts as the identity on $a_{2,j}^i$ for $1 \leq i \leq 10$ and $j \in \{0, 1, 2\}$, while $\eta^{-1}(a_{2,1}^{11}) = e_{6,1}^1 + e_{6,1}^2$ and $\eta^{-1}(a_{2,1}^{12}) = e_{6,1}^1 + e_{6,2}^2$. Applying η^{-1} to the generators of H' yields generators for an admissible isotropic subgroup H for $10G(A_2) + 2G(E_6)$:

$$\begin{aligned} & a_{2,1}^1 + a_{2,1}^2 + a_{2,1}^3 + a_{2,1}^4 + a_{2,1}^5 + a_{2,1}^6 \\ & a_{2,1}^1 + a_{2,1}^2 + a_{2,2}^3 + a_{2,2}^4 + a_{2,1}^7 + a_{2,1}^8 \\ & a_{2,1}^1 + a_{2,1}^2 + a_{2,2}^5 + a_{2,2}^6 + a_{2,1}^9 + a_{2,1}^{10} \\ & a_{2,1}^1 + a_{2,1}^2 + a_{2,2}^3 + a_{2,2}^5 + e_{6,2}^1 \\ & a_{2,1}^7 + a_{2,2}^8 + a_{2,1}^9 + a_{2,2}^{10} + e_{6,2}^2 \end{aligned}$$

If there were an admissible isotropic subgroup J of $10G(A_2) + 2G(E_6)$ not in the orbit of H , it would have to fold to an isotropic subgroup J' of $12G(A_2)$ in an orbit different from H' . Necessarily, J' contains roots, and these will have the form $a_{2,1 \text{ or } 2}^i + a_{2,1 \text{ or } 2}^j + a_{2,1 \text{ or } 2}^k$ with distinct $i, j \in \{1, \dots, 10\}$ and $k \in \{11, 12\}$. These roots can then be seen as roots of E_6 . The only root system of a complete even unimodular lattice in dimension 24 with root system containing a summand E_6 is $4E_6$. But to transform $12A_2$ to $4E_6$ would require roots as above in which $k \notin \{11, 12\}$. Applying η^{-1} to a root of this kind yields an element of norm 2 in J . Thus, there can be no admissible isotropic subgroup in an orbit different from the one containing H ; hence, there is exactly one isometry class of even unimodular lattices with root system $10A_2 + 2E_6$.

3. ELEMENTARY ISOFANS AND ISOFOLDS

In the previous section, it was shown that φ_{D_k} , $k \geq 2$, is an isofan, as was noted by Venkov [V]. Conway and Pless [CP] found several other isofans that aided them in obtaining some of their codes from already known codes. The associated isofolds for these are:

$$\begin{aligned} \eta_{2E_7}: G(2E_7) &\rightarrow G(D_6); & e_{7,1}^1 &\mapsto d_{6,1}, e_{7,1}^2 \mapsto d_{6,3}; \\ \eta_{D_6+E_7}: G(D_6 + E_7) &\rightarrow G(A_1 + D_4); & e_{7,1} &\mapsto a_{1,1} + d_{4,2}, \\ & & d_{6,j} &\mapsto a_{1,1} + d_{4,j}, j \in \{1, 3\}; \\ \eta_{2D_6}: G(2D_6) &\rightarrow G(4A_1); & d_{6,1}^1 &\mapsto a_{1,1}^1 + a_{1,1}^2 + a_{1,1}^3, d_{6,3}^1 \mapsto a_{1,1}^1 + a_{1,1}^2 + a_{1,1}^4, \\ & & d_{6,1}^2 &\mapsto a_{1,1}^1 + a_{1,1}^3 + a_{1,1}^4, d_{6,3}^2 \mapsto a_{1,1}^2 + a_{1,1}^3 + a_{1,1}^4. \end{aligned}$$

There are, however, other isofolds. The purpose of this section is to determine all possible isofolds.

DEFINITION. Let $R = I_1 + \dots + I_l$ be the concatenation of indecomposable root systems I_i , $1 \leq i \leq l$. Let $\eta: G(R) \rightarrow G(R')$ be an isofold for some root system R' . One says that the isofold η is *imprimitive* if there exists an $i \in \{1, \dots, l\}$ such that

$$\eta|_{G(I_i)}(G(I_i)) \simeq G(I_i) \quad \text{and} \quad \mathbf{n}(x) = \mathbf{n}(\eta|_{G(I_i)}(x)) \quad \text{for all } x \in G(I_i).$$

In effect, this means that I_i is a summand of R' , and η restricted to $G(I_i)$ preserves norms, although it may not be the identity.