3.7 Links of 5-spheres in S^8

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By the transversality theorem ([17], IV.(2.4)), one sees that we may assume $S_i^2 \cap g_j(S^3) = \varnothing$ for all i and j.

By Corollary 3.9, the ambient isotopy class of the embedding g_k is determined by the element $\varphi_k := [g_k] \in \pi_3(W_k)$, $W_k := W \setminus \bigcup_{j \neq k} g_j(S^3)$, $k = 1, \dots, b'$. We clearly have (compare [8])

$$\pi_3(W_k) = \pi_3\big(\underbrace{S^2 \vee \cdots \vee S^2}_{b\times} \vee \underbrace{S^3 \vee \cdots \vee S^3}_{(b'-1)\times}\big),$$

so that the Hilton-Milnor theorem yields

$$\pi_3(W_k) = \bigoplus_{i=1}^{b} \pi_3(S^2) \oplus \bigoplus_{1 \le i < j \le b} \pi_3(S^3) \oplus \bigoplus_{j \neq k} \pi_3(S^3).$$

Hence, we write φ_k as a tuple of integers:

$$\varphi_k = \big(l_i^k,\ i = 1, \dots, b;\quad l_{ij}^k,\ 1 \le i < j \le b;\ \lambda_{kj},\ j \neq k\big).$$

Observe that, for $j \neq k$, φ_k is mapped under the natural homomorphism

$$\pi_3(W_k) \longrightarrow H_3(W_k, \mathbf{Z}) \longrightarrow H_3\big(W \setminus g_j(S^3), \mathbf{Z}\big)\ (\cong \mathbf{Z})$$

to the image of the fundamental class of S^3 under g_{j*}. Thus, λ_{kj} is just the 'usual' linking number of the spheres $g_k(S^3)$ and $g_j(S^3)$ in W (compare [8]).

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Let $\mathcal{FC}_b^{\mathrm{PL}(\mathcal{C}^\infty)}$ be as before, and let $\mathcal{C}_b^{\mathrm{PL}(\mathcal{C}^\infty)}$ be the group of isotopy classes of piecewise linear (smooth) embeddings of b disjoint copies of S^5 into S^8. For $b = 1$, these groups are studied in [10], [19], and [20]. A brief summary with references of results in the case $b > 1$ is contained in Section 2.6 of [11]. We will review some of this material below.

PROPOSITION 3.15. *We have* $\mathcal{FC}_1^{\mathcal{C}^\infty} \cong \mathcal{FC}_1^{\mathrm{PL}} \cong \mathbf{Z}_2$.

Proof. Since $\pi_5(\mathrm{SO}(3)) \cong \mathbf{Z}_2$, the standard embedding of S^5 into S^8 with its two possible framings provides an injection of $\mathbf{Z}_2$ into $\mathcal{FC}_1^{\mathrm{PL}(\mathcal{C}^\infty)}$. By Zeeman's unknotting theorem 3.10, the map $\mathbf{Z}_2 \longrightarrow \mathcal{FC}_1^{\mathrm{PL}}$ is an isomorphism. As remarked in Section 2.6 of [11], $\mathcal{FC}_1^{\mathrm{PL}}$ is isomorphic to $\mathcal{F}\vartheta$, the group of h-cobordism classes of framed submanifolds of S^8 which are homotopy 5-spheres. Moreover, by [10] and [19], there is an exact sequence

$$\cdots \longrightarrow \vartheta^6 \longrightarrow \mathcal{FC}_1^{\mathcal{C}^\infty} \longrightarrow \mathcal{F}\vartheta \longrightarrow \vartheta^5 \longrightarrow \cdots.$$

As the groups ϑ^5 and ϑ^6 of exotic 5- and 6-spheres are trivial [17], our claim is settled. □

Let $\mathrm{L}_b \subset \mathcal{C}_b^{\mathcal{C}^\infty}$ be the subgroup of those embeddings for which the restriction to each component is isotopic to the standard embedding. As observed in Section 2.6 of [11], Zeeman's unknotting theorem 3.10 implies that $\mathrm{L}_b = \mathcal{C}_b^{\mathrm{PL}}$. The following result settles Proposition 2.3:

COROLLARY 3.16. $\mathcal{FC}_b^{\mathcal{C}^\infty} \cong \mathcal{FC}_b^{\mathrm{PL}} \cong \mathrm{L}_b \oplus \bigoplus_{i=1}^{b} \mathbf{Z}_2$.

For the group L_b, Theorem 1.3 of [11] provides a fairly explicit description as an extension of abelian groups. For this, consider the b-fold wedge product $\bigvee_{i=1}^{b} S^2$ of 2-spheres together with its inclusion $i\colon \bigvee_{i=1}^{b} S^2 \hookrightarrow \bigtimes_{i=1}^{b} S^2$ into the b-fold product of 2-spheres. Finally, let $p_i\colon \bigvee_{i=1}^{b} S^2 \longrightarrow S^2$ be the projection onto the i^{th} factor, $i = 1, \ldots, b$. Set, for $m = 1, 2, \ldots$,

$$\Lambda_{b,j}^m := \mathrm{Ker}\Big(\pi_m(p_j)\colon \pi_m\big(\bigvee_{i=1}^{b} S^2\big) \longrightarrow \pi_m(S^2)\Big), \quad j = 1, \ldots, b,$$

$$\Lambda_b^m := \bigoplus_{j=1}^{b} \Lambda_{b,j}^m$$

and

$$\Pi_b^m := \mathrm{Ker}\Big(\pi_m(i)\colon \pi_m\big(\bigvee_{i=1}^{b} S^2\big) \longrightarrow \bigoplus_{i=1}^{b} \pi_m(S^2)\Big),$$

and define

$$w_b^m\colon \Lambda_b^m \longrightarrow \Pi_b^{m+1}$$

on $\Lambda_{b,j}^m$ by $w_b^m(\alpha) := [\alpha, \iota_i]$. Here, $[.,.]$ stands for the Whitehead product inside the homotopy groups of $\bigvee_{i=1}^{b} S^2$ and $\iota_i\colon S^2 \hookrightarrow \bigvee_{i=1}^{b} S^2$ for the inclusion of the i^{th} factor, $i = 1, \ldots, b$. Theorem 1.3 of [11] yields in our situation

THEOREM 3.17. *There is an exact sequence of abelian groups*

$$0 \longrightarrow \mathrm{Coker}(w_b^6) \longrightarrow \mathrm{L}_b \longrightarrow \mathrm{Ker}(w_b^5) \longrightarrow 0.$$

We remark that the formulas of Steer [33] might be used for the explicit computation of Whitehead products and thus for the determination of $\operatorname{Coker}(w_b^6)$ and $\operatorname{Ker}(w_b^5)$. The free part of L_b, e.g., can be obtained quite easily. We confine ourselves to prove the following important fact.

COROLLARY 3.18. *The group* L_b *has positive rank for* $b \geq 2$.

Proof. Let $\mathbf{L}_b := \bigoplus_{l\geq 1} \mathbf{L}_{b,l}$ be the free graded Lie algebra with $\mathbf{L}_{b,1} := \bigoplus_{i=1}^{b} \mathbf{Z} \cdot e_i$. For $l = 2,3,\ldots$, let $e_1^l,\ldots,e_{d_l}^l$ be a basis for $\mathbf{L}_{b,l}$ consisting of iterated commutators of the e_i. By assigning ι_i to e_i, every iterated commutator $c \in \mathbf{L}_{b,l}$ in the e_i defines an element $\alpha(c) \in \pi_{l+1}(\bigvee_{i=1}^{b} S^2)$.

To settle our claim, it is certainly sufficient to show that $\operatorname{Coker}(w_b^6)$ has positive rank. Now, by the Hilton-Milnor theorem

$$\Pi_b^7 \cong \bigoplus_{l=3}^{7}\bigoplus_{k=1}^{d_{l-1}} \pi_7(S^l)\cdot\alpha(e_k^{l-1}).$$

Note that $\pi_7(S^l)$ is finite for $l \notin \{4,7\}$ (see [32] and [35] for the explicit description of those groups). The Hopf fibration $S^7 \longrightarrow S^4$ [32], on the other hand, yields a decomposition $\pi_7(S^4) \cong \pi_6(S^3)\oplus\pi_7(S^7) \cong \mathbf{Z}_{12}\oplus\mathbf{Z}$. Therefore, it will suffice to show that the free part of Λ_b^6 is mapped to $\bigoplus_{j=1}^{d_6} \pi_7(S^7)\cdot\alpha(e_j^6)$. For $j = 1,\ldots,b$, we have

$$\Lambda_{b,j}^6 \cong \bigoplus_{i\neq j} \pi_6(S^2)\cdot\iota_i \oplus \bigoplus_{l=3}^{6}\bigoplus_{k=1}^{d_{l-1}} \pi_6(S^l)\cdot\alpha(e_k^{l-1}).$$

The group $\pi_6(S^l)$ is finite for $l < 6$, and we obviously have $[\alpha(e_k^5),\iota_j] = \alpha([e_k^5,e_j])$. If we expand the commutator $[e_k^5,e_j]$ in the basis $e_1^6,\ldots,e_{d_6}^6$, we find an expansion for $[\alpha(e_k^6),\iota_j]$ in terms of the $\alpha(e_k^6)$. □

COROLLARY 3.19. *The set of* $\mathrm{GL}_b(\mathbf{Z})$*-equivalence classes of elements in* L_b *is infinite for* $b \geq 2$.

Proof. We have seen that the $\mathrm{GL}_b(\mathbf{Z})$-set $\mathbf{L}_{b,3}$ is contained in the $\mathrm{GL}_b(\mathbf{Z})$-set L_b. The $\mathrm{GL}_b(\mathbf{Z})$-action on $\mathbf{L}_{b,3}$ originates from a homomorphism $\mathrm{GL}_b(\mathbf{Z}) \longrightarrow \mathrm{GL}(\mathbf{L}_{b,3}) := \operatorname{Aut}_{\mathbf{Z}}(\mathbf{L}_{b,3})$. In particular, any matrix $g \in \mathrm{GL}_b(\mathbf{Z})$ preserves the absolute value of the determinant of any d_3 elements in $\mathbf{L}_{b,3}$. This implies, for instance, that $a\cdot e_1^3$ and $b\cdot e_1^3$ cannot lie in the same $\mathrm{GL}_b(\mathbf{Z})$-orbit, if $0 \leq a < b$. □