# 3.4 SOME 4-DIMENSIONAL CW-COMPLEXES

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### 3.4 Some 4-dimensional CW-complexes

By Remark 3.4, a handle decomposition of $X$ gives us a CW-complex which is homotopy equivalent to $X$. The following discussion will enable us to understand the 4-skeleton of that complex.

Let $W := S^2 \vee \cdots \vee S^2$ be the $b$-fold wedge product of 2-spheres. Suppose $X$ is the CW-complex obtained by attaching a 4-cell to $W$ via the map $g \in \pi_3(W)$. The Hilton-Milnor theorem ([30], Thm. 7.9.4) asserts that

$$\pi_3(W) = \bigoplus_{i=1}^{b} \pi_3(S^2) \oplus \bigoplus_{1 \le i < j \le b} \pi_3(S^3)\,.$$

Choosing the standard generators for $\pi_3(S^2)$ and $\pi_3(S^3)$, we can describe $g$ by a tuple $(l_i, i = 1, \ldots, b; l_{ij}, 1 \le i < j \le b)$ of integers. These integers determine the cohomology ring of $X = W \cup_g D^4$ as follows:

Proposition 3.11. *Let $y \in H^4(X, \mathbf{Z})$ be the generator of $H^4(X, \mathbf{Z})$ given by the attached 4-cell and $x_1, \ldots, x_b$ the canonical basis of $H^2(X, \mathbf{Z}) = H^2(W, \mathbf{Z})$. Then*

$$\begin{aligned} x_i \cup x_j &= l_{ij} \cdot y, \qquad & 1 \le i < j \le b\,, \\ x_i \cup x_i &= l_i \ \cdot y, \qquad & i = 1, \ldots, b\,. \end{aligned}$$

This is proved like [22], (1.5), p. 103. We recall the proof in the following example.

Example 3.12. We treat the case $b = 2$. Consider the embedding

$$\iota \colon S^2 \vee S^2 \hookrightarrow S^2 \times S^2 \hookrightarrow \mathbf{CP}^\infty \times \mathbf{CP}^\infty\,.$$

The standard basis for $H^4(\mathbf{CP}^\infty \times \mathbf{CP}^\infty, \mathbf{Z}) \cong \mathbf{Z}^{\oplus 3}$ is given by the elements $y_1$, $y_2$, $y_3$ obtained from attaching $D^4$ via $(1,0;0)$, $(0,0;1)$, and $(0,1;0)$, respectively. Let $h \colon D^4 \longrightarrow D^4 \vee D^4 \vee D^4$ be the canonical map followed by

$$\big(\vartheta \cdot x \longmapsto \vartheta \cdot m_{l_1}(x)\big) \vee \big(\vartheta \cdot x \longmapsto \vartheta \cdot m_{l_{12}}(x)\big) \vee \big(\vartheta \cdot x \longmapsto \vartheta \cdot m_{l_2}(x)\big)\,.$$

Here, $m_k$ stands for a representative of $[k \cdot \mathrm{id}_{S^3}] \in \pi_3(S^3)$ and $D^4 = \big\{\vartheta \cdot x \mid x \in S^3, \vartheta \in [0,1]\big\}$. Now, $h$ and $\iota$ glue to a map $f \colon X \longrightarrow \mathbf{CP}^\infty \times \mathbf{CP}^\infty$, and

$$\begin{aligned} f^* \colon H^4(\mathbf{CP}^\infty \times \mathbf{CP}^\infty, \mathbf{Z}) &\longrightarrow H^4(X, \mathbf{Z}) \\ a_1 y_1 + a_2 y_2 + a_3 y_3 &\longmapsto \big(a_1 l_1 + a_2 l_{12} + a_3 l_2\big) y\,, \end{aligned}$$

so that the assertion follows from the naturality of the cup-product.