2.1 The classical invariants

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and $f^*(\underline{y}') = \underline{y}$. We denote by $\mathfrak{J}^{\mathrm{PL}(C^\infty)}(b, b')$ the set of isomorphy classes of piecewise linear (smooth) based E-manifolds $(X, \underline{x}, \underline{y})$ of dimension eight with vanishing second Stiefel-Whitney class, $b_2(X) = b$, and $b_4(X) = b'$.

2.1 The classical invariants

In the terminology of [24], the classical invariants of an E-manifold consist of its cohomology ring, the Stiefel-Whitney classes, the Wu classes, the Pontrjagin classes, the Euler class, the Steenrod squares, the reduced Steenrod powers, and the Pontrjagin powers. For an eight-dimensional E-manifold X with vanishing second Stiefel-Whitney class, the main result of [24] states that the classical invariants are fully determined by the following invariants:

1. The cup product map

$$\begin{aligned} \delta_X \colon S^2H^2(X, \mathbf{Z}) &\longrightarrow H^4(X, \mathbf{Z}) \\ x \otimes x' &\longmapsto x \cup x' . \end{aligned}$$

2. The intersection form

$$\begin{aligned} \gamma_X \colon S^2H^4(X, \mathbf{Z}) &\longrightarrow \mathbf{Z} \\ y \otimes y' &\longmapsto (y \cup y')[X] . \end{aligned}$$

   Here, $[X] \in H_8(X, \mathbf{Z})$ is the fundamental class determined by the orientation.

3. The first Pontrjagin class $p_1(X) \in H^4(X, \mathbf{Q})$.

REMARK 2.1. The above invariants are not independent. By associativity of the cohomology ring, the following relation holds

$$\delta_X^*(\gamma_X) \in S^4H^2(X, \mathbf{Z})^\vee, \tag{1}$$

i.e.,

$$\gamma_X\big(\delta_X(x_1 \otimes x_2) \otimes \delta_X(x_3 \otimes x_4)\big) = \gamma_X\big(\delta_X(x_1 \otimes x_3) \otimes \delta_X(x_2 \otimes x_4)\big) ,$$

for all $x_1, x_2, x_3, x_4 \in H^2(X, \mathbf{Z})$. Furthermore, PROPOSITION ([24], Prop. A.7 or Cor. 3.14 below). *For every element $y \in H^4(X, \mathbf{Z})$ we have*

$$p_1(X)y \equiv 2y^2 \mod 4\,. \tag{2}$$

Note that this implies $p_1(X) \in H^4(X, \mathbf{Z})$. If, in addition, X is differentiable then for every integral lift $W \in H^2(X, \mathbf{Z})$ of $w_2(X)$ one has

$$3p_1(X)^2 - 14p_1(X)W^2 + 7W^4 \equiv 12 \operatorname{Sign}(\gamma_X) \mod 2688\,. \tag{3}$$

Müller has also shown [24] that these relations imply all the other relations among the classical invariants of X. Conversely, a piecewise linear manifold X whose invariants obey relation (3) admits a differentiable structure [18], [24].

We are led to the following algebraic concept: A *system of invariants of type* (b, b') is a triple (δ, γ, p), consisting of

- a homomorphism $\delta\colon S^2\mathbf{Z}^{\oplus b} \longrightarrow \mathbf{Z}^{\oplus b'}$,
- a unimodular symmetric bilinear form $\gamma\colon S^2\mathbf{Z}^{\oplus b'} \longrightarrow \mathbf{Z}$, and
- an element $p \in \mathbf{Z}^{\oplus b'}$.

We denote by $Z(b, b')$ the set of systems of invariants of type (b, b').

Now, let $(X, \underline{x}, y)$ be a based eight-dimensional E-manifold. This defines a set of invariants $Z_{(X,\underline{x},y)} := (\delta_X, \gamma_X, p_1(X))$ of type $(b_2(X), b_4(X))$. Thus, we have natural maps

$$\begin{aligned} Z^{\mathrm{PL}(C^\infty)}(b, b')\colon \mathfrak{Z}^{\mathrm{PL}(C^\infty)}(b, b') &\longrightarrow Z(b, b') \\ \left[X, \underline{x}, y\right] &\longmapsto Z_{(X,\underline{x},y)}\,. \end{aligned}$$

It will be the concern of our paper to understand the maps $Z^{\mathrm{PL}(C^\infty)}$ as well as possible. The first result can be easily derived from Wall's work [36] and deals with the case $b = 0$. It will be proved in detail in Section 4.1.

THEOREM 2.2. i) *The map $Z^{\mathrm{PL}}(0, b')$ is injective. Its image consists precisely of those elements which satisfy the relations* (1) *and* (2).

ii) *Given two smooth based E-manifolds (X, y) and (X', y') with $b_2(X) = 0 = b_2(X')$ and $Z_{(X,y)} = Z_{(X',y')}$, there exists an exotic 8-sphere Σ such that $(X \# \Sigma, y)$ and (X', y') are smoothly isomorphic. In particular, the fibres of Z^{C^∞} have cardinality at most two. The image of Z^{C^∞} consists exactly of those elements which satisfy the relations* (1), (2), *and* (3).