1. The Grothendieck formula

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and a cohomology class v in $H^2(M;\mathbf{Z}/2)$ such that v cannot be represented as the second Stiefel-Whitney class of a real vector bundle on M (it is now known that $m \geq 11$ can be replaced by $m \geq 6$, which is sharp [27]). On the other hand, for any compact nonsingular real algebraic set X, each cohomology class in $H^2(X;\mathbf{Z}/2)$, whose Poincaré dual homology class can be represented by an algebraic subset of X, is the second Stiefel-Whitney class of some algebraic vector bundle on X. Therefore the conjecture has to be false.

We call the latter part of the argument the *Grothendieck formula in real algebraic geometry*. This was proved in [7] in two steps. First a proof of the Grothendieck formula relating vector bundles and algebraic cycles on schemes over $\mathbf{R}$ was sketched (an analog of the formula from earlier papers [18, 19] for varieties over an algebraically closed field); this sketch contains some flaws. Then a connection, established in [15], between the Chern classes with values in the Chow ring and the Stiefel-Whitney classes yielded the conclusion. The appearance of [17] allowed for a shorter proof [12], based on the same principles and free from the flaws mentioned above. According to the authors' experience such proofs still present considerable difficulty for many topologically inclined mathematicians. The goal of this paper is to give a self-contained topological proof that uses only the simplest facts from algebra. Several applications of the Grothendieck formula in real algebraic geometry, besides the one discussed above, are contained in [12, 13, 22].

The paper assumes knowledge of singular homology and cohomology with coefficients in $\mathbf{Z}/2$ at the level of [26]. Real vector bundles and their Stiefel-Whitney classes, for which a good reference is [24], are also used. All smooth (of class C^∞) manifolds are assumed to be paracompact and without boundary. From real algebraic geometry we require only a few notions, recalled here and elucidated in detail in just a few pages of [5], [8], or [11]. Basic and generally well-known facts from commutative algebra that are needed can all be found in [23].

1. THE GROTHENDIECK FORMULA

REAL ALGEBRAIC VARIETIES

The Zariski topology on $\mathbf{R}^n$ is the topology for which the closed sets are precisely the algebraic subsets of $\mathbf{R}^n$. Let V be a nonempty Zariski locally closed subset of $\mathbf{R}^n$ (that is, V is the difference of two algebraic subsets of $\mathbf{R}^n$). The *dimension* $\dim V$ of V is the largest integer d for which there exist an open subset N of $\mathbf{R}^n$ (in the usual metric topology) and polynomials $P_1, \dots, P_{n-d}$ in $\mathbf{R}[T_1, \dots, T_n]$ such that $N \cap V$ is a nonempty set, $V \subset Z$, $N \cap V = N \cap Z$, where

$$Z = \{z \in \mathbf{R}^n \mid P_1(z) = \cdots = P_{n-d}(z) = 0\},$$

and the Jacobian matrix $\left[\frac{\partial P_i}{\partial T_j}(z)\right]$, $1 \leq i \leq n-d$, $1 \leq j \leq n$, has rank $n-d$ for every point z in $N \cap V$ (several other characterizations of $\dim V$ are given in [8, Sect. 3.4 and 11, Sect. 2.8]). A point x in V is said to be *nonsingular* if one can find N and $P_1, \dots, P_{n-d}$ as above, with x in $N \cap V$ and $d = \dim V$; otherwise x is called singular (this agrees with [5, 11], whereas in [8] a slightly different definition is used, with the condition $d = \dim V$ omitted). Clearly, the set of all nonsingular points of V is a smooth submanifold of $\mathbf{R}^n$ of dimension $\dim V$. Consider V endowed with the Zariski topology induced from $\mathbf{R}^n$. The set $\operatorname{Sing}(V)$ of all singular points of V is Zariski closed in V and

$$\dim \operatorname{Sing}(V) < \dim V$$

[5, p. 28 or 8, p. 137 or 11, p. 69]. If $\operatorname{Sing}(V)$ is empty, V is said to be nonsingular.

Recall that V is irreducible if it cannot be represented as the union of two Zariski closed subsets of V, distinct from V. Assuming that V is irreducible, one has $\dim W < \dim V$ for every Zariski closed subset W of V, $W \neq V$ [5, p. 28 or 8, p. 136 or 11, p. 50]. If V is not irreducible, then $V = V_1 \cup \ldots \cup V_k$, where $V_1, \dots, V_k$ are irreducible Zariski closed subsets of V, with V_i not contained in V_j for $i \neq j$; the sets $V_1, \dots, V_k$ are uniquely determined and called the irreducible components of V [5, p. 20 or 8, p. 119 or 11, p. 50].

A function $f\colon V \to \mathbf{R}$ is said to be *regular* if for every point x in V, there exist an open neighborhood (in the Zariski topology) U_x of x in V and two polynomials P and Q in $\mathbf{R}[T_1, \dots, T_n]$ such that $Q(y) \neq 0$ and $f(y) = P(y)/Q(y)$ for all y in U_x. In fact, one can take $U_x = V$, and hence f is always a quotient of two polynomials $f = P/Q$ with $Q(y) \neq 0$ for all y in V [5, p. 19 or 8, p. 121 or 11, p. 62]. The set $\mathcal{R}(V)$ of all regular functions on V forms a ring under pointwise addition and multiplication.

Throughout this paper, a *real algebraic variety* is, by definition, a Zariski locally closed subset of $\mathbf{R}^n$, for some n. A map $\varphi\colon V \to W$ between real algebraic varieties, $W \subset \mathbf{R}^p$, is called *regular* if each component φ_i of $\varphi = (\varphi_1, \dots, \varphi_p)$ is in $\mathcal{R}(V)$. If, moreover, φ is bijective and φ^{-1} is regular, we call φ *biregular.* One easily sees that nonsingular points and dimension are invariant under biregular maps [5, p. 28, or 8, p. 126 or 11, p. 67].

Unless explicitly stated otherwise, all topological notions related to real algebraic varieties will refer to the usual metric topology.

COMBINATORIAL PROPERTIES OF REAL ALGEBRAIC VARIETIES

Recall that the semialgebraic subsets of $\mathbf{R}^n$ form the smallest family of subsets containing all sets of the form

$$\{x \in \mathbf{R}^n \mid P(x) > 0\}, \text{ where } P \text{ is in } \mathbf{R}[T_1, \dots, T_n],$$

and closed under taking finite unions, finite intersections, and complements. Obviously, any algebraic subset of $\mathbf{R}^n$ is semialgebraic.

We shall make use of the following important result (for its proof cf. [8, Theorem 2.6.12] or [11, Theorem 9.2.1]):

THEOREM 1.1. *Let T be a compact semialgebraic set. Given a finite family $\mathcal{F}$ of semialgebraic subsets of T, there exists a semialgebraic triangulation of T compatible with $\mathcal{F}$.*

In other words, there exist a simplicial complex K and a homeomorphism $\Phi\colon |K| \to T$, where $|K|$ is the polyhedron determined by K, such that for each open simplex σ of K and each set S in $\mathcal{F}$, the image $\Phi(\sigma)$ is a semialgebraic subset of T, which is either contained in or disjoint from S.

For any pair (X, A) of topological spaces, the Euler-Poincaré characteristic $\chi(X, A)$ is defined by

$$\chi(X, A) = \sum_{r \geq 0} (-1)^r \dim_{\mathbf{Z}/2} H_r(X, A; \mathbf{Z}/2),$$

provided that $\dim_{\mathbf{Z}/2} H_r(X, A; \mathbf{Z}/2)$ is finite for all $r \geq 0$ and equals 0 for all r large enough (if the homology group $H_*(X, A; \mathbf{Z})$ is finitely generated, then this defintion coincides with the usual one [16, Proposition VI.7.21]). If $\chi(X)$ and $\chi(A)$ are defined, then $\chi(X, A)$ is also defined and $\chi(X, A) = \chi(X) - \chi(A)$ [16, Proposition V.5.7]. If K is a finite simplicial complex and ℓ_r is the number of r-simplices in K, then

$$\chi(|K|) = \sum_{r \geq 0} (-1)^r \ell_r .$$

Note that for any compact real algebraic variety V, the Euler-Poincaré characteristic $\chi(V, V \setminus \{x\})$ is defined for every point x of V. Indeed, by Theorem 1.1, there exists a triangulation $\Phi\colon |K| \to V$ of V such that $\Phi(v) = x$ for some vertex v of K. If L is the subcomplex of K of all simplices that do not have v as a vertex, then $|L|$ is a deformation retract of $|K| \setminus \{v\}$, and hence

$$\chi(V, V\setminus\{x\}) = \chi(|K|, |K|\setminus\{v\}) = \chi(|K|, |L|) = \chi(|K|) - \chi(|L|).$$

It follows that

$$\chi(V, V\setminus\{x\}) = \sum_{r\geq 0}(-1)^r m_r, \tag{1.2}$$

where m_r is the number of r-simplices of K having v as a vertex.

THEOREM 1.3. *Let V be a compact real algebraic variety. Then for every point x in V, the Euler-Poincaré characteristic $\chi(V, V\setminus\{x\})$ is an odd integer.*

Reference for the proof. It is proved in [8, Theorem 3.10.4], by a nice topological argument, that

$$\sum_{r\geq 0} \dim_{\mathbf{Z}/2} H_r(V, V\setminus\{x\}; \mathbf{Z}/2)$$

is an odd integer. This is equivalent to Theorem 1.3. □

COROLLARY 1.4. *Let V be a compact d-dimensional real algebraic variety and let $\Phi\colon |K| \to V$ be a triangulation of V. Then for any $(d-1)$-simplex σ of K, the number $n(\sigma)$ of d-simplices of K having σ as a face is even.*

Proof. Let $\tau_1, \dots, \tau_{n(\sigma)}$ be the d-simplices of K having σ as a face. Let K' be the barycentric subdivision of K and let b be the barycenter of σ. Denote by n_i the number of simplices s of the barycentric subdivision of τ_i such that b is a vertex of s and s is not in the barycentric subdivision of σ. One readily sees that n_i is odd. Let n be the number of simplices in the barycentric subdivision of σ having b as a vertex. Clearly, n is odd. Note that $n + n_1 + \cdots + n_{n(\sigma)}$ is the number of simplices of K' having b as a vertex. In view of (1.2) and Theorem 1.3, $n(\sigma)$ has to be even. Hence the proof is complete. □

ALGEBRAIC CYCLES

Given a compact d-dimensional real algebraic variety V, we shall now define a homology class $[V]$ in $H_d(V, \mathbf{Z}/2)$ playing a special role in various problems concerning geometry and topology of varieties.

Choose a semialgebraic triangulation of V (Theorem 1.1). By Corollary 1.4, the sum of all d-simplices of this triangulation is a cycle with coefficients in $\mathbf{Z}/2$. The homology class $[V]$ in $H_d(V, \mathbf{Z}/2)$ represented by this cycle is independent of the choice of the triangulation. Indeed, taking any two semialgebraic triangulations of V we can, using Theorem 1.1, find a third one, which is a common subdivision of the two. The uniqueness of $[V]$ follows immediately.

The excision property implies that for each nonsingular point x of V, the image of $[V]$ by the canonical homomorphism

$$H_d(V; \mathbf{Z}/2) \longrightarrow H_d(V, V \setminus \{x\}; \mathbf{Z}/2) \cong \mathbf{Z}/2$$

is nonzero. The class $[V]$ is called the *fundamental class* of V. If V is nonsingular, then $[V]$ coincides with the fundamental class of V regarded as a manifold. For other, equivalent, definitions of the fundamental class, cf. [10, 14, 15].

Let X be a compact real algebraic variety. For any d-dimensional Zariski closed subset V of X, we call the element $[V]_X = i_*([V])$ of $H_d(X; \mathbf{Z}/2)$, where $i\colon V \hookrightarrow X$ is the inclusion map, the *homology class of X represented by V*. Denote by

$$H_d^{\mathrm{alg}}(X; \mathbf{Z}/2)$$

the subgroup of $H_d(X; \mathbf{Z}/2)$ generated by all homology classes of X represented by d-dimensional Zariski closed subsets of X. Given two d-dimensional Zariski closed subsets V_1 and V_2 of X, we have $[V_1]_X + [V_2]_X = [W]_X$, where W is the union of the irreducible d-dimensional components of $V_1 \cup V_2$ not contained in $V_1 \cap V_2$. It follows that every element of $H_d^{\mathrm{alg}}(X; \mathbf{Z}/2)$ is of the form $[V]_X$ for some d-dimensional Zariski closed subset V of X.

Assuming that X is compact and nonsingular, we set

$$H^c_{\mathrm{alg}}(X; \mathbf{Z}/2) = D_X^{-1}(H_d^{\mathrm{alg}}(X; \mathbf{Z}/2)),$$

where $c + d = \dim X$ and $D_X\colon H^c(X; \mathbf{Z}/2) \to H_d(X; \mathbf{Z}/2)$ is the Poincaré duality isomorphism, $D_X(u) = u \cap [X]$ for every u in $H^c(X; \mathbf{Z}/2)$. The groups $H_d^{\mathrm{alg}}(X; \mathbf{Z}/2)$ and $H^c_{\mathrm{alg}}(X; \mathbf{Z}/2)$ are important invariants of compact nonsingular real algebraic varieties. The reader can refer to [14] for a short survey of their properties and applications, and for a more extensive list of references. These groups have the expected functorial properties, which however will neither be proved nor used here.

THE GROTHENDIECK FORMULA

In order to state the Grothendieck formula, we have to recall the definition of an algebraic vector bundle.

An *algebraic vector bundle* on a real algebraic variety X is a triple $\xi = (E, \pi, X)$, where E is a real algebraic variety, $\pi: E \to X$ is a regular map, and the following conditions are satisfied:

(i) for every point x in X, the fiber $E_x = \pi^{-1}(x)$ is a real vector space,

(ii) there exist a finite cover $\{U_\lambda\}_{\lambda \in \Lambda}$ of X by Zariski open sets, and for each λ in Λ, a nonnegative integer k and a biregular map $\varphi: \pi^{-1}(U_\lambda) \to U_\lambda \times \mathbf{R}^k$ such that $\varphi(E_x) = \{x\} \times \mathbf{R}^k$ and the restriction $E_x \to \{x\} \times \mathbf{R}^k$ of φ is a linear isomorphism for every x in U_λ,

(iii) ξ is an algebraic subbundle of the trivial vector bundle $X \times \mathbf{R}^p$, for some p.

Condition (iii) means that there exists a regular map $i: E \to X \times \mathbf{R}^p$ such that $i(E_x) \subseteq \{x\} \times \mathbf{R}^p$ and the restriction $E_x \to \{x\} \times \mathbf{R}^p$ of i is an injective linear map for every x in X.

Basic properties of algebraic vector bundles can be found in [11, Chapter 12]. The reader should keep in mind that algebraic vector bundles considered here are sometimes called strongly algebraic vector bundles in the literature [9, 10, 13].

Our main goal is to give a self-contained proof of the following Grothendieck formula.

THEOREM 1.5. *Let X be a compact nonsingular real algebraic variety. For every cohomology class v in $H^2_{\mathrm{alg}}(X; \mathbf{Z}/2)$, there exists an algebraic vector bundle ξ on X with $w_1(\xi) = 0$ and $w_2(\xi) = v$.*

Here $w_k(-)$ stands for the k^{th} Stiefel-Whitney class.

We end this section by stating two results whose proofs use, in an essential way, the Grothendieck formula.

Given a compact smooth manifold M, let us denote by $\mathrm{Vect}(M)$ the set of isomorphism classes of topological real vector bundles on M and define

$$W^2(M) = \{v \in H^2(M; \mathbf{Z}/2) \mid v = w_2(\xi) \quad \text{for some } \xi \text{ in } \mathrm{Vect}(M)\}.$$

One easily sees that $W^2(M)$ is a subgroup of $H^2(M; \mathbf{Z}/2)$. As mentioned in the introduction, in general, $W^2(M) \neq H^2(M; \mathbf{Z}/2)$ for $\dim M \geq 6$. The group $W^2(M)$ plays a crucial role in the problem of representation of homology classes in codimension 2 by Zariski closed subsets. More precisely, we have the following result.

THEOREM 1.6. *Let M be a compact orientable smooth manifold of dimension at least 5 and let G be a subgroup of $H^2(M;\mathbf{Z}/2)$. Then the following conditions are equivalent:*

(a) There exist a nonsingular real algebraic variety X and a diffeomorphism $\varphi: X \to M$ such that $\varphi^*(G) = H^2_{\mathrm{alg}}(X;\mathbf{Z}/2)$.

(b) $w_2(M) \in G \subseteq W^2(M)$, where $w_2(M)$ is the second Stiefel-Whitney class of M.

Proof. See [13].

Another application concerns the problem of approximation of smooth curves (that is, one-dimensional smooth submanifolds) by algebraic curves. First recall that a compact smooth submanifold N of a nonsingular real algebraic variety X is said to *admit an algebraic approximation* in X if for each neighborhood $\mathcal{U}$ of the inclusion map $N \hookrightarrow X$ (in the $\mathcal{C}^\infty$ topology on the set $\mathcal{C}^\infty(N,X)$ of smooth maps from N into X), there exists a smooth embedding $e: N \to X$ such that e is in $\mathcal{U}$ and $e(N)$ is a nonsingular Zariski closed subset of X.

THEOREM 1.7. *Let X be a compact nonsingular real algebraic variety of dimension 3 and let C be a compact smooth curve in X. Then C admits an algebraic approximation in X if and only if the $\mathbf{Z}/2$-homology class represented by C is in $H_1^{\mathrm{alg}}(X;\mathbf{Z}/2)$.*

The proof of Theorem 1.7 will be given elsewhere. Under the extra assumption that C is connected and homologous to the union of finitely many nonsingular real algebraic curves in X the theorem is proved in [4].

2. PROOF OF THE GROTHENDIECK FORMULA

We shall use homology and cohomology groups with coefficients exclusively in $\mathbf{Z}/2$ and therefore we shall suppress the coefficient group in our notation.