7. Concentration to a non-trivial space

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a universality property: it contains an isomorphic copy of every separable metric group [Usp]. See also [Gr3].

Using concentration of measure, one can prove that the group Iso$(\mathbf{U})$ is extremely amenable. The Ramsey–Dvoretzky–Milman property leads to the following Ramsey-type result:

Let F be a finite metric space, and let all isometric embeddings of F into $\mathbf{U}$ be coloured using finitely many colours. Then for every finite metric space G and every $\varepsilon > 0$ there is an isometric copy $G' \subset \mathbf{U}$ of G such that all isometric embeddings of F into $\mathbf{U}$ that factor through G are monochromatic to within ε.

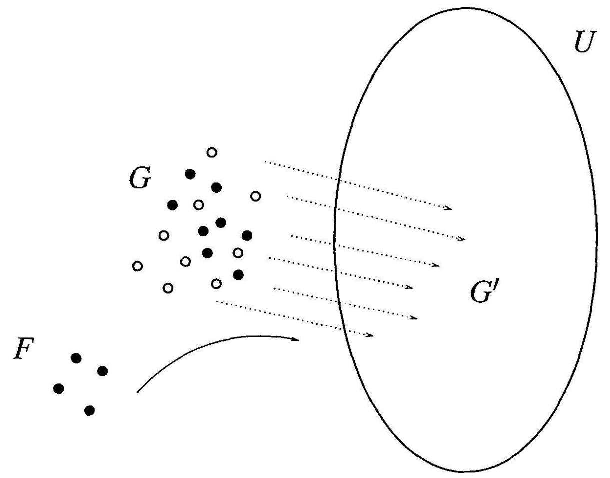

FIGURE 5
A Ramsey-type result for metric spaces

Here we say that a set A is *monochromatic to within* ε if there is a monochromatic set A' at a Hausdorff distance $< \varepsilon$ from A. In our case, the Hausdorff distance is formed with regard to the uniform metric on $\mathbf{U}^F$.

One can also obtain similar results, for example, for the separable Hilbert space ℓ_2 and for the unit sphere $\mathbf{S}^\infty$ in ℓ_2 [P3].

7. CONCENTRATION TO A NON-TRIVIAL SPACE

Let f be a Borel measurable real-valued function on an *mm*-space $X = (X, d, \mu)$. A number $M = M_f$ is called a *median* (or *Lévy mean*) of f if both $f^{-1}[M, +\infty)$ and $f^{-1}(-\infty, M]$ have measure $\geq \frac{1}{2}$.

EXERCISE 13. Show that the median M_f always exists, though it need not be unique.

EXERCISE 14. Assume that a function f as above is 1-Lipschitz, that is, $|f(x)-f(y)| \leq d(x,y)$ for all $x,y \in X$. Prove that for every $\varepsilon > 0$,

$$\mu\{|f(x) - M_f| > \varepsilon\} \leq 2\alpha_X(\varepsilon).$$

Thus, one can express the phenomenon of concentration of measure by stating that on a 'high-dimensional' *mm*-space, every Lipschitz (more generally, uniformly continuous) function is, probabilistically, almost constant.

Following Gromov [Gr3, $3\frac{1}{2}$.45], let us recast the concentration phenomenon yet again.

On the space $L(0,1)$ of all measurable functions define the metric me_1, generating the topology of convergence in measure, by letting $\mathrm{me}_1(h_1, h_2)$ stand for the infimum of all $\lambda > 0$ with the property

$$\mu^{(1)}\{|h_1(x) - h_2(x)| > \lambda\} < \lambda.$$

(Here $\mu^{(1)}$ denotes the Lebesgue measure on the unit interval $\mathbf{I} = [0,1]$.)

Now let $X = (X, d_X, \mu_X)$ and $Y = (Y, d_Y, \mu_Y)$ be two Polish *mm*-spaces. There exist measurable maps $f\colon \mathbf{I} \to X$, $g\colon \mathbf{I} \to Y$ such that $\mu_X = f_* \mu^{(1)}$ and $\mu_Y = g_* \mu^{(1)}$. Denote by L_f the set of all functions of the form $h = h_1 \circ f$, where $h_1\colon X \to \mathbf{R}$ is 1-Lipschitz, having the property $h(0) = 0$. Similarly, define the set L_g. Now define a non-negative real number $\underline{H}_1\mathcal{L}\iota(X,Y)$ as the infimum of Hausdorff distances between L_f and L_g (formed using the metric me_1 on the space of functions), taken over all parametrizations f and g as above.

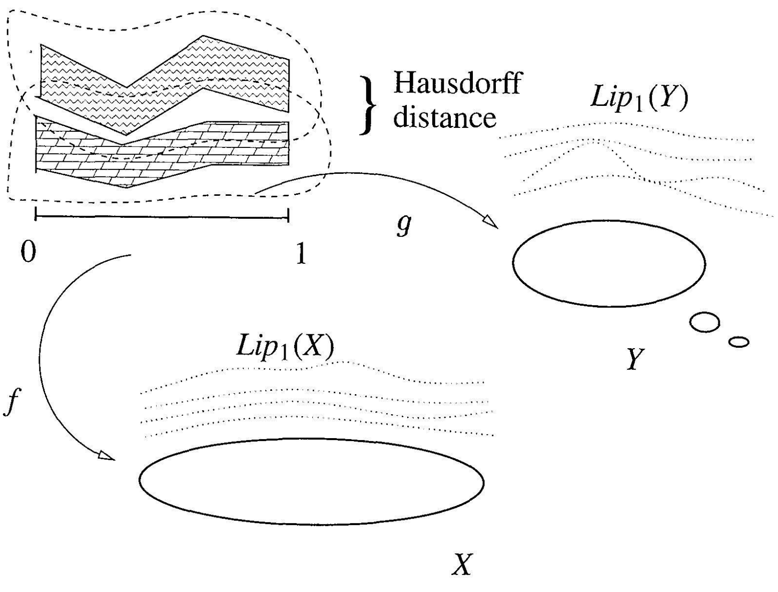

FIGURE 6

Gromov's distance $\underline{H}_1\mathcal{L}\iota$ between *mm*-spaces

EXERCISE 15. Prove that $\underline{H}_1\mathcal{L}\iota$ is a metric on the space of (isomorphism classes of) all Polish mm-spaces.

EXERCISE 16. Prove that a sequence of mm-spaces $X_n = (X_n, d_n, \mu_n)$ forms a Lévy family if and only if it converges to the trivial mm-space in the metric $\underline{H}_1\mathcal{L}\iota$:

$$X_n \xrightarrow{\underline{H}_1\mathcal{L}\iota} \{*\}\,.$$

If one now replaces the trivial space on the right hand side with an arbitrary mm-space[6]), one obtains the concept of *concentration to a non-trivial space.*

According to Gromov, this type of concentration commonly occurs in statistical physics. At the same time, there are very few known non-trivial examples of this kind in the context of transformation groups.

Here is just one problem in this direction, suggested by Gromov. Every probability measure ν on a group G determines a random walk on G. How can one associate to (G, ν) in a natural way a sequence of mm-spaces which would concentrate to the boundary [Fur] of the random walk ?

8. READING SUGGESTIONS

The 2001 Borel seminar was based on Chapter $3\frac{1}{2}$ of the green book [Gr3], which contains a wealth of ideas and concepts and can be complemented by [Gr4]. The survey [M3] by Vitali Milman, to whom we owe the present status of the concentration of measure phenomenon, is highly relevant and rich in material, especially if read in conjunction with a recent account of the subject by the same author [M4]. The book [M-S] is, in a sense, indispensable and should always be within one's reach. Talagrand's fundamental paper [Ta1] has to be at least browsed by every learner of the subject, while the paper [Ta2] of the same author offers an independent introduction to the subject of concentration of measure. The newly-published book by Ledoux [Led], apparently the first ever monograph devoted exclusively to concentration, is highly readable and covers a wide range of topics. Don't miss the introductory survey by Schechtman [Sch]. The modern setting for concentration was designed in the important paper [Gr-M1] by Gromov and Milman, which had also introduced the subject of this lecture and from which many results (perhaps with slight modifications) have been taken.

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[6]) Or, more generally, a uniform space — for instance, a non-metrizable compact space — with measure.