

# 7. Concentration to a non-trivial space

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a universality property: it contains an isomorphic copy of every separable metric group [Usp]. See also [Gr3].

Using concentration of measure, one can prove that the group  $\text{Iso}(\mathbf{U})$  is extremely amenable. The Ramsey–Dvoretzky–Milman property leads to the following Ramsey-type result:

*Let  $F$  be a finite metric space, and let all isometric embeddings of  $F$  into  $\mathbf{U}$  be coloured using finitely many colours. Then for every finite metric space  $G$  and every  $\varepsilon > 0$  there is an isometric copy  $G' \subset \mathbf{U}$  of  $G$  such that all isometric embeddings of  $F$  into  $\mathbf{U}$  that factor through  $G$  are monochromatic to within  $\varepsilon$ .*

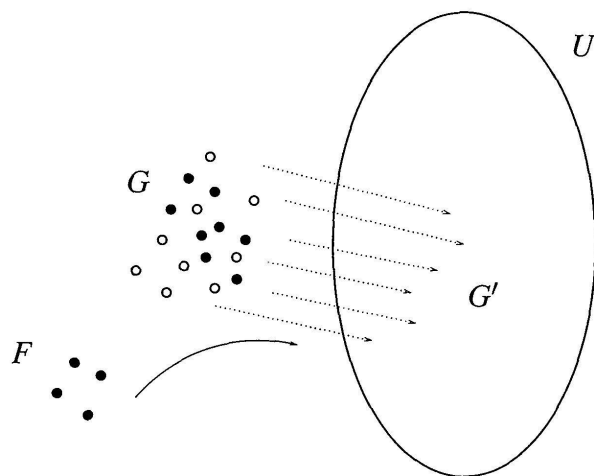


FIGURE 5

A Ramsey-type result for metric spaces

Here we say that a set  $A$  is *monochromatic to within  $\varepsilon$*  if there is a monochromatic set  $A'$  at a Hausdorff distance  $< \varepsilon$  from  $A$ . In our case, the Hausdorff distance is formed with regard to the uniform metric on  $\mathbf{U}^F$ .

One can also obtain similar results, for example, for the separable Hilbert space  $\ell_2$  and for the unit sphere  $\mathbf{S}^\infty$  in  $\ell_2$  [P3].

## 7. CONCENTRATION TO A NON-TRIVIAL SPACE

Let  $f$  be a Borel measurable real-valued function on an *mm*-space  $X = (X, d, \mu)$ . A number  $M = M_f$  is called a *median* (or *Lévy mean*) of  $f$  if both  $f^{-1}[M, +\infty)$  and  $f^{-1}(-\infty, M]$  have measure  $\geq \frac{1}{2}$ .

**EXERCISE 13.** Show that the median  $M_f$  always exists, though it need not be unique.

EXERCISE 14. Assume that a function  $f$  as above is 1-Lipschitz, that is,  $|f(x) - f(y)| \leq d(x, y)$  for all  $x, y \in X$ . Prove that for every  $\varepsilon > 0$ ,

$$\mu\{|f(x) - M_f| > \varepsilon\} \leq 2\alpha_X(\varepsilon).$$

Thus, one can express the phenomenon of concentration of measure by stating that on a ‘high-dimensional’  $mm$ -space, every Lipschitz (more generally, uniformly continuous) function is, probabilistically, almost constant.

Following Gromov [Gr3, 3  $\frac{1}{2}$ .45], let us recast the concentration phenomenon yet again.

On the space  $L(0, 1)$  of all measurable functions define the metric  $me_1$ , generating the topology of convergence in measure, by letting  $me_1(h_1, h_2)$  stand for the infimum of all  $\lambda > 0$  with the property

$$\mu^{(1)}\{|h_1(x) - h_2(x)| > \lambda\} < \lambda.$$

(Here  $\mu^{(1)}$  denotes the Lebesgue measure on the unit interval  $\mathbf{I} = [0, 1]$ .)

Now let  $X = (X, d_X, \mu_X)$  and  $Y = (Y, d_Y, \mu_Y)$  be two Polish  $mm$ -spaces. There exist measurable maps  $f: \mathbf{I} \rightarrow X$ ,  $g: \mathbf{I} \rightarrow Y$  such that  $\mu_X = f_* \mu^{(1)}$  and  $\mu_Y = g_* \mu^{(1)}$ . Denote by  $L_f$  the set of all functions of the form  $h = h_1 \circ f$ , where  $h_1: X \rightarrow \mathbf{R}$  is 1-Lipschitz, having the property  $h(0) = 0$ . Similarly, define the set  $L_g$ . Now define a non-negative real number  $\underline{H}_1 \mathcal{L} \iota(X, Y)$  as the infimum of Hausdorff distances between  $L_f$  and  $L_g$  (formed using the metric  $me_1$  on the space of functions), taken over all parametrizations  $f$  and  $g$  as above.

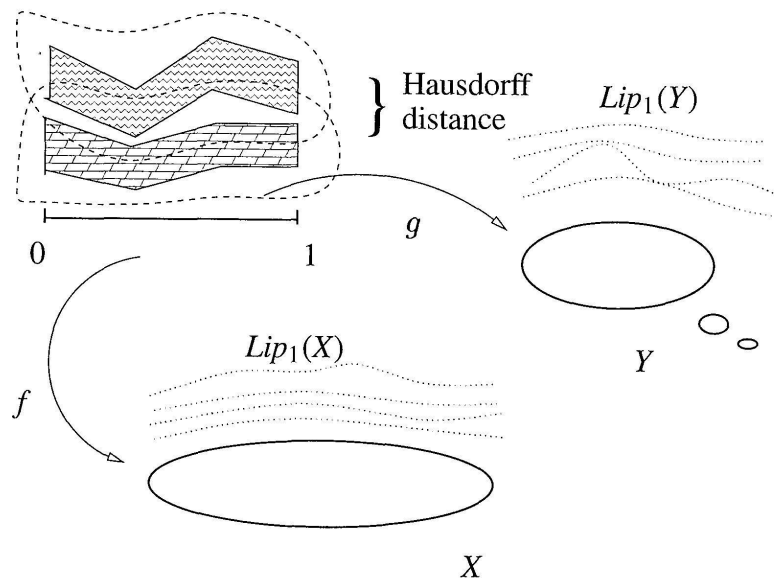


FIGURE 6

Gromov's distance  $\underline{H}_1 \mathcal{L} \iota$  between  $mm$ -spaces

EXERCISE 15. Prove that  $\underline{H}_1\mathcal{L}_\nu$  is a metric on the space of (isomorphism classes of) all Polish  $mm$ -spaces.

EXERCISE 16. Prove that a sequence of  $mm$ -spaces  $X_n = (X_n, d_n, \mu_n)$  forms a Lévy family if and only if it converges to the trivial  $mm$ -space in the metric  $\underline{H}_1\mathcal{L}_\nu$ :

$$X_n \xrightarrow{\underline{H}_1\mathcal{L}_\nu} \{*\}.$$

If one now replaces the trivial space on the right hand side with an arbitrary  $mm$ -space<sup>6</sup>), one obtains the concept of *concentration to a non-trivial space*.

According to Gromov, this type of concentration commonly occurs in statistical physics. At the same time, there are very few known non-trivial examples of this kind in the context of transformation groups.

Here is just one problem in this direction, suggested by Gromov. Every probability measure  $\nu$  on a group  $G$  determines a random walk on  $G$ . How can one associate to  $(G, \nu)$  in a natural way a sequence of  $mm$ -spaces which would concentrate to the boundary [Fur] of the random walk?

## 8. READING SUGGESTIONS

The 2001 Borel seminar was based on Chapter 3 $\frac{1}{2}$  of the green book [Gr3], which contains a wealth of ideas and concepts and can be complemented by [Gr4]. The survey [M3] by Vitali Milman, to whom we owe the present status of the concentration of measure phenomenon, is highly relevant and rich in material, especially if read in conjunction with a recent account of the subject by the same author [M4]. The book [M-S] is, in a sense, indispensable and should always be within one's reach. Talagrand's fundamental paper [Ta1] has to be at least browsed by every learner of the subject, while the paper [Ta2] of the same author offers an independent introduction to the subject of concentration of measure. The newly-published book by Ledoux [Led], apparently the first ever monograph devoted exclusively to concentration, is highly readable and covers a wide range of topics. Don't miss the introductory survey by Schechtman [Sch]. The modern setting for concentration was designed in the important paper [Gr-M1] by Gromov and Milman, which had also introduced the subject of this lecture and from which many results (perhaps with slight modifications) have been taken.

<sup>6</sup>) Or, more generally, a uniform space — for instance, a non-metrizable compact space — with measure.