6.3 Ramsey's theorem

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **48 (2002)**

Heft 3-4: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **24.09.2024**

The studies of the phenomenon of concentration of measure were given a boost by Vitali Milman's new proof of the Dvoretzky theorem [M1], based on a suitable finite-dimensional approximation to the lemma which follows directly from results that we have previously stated:

LEMMA (Milman). *The pair* $(\mathrm{U}(\mathcal{H}), \mathbf{S}^\infty)$ *has the R–D–M property, where* $\mathbf{S}^\infty$ *is the unit sphere of an infinite-dimensional Hilbert space* $\mathcal{H}$.

6.3 RAMSEY'S THEOREM

Let r be a positive natural number. By $[r]$ one denotes the set $\{1,2,\dots,r\}$. A *colouring* of a set X with r colours, or simply *r-colouring,* is any map $\chi\colon X \to [r]$. A subset $A \subseteq X$ is *monochromatic* if for every $a,b \in A$ one has $\chi(a) = \chi(b)$.

Put otherwise, a finite colouring of a set X is nothing but a partition of X into finitely many (disjoint) subsets.

Let X be a set, and let k be a natural number. Denote by $[X]^k$ the set of all k-subsets of X, that is, all (unordered!) subsets containing exactly k elements.

INFINITE RAMSEY THEOREM. *Let* X *be an infinite set, and let* k *be a natural number. For every finite colouring of* $[X]^k$ *there exists an infinite subset* $A \subseteq X$ *such that the set* $[A]^k$ *is monochromatic.*

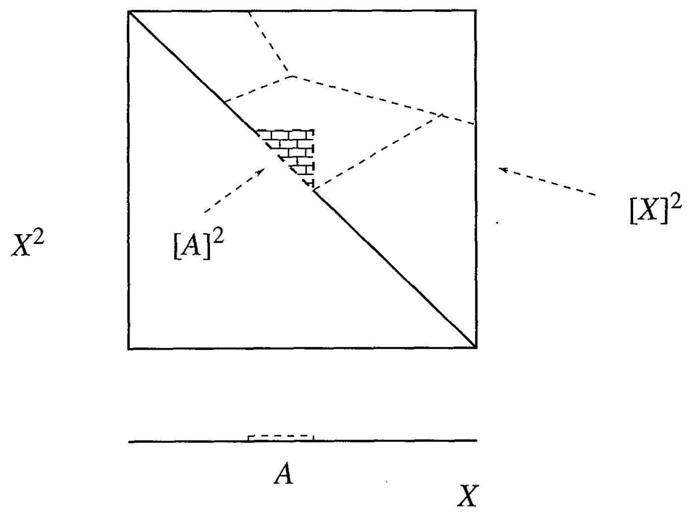

FIGURE 4
Ramsey theorem for $k = 2$

REMARK 10. For $k = 1$ the statement is simply the pigeonhole principle. Here is a popular interpretation of the result in the case $k = 2$. Among infinitely many people, either there is an infinite subset of people every two of whom know each other, or there is an infinite subset no two members of which know each other.

FINITE RAMSEY THEOREM. *For every triple of natural numbers, k, l, r, there exists a natural number $R(k, l, r)$ with the following property. If $N \geq R(k, l, r)$ and the set of all k-subsets of $[N]$ is coloured using r colours, then there is a subset $A \subseteq [N]$ of cardinality $|A| = l$ such that all k-subsets of A have the same colour.*

REMARK 11. The Infinite Ramsey Theorem implies the finite version through a simple compactness argument. At the same time, the infinite version does not seem quite to follow from the finite one. The finite version is equivalent to the following statement:

Let X be an infinite set, and let k be a natural number. For every finite colouring of $[X]^k$ and every natural n there exists a subset $A \subseteq X$ of cardinality n such that $[A]^k$ is monochromatic.

A good introductory reference to Ramsey theory is [Gra].

Denote by $\mathrm{Aut}(\mathbf{Q})$ the group of all order-preserving bijections of the set of rational numbers, equipped with the topology of pointwise convergence on the discrete set $\mathbf{Q}$. In other words, we regard $\mathrm{Aut}(\mathbf{Q})$ as a (closed) topological subgroup of S_∞. A basic system of neighbourhoods of identity is formed by open subgroups each of which stabilizes elements of a given finite subset of $\mathbf{Q}$.

EXERCISE 10. Use Corollary 7 to prove that the finite Ramsey theorem is equivalent to the statement:

The topological group $\mathrm{Aut}(\mathbf{Q})$ *is extremely amenable.*

[*Hint.* For a finite subset $M \subset \mathbf{Q}$, the left factor space of $\mathrm{Aut}(\mathbf{Q})$ by the stabilizer of M can be identified with the set $[\mathbf{Q}]^n$, where $n = |M|$, equipped with the discrete uniformity (or $\{0, 1\}$-valued metric). Cover $[\mathbf{Q}]^n$

with finitely many sets on each of which the given function f has oscillation $< \varepsilon$, and apply Ramsey's theorem. Use Remark 11.]

6.4 Extreme amenability and minimal flows

Corollary 8. *The group of orientation-preserving homeomorphisms of the closed unit interval,* $\mathrm{Homeo}_+(\mathbf{I})$, *equipped with the compact-open topology, is extremely amenable.*

Proof. Indeed, the extremely amenable group $\mathrm{Aut}(\mathbf{Q})$ admits a continuous monomorphism with a dense image into the group $\mathrm{Homeo}_+(\mathbf{I})$.

Remark 12. Thompson's group F consists of all piecewise-linear homeomorphisms of the interval whose points of non-smoothness are finitely many dyadic rational numbers, and the slopes of any linear part are powers of 2. (See [CFP].) It is a major open question in combinatorial group theory whether the Thompson group is amenable. Since F is everywhere dense in $\mathrm{Homeo}_+(\mathbf{I})$, our Corollary 8 does not contradict the possible amenability of F.

Using the extreme amenability of the topological groups $\mathrm{Aut}(\mathbf{Q})$ and $\mathrm{Homeo}_+(\mathbf{I})$, one is able to compute explicitly the universal minimal flows of some larger topological groups as follows.

Corollary 9. *The circle* $\mathbf{S}^1$ *forms the universal minimal* $\mathrm{Homeo}_+(\mathbf{S}^1)$-*space.*

Proof. Let $\theta \in \mathbf{S}^1$ be an arbitrary point. The isotropy subgroup St_θ of θ is isomorphic to $\mathrm{Homeo}_+(\mathbf{I})$. Because of that, whenever the topological group $\mathrm{Homeo}_+(\mathbf{S}^1)$ acts continuously on a compact space X, the subgroup St_θ has a fixed point, say $x' \in X$. The mapping $\mathrm{Homeo}_+(\mathbf{S}^1) \ni h \mapsto h(x') \in X$ is constant on the left St_θ-cosets and therefore gives rise to a continuous equivariant map $\mathrm{Homeo}_+(\mathbf{S}^1)/\mathrm{St}_\theta \cong \mathbf{S}^1 \to X$.

For the above results concerning groups $\mathrm{Aut}(\mathbf{Q})$, $\mathrm{Homeo}_+(\mathbf{I})$, and $\mathrm{Homeo}_+(\mathbf{S}^1)$, see [P1].

Now denote by LO the set of all linear orders on $\mathbf{Z}$, equipped with the (compact) topology induced from $\{0,1\}^{\mathbf{Z}\times\mathbf{Z}}$. The group S_∞ acts on LO by double permutations.