

## §4. Numerical results

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TABLE 2

$q$	32	128	512	2048	8192
interval	[0, 6]	[0, 12]	[10, 34]	[64, 108]	[300, 384]

§4. NUMERICAL RESULTS

In order to obtain numerical results on  $N(A, B)$  to test our heuristics the first remark is that  $N(A_1, B) = N(A_2, B)$  if  $\text{Tr}(A_1) = \text{Tr}(A_2)$ . So we have to distinguish only between  $\text{Tr}(A) = 0$  and  $\text{Tr}(A) = 1$ . We shall compute the trace of Frobenius for the seven factors of our Jacobian. We shall write  $f_4 = f_1 + f_2$ ,  $f_5 = f_1 + f_3$ ,  $f_6 = f_2 + f_3$  and  $f_7 = f_1 + f_2 + f_3$ . The Jacobians of the curves  $C_{f_i}$  given by  $y^2 + y = f_i$  for  $i = 1, \dots, 7$  constitute the seven factors of  $\text{Jac}(C_{A,B})$ . We write

$$n_{f_i} = \#\{x \in \mathbf{F}_q^* : \text{Tr}(f_i(x)) = 0\}.$$

(4.1) PROPOSITION. *The number of solutions  $N(A, B)$  over  $\mathbf{F}_{q=2^m}$  with  $m$  odd of the system (1) with  $\lambda = A^2 + A + 1 + B \neq 0$  is given by*

$$N(A, B) = \frac{2q - 2 - 2(n_{f_1} + n_{f_2} + n_{f_3} - n_{f_4} - n_{f_5} - n_{f_6} + n_{f_7})}{24} \quad \text{if } \text{Tr}(A) = 0,$$

and

$$N(A, B) = \frac{-6q - 2 + 2 \sum_{i=1}^7 n_{f_i}}{24} \quad \text{if } \text{Tr}(A) = 1.$$

*Proof.* As just explained we may take  $A = 0$  or  $A = 1$ . Then  $\lambda = B + 1 \neq 0$  and we set  $f_1 = (B + 1)(x^3 + x)$ ,  $f_2 = (B + 1)(1/x^3 + 1/x)$  and  $f_3 = (B + 1)(x + 1/x)$ . Then  $C_{1,B} = C_{f_1} \times_{\mathbf{P}^1} C_{f_2} \times_{\mathbf{P}^1} C_{f_3}$  and  $C_{0,B} = C_{f_1+1} \times_{\mathbf{P}^1} C_{f_2+1} \times_{\mathbf{P}^1} C_{f_3+1}$ . As in Theorem (2.3) the curves  $C_{f_i}$  for  $i = 4, \dots, 7$  give the remaining traces of Frobenius.

The trace of Frobenius  $t(C_{f_i})$  is of the form

$$t(C_{f_i}) = q + 1 - 2n_{f_i} - a_i,$$

where  $a_i$  is the contribution of  $x = 0, \infty$ , while the trace of Frobenius of  $C_{f_i+1}$  is

$$t(C_{f_i}) = -q + 3 + 2n_{f_i} - b_i,$$

where  $b_i$  is the contribution of  $x = 0, \infty$ . By analyzing these contributions from 0 and  $\infty$  one gets the proposition.

We now give tables with the distribution of the numbers  $N(A, B)$  for  $q = 2^m$  with  $m$  odd and  $5 \leq m \leq 13$ . These tables are obtained by computing the numbers  $n_{f_i}$  and they solve the coset weight distribution problem for the corresponding  $BCH(3)$  codes. The first unknown case up to now was  $q = 2^9$ , see [C-Z]. Moreover, the tables confirm our heuristics. We list the frequencies divided by  $q/2$ .

TABLE 3

$q = 2^5 :$

$N(A, B)$	0	2
frequency	27	35

$q = 2^7 :$

$N(A, B)$	0	2	4	6	8	10
frequency	2	28	98	84	35	7

$q = 2^9 :$

$N(A, B)$	12	14	16	18	20	22	24	26	28	30	32
frequency	18	21	117	180	148	195	199	81	36	18	9

$q = 2^{11} :$

$N(A, B)$	66	68	70	72	74	76	78	80	82	84	86
frequency	22	66	88	55	176	264	187	374	374	374	451
$N(A, B)$	88	90	92	94	96	98	100	102	104	106	108
frequency	365	341	275	341	154	44	55	33	11	22	22

$q = 2^{13} :$

In this case we encounter a new phenomenon. The function  $N(A, B)$  assumes even values in the interval  $[290, 390]$ , but not all even values are taken. This contradicts the expectation of [C-Z] that the values form a sequence

of even integers without gaps. The frequency divided by  $q/2$  of the value  $290 + 2\ell$  with  $0 \leq \ell \leq 50$  is given by

$$13 \gamma_\ell + \begin{cases} 1 & \text{if } \ell = 11, \\ 1 & \text{if } \ell = 37, \\ 0 & \text{else,} \end{cases}$$

where  $\gamma = (\gamma_0, \dots, \gamma_{50})$  is the vector

$$\gamma = (1, 0, 1, 0, 1, 0, 6, 3, 5, 5, 12, 7, 19, 15, 22, 25, 37, 40, 43, 37, 35, 60, 54, 72, 72, 58, 65, 61, 57, 57, 63, 48, 35, 44, 34, 34, 25, 29, 25, 15, 9, 7, 2, 3, 7, 3, 3, 1, 0, 1, 2).$$

In accordance with our heuristics less than 1% of the  $N(A, B)$  lie outside the interval  $[300, 384]$ .

### §5. THE COVERING RADIUS

A problem in coding theory that precedes the coset weight distribution problem is the determination of the covering radius. It is defined for a binary linear code  $\mathcal{C}$  of length  $n$  as the smallest integer  $\rho$  such that the spheres of radius  $\rho$  around the codewords cover  $\mathbf{F}_2^n$ . Equivalently, it is the maximum weight of a coset leader (by which we mean a vector of minimum weight in a coset of  $\mathcal{C}$  in  $\mathbf{F}_2^n$ ). It is an interesting parameter of a code since it provides information on the performance of the code when used in data compression.

In a series of papers [H-B], [A-M] and [H], of which [H-B] and [H] treat the case  $m$  even and [A-M] the case  $m$  odd, it was proved that the  $BCH(3)$  code of length  $n = 2^m - 1$  has covering radius

$$\rho(BCH(3)) = 5 \quad \text{for } m \geq 4.$$

The proofs for the various cases are very different. Using algebraic geometry we can give a unified proof.

In order to prove that  $\rho(BCH(3)) = 5$  we have to show that for every  $(A, B, C) \in \mathbf{F}_q^3$  the system of equations:

$$(15) \quad \begin{aligned} x_1 + \dots + x_5 &= A, \\ x_1^3 + \dots + x_5^3 &= B, \\ x_1^5 + \dots + x_5^5 &= C, \end{aligned}$$

has a solution  $(x_1, \dots, x_5) \in \mathbf{F}_q^5$ . On replacing  $x_i$  by  $x_i + A$  we may assume without loss of generality that  $A = 0$  and  $(B, C) \neq (0, 0)$ . If we then