1. Introduction

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ALMOST COMPLEX STRUCTURES ON 8-MANIFOLDS

by Stefan MÜLLER and Hansjörg GEIGES

1. INTRODUCTION

Throughout this paper, M denotes a closed, connected, smooth and oriented manifold of even dimension $2n$. Our main goal is to give a criterion for the existence of almost complex structures on 8-dimensional manifolds.

Recall that an almost complex structure (a.c.s.) on M is an endomorphism J of the tangent bundle TM satisfying $J^2 = -1$. This gives TM the structure of a complex vector bundle, and we write $c_i(J)$ for its Chern classes. The orientation of M is required to coincide with the orientation induced by this complex vector bundle structure on TM.

Equivalently, we may think of an almost complex structure as a reduction of the structure group from the special orthogonal group SO_{2n} to the unitary group U_n. In enumeration questions one is of course interested in the classification of almost complex structures up to homotopy. As is well-known [15, §9.5], another equivalent way to think about an almost complex structure is as a cross-section of the $\mathrm{SO}_{2n} / \mathrm{U}_n$-bundle associated to TM. This viewpoint will be particularly relevant in Section 3.

Similarly, a stable almost complex structure on M is a reduction of the structure group of the stable tangent bundle of M from SO to U.

Thus, necessary conditions for the existence of a stable a.c.s. are the existence of integral lifts $c_i \in H^{2i}(M; \mathbf{Z})$ of the even Stiefel-Whitney classes $w_{2i}(M) \in H^{2i}(M; \mathbf{Z}_2)$, that is, $w_{2i}(M) = \rho_2 c_i$, with ρ_2 denoting mod 2 reduction. Given an a.c.s. J, the class $c_n(J)$ will be the Euler class of TM (which we may identify with the Euler characteristic $\chi(M)$). Indeed, a stable a.c.s. $\widetilde{J}$ induces an a.c.s. if and only if $c_n(\widetilde{J}) = \chi(M)$, see [17].

An oriented surface clearly carries a unique a.c.s. since $SO_2 = U_1$. For $\dim M = 4$, the existence of an integral lift of $w_2(M)$ is necessary and sufficient for the existence of a stable a.c.s., and this condition is always satisfied, see [11]. Furthermore, the integral lifts c_1 of $w_2(M)$ completely classify the stable a.c.s. With $\tau(M)$ denoting the signature of M, a well-known result of Wu (cf. [11]) asserts that a.c.s. on M are classified by those integral lifts c_1 that satisfy the signature formula

$$\langle c_1^2, [M] \rangle = 3\tau(M) + 2\chi(M).$$

A reformulation of this theorem was found independently by Dessai [2] and the first author (unpublished). Write b_i for the Betti numbers of M, and b_2^+ (resp. b_2^-) for the dimension of the positive (resp. negative) eigenspace of the intersection form Q on $H_2(M; \mathbf{Z})$. Then, observing that $\chi(M) + \tau(M) = 2(1 - b_1 + b_2^+)$, Dessai's Theorem 1.4 can be stated as follows.

THEOREM 1 (Dessai). *An oriented 4-manifold M admits an almost complex structure if and only if $\chi(M) + \tau(M) \equiv 0 \mod 4$ and one of the following conditions is satisfied:*

(i) *Q is indefinite.*

(ii) *Q is positive definite and $b_1 - b_2 \leq 1$.*

(iii) *Q is negative definite and, in case $b_2 \leq 2$, $4(b_1 - 1) + b_2$ is the sum of b_2 integer squares.*

REMARK. Observe that in case (iii) with b_2 equal to 1 or 2, the condition $\chi(M) + \tau(M) \equiv 0 \mod 4$ is implied by the other conditions stated.

The advantage of this formulation over that of Wu lies in the fact that the existence of an a.c.s. is expressed solely in terms of topological invariants of M rather than by requiring the existence of a solution to the signature formula in the potentially infinite set $\rho_2^{-1}(w_2(M)) \subset H^2(M; \mathbf{Z})$.

Note that for manifolds of dimension $4k$, the condition

$$\chi(M) \equiv (-1)^k \tau(M) \mod 4$$

is necessary for the existence of an a.c.s., as was observed by Hirzebruch [10, p. 777]. This follows from an integrality argument involving the Todd genus.

Dessai also gives a finiteness criterion for a.c.s. The following is a direct consequence of [2, Thm. 2.2].

PROPOSITION 2 (Dessai). *Let M be a 4-manifold admitting an a.c.s. There exist only finitely many a.c.s. on M if and only if one of the following conditions holds:*

(i) *The intersection form Q of M is definite.*

(ii) *Q is indefinite, $b_1 \neq 2$, $b_2 = 2$.*

Observe that the obstructions to the existence of an a.c.s. on M (with $\dim M = 2n$) lie in $H^k(M;\pi_{k-1}(\mathrm{SO}_{2n}/\mathrm{U}_n))$. For $\dim M = 6$, the only non-zero coefficient group here is $\pi_2(\mathrm{SO}_6/U_3) \cong \mathbf{Z}$ (cf. [13]). The obstruction to the existence of an a.c.s. on a 6-manifold has been identified as the third integral Stiefel-Whitney class $W_3(M) = \beta w_2(M) \in H^3(M;\mathbf{Z})$, where β denotes the Bockstein homomorphism induced by the coefficient sequence $\mathbf{Z} \overset{2}{\rightarrowtail} \mathbf{Z} \twoheadrightarrow \mathbf{Z}_2$ (notice that $W_3(M) = 0$ is equivalent to the existence of an integral lift of $w_2(M)$). Indeed, $W_3(M)$ is the first obstruction to the existence of an a.c.s. in any dimension, see [13]. Furthermore, homotopy classes of a.c.s. on a 6-manifold are classified by the integral lifts c_1 of $w_2(M)$, cf. [4].

The corresponding existence result for 8-dimensional manifolds is due to Heaps [6]. Write Sq^2 for the Steenrod square and $p_i(M)$ for the Pontrjagin classes of M. In the sequel, cohomology classes in $H^8(M;\mathbf{Z})$ are usually interpreted as integers under the evaluation on the fundamental cycle $[M]$.

THEOREM 3 (Heaps). (a) *An oriented 8-dimensional manifold M admits a stable almost complex structure if and only if $\beta w_2(M) = 0$ and $w_8(M) \in \mathrm{Sq}^2 H^6(M;\mathbf{Z})$. In this case, any integer lift of $w_2(M)$ can be realized as $c_1(J)$ of some stable almost complex structure J. Furthermore, any pair $(u,v) \in H^2(M;\mathbf{Z}) \times H^6(M;\mathbf{Z})$ with $(\rho_2(u), \rho_2(v)) = (w_2(M), w_6(M))$ can be realized as $(c_1(J), c_3(J))$ for some stable almost complex structure J provided* $2\chi(M) + uv \equiv 0 \mod 4$.

(b) *M admits an almost complex structure if and only if the conditions in* (a) *are satisfied and if there is a pair (u,v) as in* (a) *with*

$$8\chi(M) - 4p_2(M) + p_1^2(M) = 8uv - u^4 + 2u^2 p_1(M).$$

Implicit in statement (a) is a result of Massey that $\beta w_6(M) = 0$ for any 8-dimensional M. Heaps also proves a corresponding statement for 10-manifolds under some additional topological assumptions, see also [2].

Our following main result stands in the same relation to that of Heaps as Dessai's theorem to that of Wu. Write $TH^2(M)$ for the torsion subgroup of $H^2(M;\mathbf{Z})$. We distinguish three cases:

- case I: $\rho_2^{-1}(w_2(M)) \cap TH^2(M) = \varnothing$.
- case II_+: $\rho_2^{-1}(w_2(M)) \cap TH^2(M) \neq \varnothing$ and $b_2(M) > 0$.
- case II_0 : $\rho_2^{-1}(w_2(M)) \cap TH^2(M) \neq \varnothing$ and $b_2(M) = 0$.

Notice that case I implies that $w_2(M) \neq 0$. An example that the converse is false is provided by $M = E \times S^4$, with E denoting an Enriques surface. Here $w_2(M) \neq 0$, but we are in case II_+.

THEOREM 4. (a) *An oriented 8-dimensional manifold M admits a stable almost complex structure if and only if $\beta w_2(M) = 0$ and, in case* II, $\chi(M) \equiv 0 \mod 2$.

(b) *M admits an almost complex structure if and only if the conditions of* (a) *hold and*

(i) $\chi(M) \equiv \tau(M) \mod 4$ *in cases* I *and* II_+,

(ii) $8\chi(M) - 4p_2(M) + p_1^2(M) = 0$ *in case* II_0.

REMARK. By the observation of Hirzebruch mentioned above, the condition $\chi(M) \equiv \tau(M) \mod 4$ is certainly necessary for the existence of an a.c.s. on an 8-manifold. This condition is implied by (b) (ii).

Here is a simple example. Give $\mathbf{C}P^4$ its natural orientation induced by the complex structure, and write $\overline{\mathbf{C}P^4}$ for the same manifold with the opposite orientation. Consider the connected sum $M = \#_r \mathbf{C}P^4 \#_s \overline{\mathbf{C}P^4}$. Then we are in case I and $\chi(M) - \tau(M) = 2r + 4s + 2$. So M admits an a.c.s. if and only if r is odd. There is an analogous statement for $\#_r \mathbf{C}P^2 \#_s \overline{\mathbf{C}P^2}$, see [1].

Theorem 4 will be derived from some explicit calculations of obstruction classes. Only Theorem 3(a), but not 3(b), will be used for the proof of Theorem 4. The following is an immediate corollary.

COROLLARY 5. *On 8-dimensional manifolds M with $b_2 > 0$ the existence of an a.c.s. depends only on the oriented homotopy type of M.* □

This is false if $b_2 = 0$.

PROPOSITION 6. *The manifold* $M = \mathbf{H}P^2 \# \mathbf{H}P^2 \# S^4 \times S^4$ *admits an almost complex structure. There are 8-manifolds homotopy equivalent to* M *which do not admit any almost complex structures.*

More generally, Kahn [12, Cor. 6] has shown, for every $k \geq 1$, the existence of pairs of closed, connected, oriented manifolds M_1, M_2 of dimension $8k$ such that M_1 and M_2 have the same oriented homotopy type, but only one of them admits an almost complex structure. According to the results already mentioned, dimension 8 is indeed the smallest dimension where this phenomenon can occur.

A further simple corollary concerns the compatibility with different choices of orientation.

COROLLARY 7. *In case* I, *if* M *admits an a.c.s., then* $\overline{M}$ (*i.e.* M *with reversed orientation*) *admits an a.c.s. if and only if* $\chi(M) \equiv 0 \mod 2$.

In case II_+, *either both* M *and* $\overline{M}$ *or none of them admits an a.c.s.*

In case II_0, *if* M *admits an a.c.s., then* $\overline{M}$ *admits an a.c.s. if and only if* $\chi(M) = 0$. □

See [1] for related statements in the 4-dimensional situation.

Using the first author's original method of proof (cf. the acknowledgements below), it is possible to determine the set of all pairs (u, v) satisfying the conditions of Theorem 3; see [14] for details in the torsion-free case. These results lead to the following finiteness criterion, which is equivalent to the corresponding case in Theorems 2.2 and 2.3 of [2].

PROPOSITION 8. *Assume the oriented* 8*-manifold admits an almost complex structure. Then the number of a.c.s. on* M *is finite if and only if* $b_2 = 0$, *or* $b_1 = 1$ *and* $8\chi(M) - 4p_2(M) + p_1^2(M) \neq 0$. □

We note in passing that applying this result to complete intersections corrects an error in [7]; see [14, §4.5.2.] and [2, §2.3].

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Theorem 4 in the general case is also due to the first author. The present joint note grew from the second author's observation that the original proof can be simplified and be made more geometric by appealing to considerations in [5].

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2. Stable almost complex structures

To derive Theorem 4(a) from Theorem 3(a) we merely have to show that the condition $w_8(M) \in \mathrm{Sq}^2 H^6(M;\mathbf{Z})$ is void in case I and equivalent to $\chi(M) \equiv 0 \mod 2$ (i.e. $w_8(M) = 0$) in case II.

Indeed, in case I we find an integral lift u of $w_2(M)$ whose free part is indivisible. Then there is a dual element $u' \in H^6(M;\mathbf{Z})$ such that $uu' = 1 \in H^8(M;\mathbf{Z})$. By the Wu formula it follows that

$$\mathrm{Sq}^2 H^6(M;\mathbf{Z}) = w_2(M)\, H^6(M;\mathbf{Z}) = \rho_2(u\, H^6(M;\mathbf{Z})) = H^8(M;\mathbf{Z}_2)\,.$$

In case II, on the other hand, we can lift $w_2(M)$ to a torsion class $u \in H^2(M;\mathbf{Z})$, thus

$$\mathrm{Sq}^2 H^6(M;\mathbf{Z}) = \rho_2(u\, H^6(M;\mathbf{Z})) = 0\,.$$

3. The top-dimensional obstruction

Assume we have an almost complex structure J_0 over M with a disc D^8 removed (which is homotopy equivalent to the 7-skeleton of M). Thinking of an almost complex structure J as a section of the SO_8/U_4-bundle associated to TM, we may interpret J_0 as such a section defined only over the 7-skeleton of M. The obstruction $\mathfrak{o}(M, J_0)$ to extending J_0 to an almost complex structure on M then lives in

$$H^8(M;\pi_7(\mathrm{SO}_8/U_4)) \cong \pi_7(\mathrm{SO}_8/U_4) \cong \mathbf{Z} \oplus \mathbf{Z}_2\,.$$

See [13] for references to the computations of the homotopy group above and others used below. The homotopy group involved here is in fact the