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FINITE GROUP ACTIONS ON THE 7-SPHERE

by Hansjörg GEIGES and Charles B. THOMAS

The aim of this note is to prove the following theorem.

THEOREM. (i) *Let π be a finite group acting freely and topologically on S^7. Then π can also act freely and linearly on S^7.*

(ii) *For odd natural numbers $2s-1$ different from 1, 3 or 7, one can always find a free smooth action on S^{2s-1} by a finite group which cannot act freely and linearly.*

A weaker version of this theorem was announced in [3]. The proof of the theorem uses only classical results about finite group actions on spheres, mostly from [15], and it is a little surprising that this theorem has not been observed before. The interest of this result in the context of more recent investigations on geometric structures on spherical space forms is explained in [3] and [4].

The (as yet unproven) analogue of part (i) of our theorem in dimension 3 forms an essential step in Thurston's geometrisation programme [13]. According to Thurston's conjecture, any 3-manifold covered by S^3 is actually a quotient of S^3 under a free linear action of some finite group π. This splits into proving the said analogue of our theorem, and then showing that the only possible actions of π are indeed linear, i.e. given by fixed-point free representations of π in SO(4). A proof of the former was announced in [10], but at present it is not clear whether all details of the argument can be filled in. For some groups acting on S^3 it is known that they can only act linearly, cf. [4]. See [5, Problems 3.37 and 3.45] for further references on this issue.

The only finite groups acting freely and topologically on S^1 are the cyclic groups, and any such action is conjugate to a linear one.

Notice that part (i) of our theorem does not make any statement about the possible actions of the groups π on S^7. As is well-known, and contrary to what one hopes to be true in dimension 3, there are many exotic smooth actions (i.e. actions not conjugate to a linear one) on spheres of dimension ≥ 5, even by cyclic groups, see [14, Chapter 14].

The proof of part (i) of the theorem is case by case and exploits the discussion of groups with periodic cohomology (or periodic groups, for short) in [15]. First we collect some fundamental and well-known facts. Recall that a finite group π is said to satisfy the pq-condition (p, q prime) if all subgroups of π of order pq are cyclic.

Since the group π acts freely on S^7, it has periodic cohomology, and the (minimal) cohomological period divides 8. As a periodic group, π satisfies all p^2-conditions, cf. [1, Thm. VI.9.5]. By [8] the fact that π acts freely on some sphere implies that all $2p$-conditions are satisfied; there are no free actions by a dihedral group on S^{2s-1}.

A group of order pq with p and q distinct odd primes must be either the cyclic group C_{pq} or the metacyclic group $\mathsf{D}_{p,q}$ given as a nontrivial extension of C_p by C_q. But $\mathsf{D}_{p,q}$ has cohomological period equal to $2q$, cf. [2, p. 229], so a group π as in part (i) of our theorem also satisfies all pq-conditions. Recall from [15, Thm. 5.3.1] that the pq-conditions are necessary for the existence of a free linear action on some sphere.

We now distinguish between the solvable and non-solvable cases.

A. If π is solvable, then Theorem 6.1.11 of Wolf's monograph [15] applies. This theorem gives a complete list of the finite solvable groups with periodic cohomology, separated into four classes, and states that such a group satisfies all pq-conditions if and only if π can act freely and linearly on some sphere. As it stands, however, it is not strong enough to guarantee the existence of a free linear action on S^7, so we need to take a closer look at the four classes in Wolf's classification theorem.

I. The first class consists of the metacyclic groups of order mn with presentation

$$\{A, B \mid A^m = B^n = 1\,, \quad BAB^{-1} = A^r\}\,,$$

where $m \geq 1$, $n \geq 1$, $\gcd(n(r-1), m) = 1$, and $r^n \equiv 1 \mod m$. Furthermore, the fact that π satisfies all pq-conditions is equivalent to the following: if d

is the smallest natural number such that $r^d \equiv 1 \mod m$, then n/d is divisible by every prime divisor of d. This numerical condition will be satisfied in all solvable cases **II** to **IV** below.

The cohomological period of π is $2d$ (cf. [2, p. 229]), so we must have $d \in \{1, 2, 4\}$. Furthermore, Wolf [15, Thm. 5.5.1] gives explicit fixed-point free linear representations of (real) degree $2d$. Recall that a linear representation $\rho\colon \pi \to \mathrm{GL}(2d)$ is called *fixed-point free* if $\rho(g)$ does not have 1 for an eigenvalue for $1 \neq g \in \pi$. Furthermore, a real linear representation is always equivalent to an orthogonal representation, so fixed-point free representations of degree $2d$ induce a free linear action on S^{2d-1} (and of course also on S^{2dk-1} for any positive integer k). In fact, one can always obtain unitary representations, cf. [16].

Notice that if π is a metacyclic group of odd order, then the only possibility for d is that it equal 1, which means that π is actually a cyclic group.

II. The second class of solvable groups π admit a presentation with generators A, B, R, relations as in **I**, and additional relations

$$R^2 = B^{n/2}, \quad RAR^{-1} = A^l, \quad RBR^{-1} = B^k,$$

subject to the numerical conditions as in **I** and the further conditions

$$l^2 \equiv r^{k-1} \equiv 1 \mod m, \qquad n = 2^u v, \quad u \geq 2, \quad v \text{ odd},$$
$$k \equiv -1 \mod 2^u, \qquad k^2 \equiv 1 \mod n.$$

The order of this group is $2mn$. With d defined as in **I**, again we must have $d \in \{1, 2, 4\}$, since the subgroup generated by A and B is of type **I** with cohomological period $2d$. According to [15, p. 180], the group π admits a fixed-point free representation of real dimension $4d$. So we only need to show that the case $d = 4$ cannot arise.

Indeed, on the one hand the numerical conditions stipulate $2^u \mid k+1$, so $4 \mid k+1$. On the other hand, since $r^{k-1} \equiv 1 \mod m$, we also know that $d \mid k-1$. Clearly this is impossible for $d = 4$.

We now want to identify such a group π of type **II** with one of the groups $\mathrm{C}_{m_1} \times \mathrm{Q}(2^{u+1}m_2, m_3, m_4)$ described by Milnor [8], with the m_i odd and pairwise coprime. We only summarise the observations necessary to make this identification.

The subgroup of π generated by R and B^v is isomorphic to the quaternion group $\mathrm{Q}(2^{u+1})$. The normal subgroup of π generated by A and B^{2^u} is cyclic of odd order mv, and we have a split extension

$$\mathrm{C}_{mv} \rightarrowtail \pi \twoheadrightarrow \mathrm{Q}(2^{u+1}).$$

The generators R and B^v of $\mathrm{Q}(2^{u+1})$ act on C_{mv} as a pair of commuting involutions, so we get a splitting

$$\mathrm{C}_{mv} = \mathrm{C}_{m_1} \times \mathrm{C}_{m_2} \times \mathrm{C}_{m_3} \times \mathrm{C}_{m_4}$$

with the m_i pairwise coprime and such that

- R and B^v act trivially on C_{m_1},
- B^v inverts elements in C_{m_2} and C_{m_3},
- R inverts elements in C_{m_2} and C_{m_4}.

This identifies π as a Milnor group as described above, cf. [2, p. 229].

Fixed-point free representations of the groups $\mathrm{Q}(2^{u+1}m_2, m_3, m_4)$ in SU(4) are described in [2, p. 255]. Taking the product with a group of coprime order never constitutes a problem: if ρ is a fixed-point free unitary representation of some group π, then

$$\widetilde{\rho}(t, g) = \exp(2\pi i/m)\rho(g)$$

defines a fixed-point free representation of $\mathrm{C}_m^t \times \pi$ if m is coprime to $|\pi|$.

III. The third class of solvable groups described by Wolf has a presentation with generators A, B, P, Q, relations and numerical conditions as in **I**, and further relations

$$P^4 = 1, \quad P^2 = Q^2 = (PQ)^2, \quad AP = PA, \quad AQ = QA,$$
$$BPB^{-1} = Q, \quad BQB^{-1} = PQ.$$

Furthermore, n has to be odd and divisible by 3.

As in the previous cases we have $d \in \{1, 2, 4\}$, and from $d \mid n$ we conclude $d = 1$. This implies the relation $BAB^{-1} = A$. Write n in the form $n = 3^u v$ with $u \geq 1$ and v not divisible by 3. Then the subgroup of π generated by P, Q and B^v is the generalised binary tetrahedral group T_u^* of order $8 \cdot 3^u$. With C_{mv} denoting the cyclic group of order mv generated by AB^{3^u} we have the split extension

$$\mathrm{C}_{mv} \rightarrowtail \pi \twoheadrightarrow \mathrm{T}_u^*,$$

and one easily verifies that AB^{3^u} commutes with the generators of T_u^*. So π is actually a direct product

$$\mathsf{C}_{mv} \times \mathsf{T}_u^* .$$

Notice that m must be odd (and also coprime to 3 by the conditions from **I**), otherwise π would contain a subgroup $\mathsf{C}_2 \times \mathsf{C}_2$ and would not be periodic. The group T_u^* is well known to act freely and linearly on S^3, and hence also on S^7, and the same is true for $\mathsf{C}_{mv} \times \mathsf{T}_u^*$.

IV. For the fourth class of solvable groups we only need to observe from Wolf's explicit presentation that they contain a normal subgroup of type **III** of index 2. Thus one may take a 4-dimensional fixed-point free representation of this normal subgroup, and then the induced 8-dimensional representation of the full group will also be fixed-point free (see [15, Lemma 5.5.3]), since the order of the normal subgroup is divisible by every prime divisor of the order of the full group. The groups in this class can be identified as extensions of a cyclic group (of order coprime to 6) by a generalised binary octahedral group O_u^*. As observed by Milnor [8], the latter have cohomological period 4 and fixed-point free representation in dimension 8, but none in dimension 4 unless $u = 1$.

This concludes the discussion of the solvable cases.

B. We now turn to the case that π is non-solvable. Notice that in this case the order of π must be even; otherwise, π being a periodic group, all Sylow subgroups are cyclic, and this would imply that π is solvable of type **I** by an old result of Burnside, cf. [15, Thm. 5.4.1].

By Suzuki's classification of periodic non-solvable groups [11, Thm. C], our π contains a subgroup isomorphic to $\mathrm{SL}_2(r)$ with $r \geq 5$ prime, the multiplicative group of (2×2)-matrices of determinant 1 with coefficients in the field of r elements (recall that $\mathrm{SL}_2(3) = \mathsf{T}_1^*$ is solvable). The cohomological period of $\mathrm{SL}_2(r)$ equals $\mathrm{lcm}(4, r-1)$, cf. [6, Lemma 1.3]. Since for a group π in our theorem this has to divide 8, the only possibility is $r = 5$. According to Suzuki, there are two types to consider.

V. The group π is the direct product of $\mathrm{SL}_2(5)$ (which has order 120) and a metacyclic group of order coprime to 30. By our comment at the end of **I**, this metacyclic group has to be cyclic, that is, $\pi = C_m \times \mathrm{SL}_2(5)$ with $\gcd(m, 30) = 1$. The group $\mathrm{SL}_2(5)$ is isomorphic to the binary icosahedral group I^* and has fixed-point free representations of real degree 4, and so the same holds for π.

VI. The final possibility is that π is an extension of a group of type **V** by a cyclic group of order 2. So we can argue as in **IV** to get a fixed-point free representation of real degree 8.

This concludes the proof of part (i) of the theorem.

Much of this discussion extends to free actions by finite groups π on S^{2^t-1}, in which case π must have period dividing 2^t. The result for solvable groups holds unchanged. To be more precise, in case **I** we obtain metacyclic groups of period $2d$ admitting fixed-point free representations of degree $2d$, with $d \in \{1, 2, 4, \ldots, 2^{k-1}\}$. In the cases **II**, **III** and **IV**, we get exactly the same groups as before (i.e. no higher values of d occur).

For non-solvable groups the restriction that the cohomological period divide 2^t implies that we only get groups $\mathrm{SL}_2(r)$ with r a prime of the form $r = 2^w + 1$ with $2 \leq w \leq t$, as well as extensions of these groups as described in **V** and **VI**. According to [12] these groups admit a free topological action on S^{2^w-1} (and hence on S^{2^t-1} by the join construction) which is conjugate to a free linear action when restricted to any proper subgroup; cf. the discussion in [4].

We now turn to the proof of part (ii). First suppose that $2s$ is divisible by some odd prime q and write $2s = 2kq$. Choose an odd prime p that divides $2^q - 1$. By the Fermat-Euler theorem we know that $2^{p-1} - 1$ is also divisible by p, so q has to be a divisor of $p - 1$, and in particular p and q will be coprime.

We can therefore define a metacyclic (but not cyclic) group $\mathrm{D}_{p,q}$ as in **I** with $m = p$, $n = q$, and $r = 2$. By [9] this group acts freely and smoothly on S^{2q-1}, and hence also on S^{2s-1}, the join of k copies of S^{2q-1}.

Finally, if $2s = 2^k$, $k \geq 4$, we appeal to the result of [7] that $\mathrm{SL}_2(r)$ acts freely and smoothly on S^{2s-1} with $2s = \mathrm{lcm}(4, r-1)$, that is, on a sphere of the dimension predicted by the cohomological period. In the particular case at hand, we can choose $r = 17$. The group $\mathrm{SL}_2(17)$ admits a free smooth action on S^{16-1} and hence on the join S^{2s-1} of 2^{k-4} copies of S^{15}. But the only groups $\mathrm{SL}_2(r)$ that can act freely and linearly on some sphere are $\mathrm{SL}_2(3)$ and $\mathrm{SL}_2(5)$, see [15, Thm. 6.3.1] – even though the non-solvable groups $\mathrm{SL}_2(r)$, $r \geq 5$ prime, satisfy all pq-conditions for primes r of the form $r = 2^l + 1$ (and only for these), cf. [15, p. 197].

This concludes the proof of part (ii).

The last comment in the proof above and our earlier remarks about actions on S^{2^t-1} also imply the following: If π acts freely and topologically on S^{2^t-1}, then π satisfies all pq-conditions. On S^{2s-1} with $2s \neq 2^t$ there are always free smooth actions by metacyclic groups $D_{p,q}$ violating the pq-conditions.

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