3. A REDUCTION

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To prove the Spectral Mapping Theorem it suffices to verify that it holds for the polynomial ring $k[t_0, \ldots, t_m]$ in variables $t_0, \ldots, t_m$ over k, and for the polynomial $F(x) = t_0 + t_1 x + \cdots + t_m x^m$. This is because any polynomial $G(x) = b_0 + b_1 x + \cdots + b_m x^m$ with coefficients in a ring R containing k as a subring is the image of $F(x)$ by the map $g \colon k[t_0, \ldots, t_m][x] \to R[x]$ defined by $g(a) = a$ for $a \in k$, by $g(t_i) = b_i$ for $i = 0, \ldots, m$, and by $g(x) = x$. If we can prove the equality $\det F(M) = \prod_{i=1}^n F(\lambda_i)$ in $k[t_0, \ldots, t_m]$ we obtain that $\det G(M) = g(\det F(M)) = \prod_{i=1}^n g(F(\lambda_i)) = \prod_{i=1}^n G(\lambda_i)$ in R.

4. THE PROOF

Clearly (2.1) holds when F is a constant a where it simply states that $\det(aI_n) = a^n$. We shall prove (2.1) for polynomials F of degree $m > 0$ by induction on m.

We first note that if $F(x)$ has a root λ in R, so that $F(x) = (x - \lambda)G(x)$ in $R[x]$, then (2.1) holds for $F(x)$. Indeed, $G(x)$ is of degree $m - 1$ so it follows from the induction hypothesis that $\det G(M) = \prod_{i=1}^n G(\lambda_i)$. Since $F(M) = (M - \lambda I_n)G(M)$ we obtain:

$$\begin{aligned} \det F(M) &= \det(M - \lambda I_n) \det G(M) \\ &= \prod_{i=1}^n (\lambda_i - \lambda) \prod_{i=1}^n G(\lambda_i) = \prod_{i=1}^n (\lambda_i - \lambda) G(\lambda_i) = \prod_{i=1}^n F(\lambda_i). \end{aligned}$$

As we saw in Section 3 it suffices to prove the result for the ring $Q = k[t_0, \ldots, t_m]$ and the polynomial $F(x) = t_0 + t_1 x + \cdots + t_m x^m$. Let x and y be independent variables over the ring Q. The polynomial $F(x) - F(y)$ in x with coefficients in $Q[y]$ has the root $x = y$. Hence, as we just observed, (2.1) holds for the polynomial $F(x) - F(y)$. We obtain the equation:

$$\det(F(M) - F(y)I_n) = \prod_{i=1}^n (F(\lambda_i) - F(y)) \tag{4.1}$$

in $Q[y]$.

The equation (2.1) is a consequence of (4.1). To see this we observe that $F(y)$ in $Q[y]$ is transcendent over Q, that is the element $F(y)$ in $Q[y]$ does not satisfy a polynomial relation $a_0 + a_1 F(y) + \cdots + a_l F(y)^l = 0$ with coefficients a_i in Q and $a_l \neq 0$, because the coefficient $a_l t_m^l$ of the highest