6. Replacing finite type geodesics by curve diagrams

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **46 (2000)**

Heft 3-4: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **18.04.2024**

For every total curve diagram in D_n there are two possibilities:

(a) the first arc of the curve diagram ends in a puncture or can be pulled tight so as to end in a puncture;

(b) the first arc cuts D_n into two disks, each of which contains at least two punctures.

For case (a) we notice that the first arc can be turned into the horizontal arc from -1 to the leftmost puncture, by an action of some appropriate element of B_n. There are now precisely M_{n-1} orbits of loose isotopy classes of curve diagrams of the remaining $n-2$ arcs in the $n-1$-punctured disk $D_n \backslash$(the first arc). So case (a) gives a contribution of M_{n-1} orbits.

The argument for case (b) is similar: the action of an appropriate element of B_n will turn the first arc of any curve diagram of type (b) into the vertical arc, oriented from bottom to top, having k punctures on its left and $n-k$ on its right, for some $k \in \{2, \ldots, n-2\}$. In this case, there should be $k-1$ arcs on the left and $n-k-1$ arcs on the right of the first arc, so there are $\binom{n-2}{k-1}$ ways to distribute the remaining $n-2$ arcs over the two sides. Finally, there are M_k respectively M_{n-k} orbits of loose isotopy classes of total curve diagrams on the disk on the left respectively on the right. □

6. Replacing finite type geodesics by curve diagrams

In this section we prove the main theorems on orderings of finite type. The strategy is to associate to every geodesic of finite type a curve diagram such that the (possibly partial) orderings arising from the geodesic and the curve diagram agree. Thus we obtain, via curve diagram orderings, a good understanding of finite type orderings.

Proof of Theorem 3.3 (a). If $D_n \backslash \gamma_\alpha$ has a path component which contains at least two holes, then we can push the boundary curve of this path component slightly into its interior, to make it disjoint from γ_α. A Dehn twist along such a curve will be a nontrivial element of B_n, and act trivially on γ_α. □

We now define *the curve diagram $C(\gamma_\alpha)$ associated to a geodesic γ_α* of finite type. It is a subset of γ_α, more precisely a union of segments of γ which start and end at self-intersection points. The diagram will be disjoint from the punctures, except that the last arc may fall into a puncture. For simplicity we shall write Γ for $C(\gamma_\alpha)$ and, as before, $\Gamma_{0\cup\ldots\cup i-1}$ for $\bigcup_{k=0}^{i-1} \Gamma_k$.

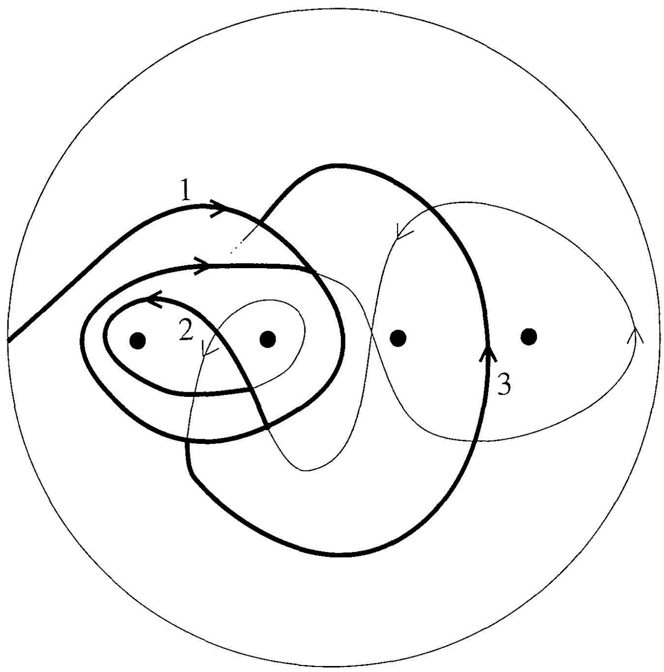

FIGURE 7
A geodesic and (in bold line) its associated curve diagram

The definition is inductive. We define $\Gamma_0 = \partial D_n$. Now suppose that we have already found $\Gamma_0, \ldots, \Gamma_{i-1}$. So every path component of $D_n \backslash \Gamma_{0\cup\cdots\cup i-1}$ is a disk containing at least one puncture. We put down a pencil at the end point of Γ_{i-1}, start tracing out γ_α, drawing an arc Γ_i^p (with "p" standing for "potential", because Γ_i^p is potentially the new arc Γ_i). We continue drawing either up to the next intersection with $\Gamma_{0\cup\cdots\cup i-1}$, or up to the first self intersection of Γ_i^p, or until γ_α falls into a puncture, whichever comes first. We now decide whether or not Γ_i^p has cut one of the components of $D_n \backslash \Gamma_{0\cup\cdots\cup i-1}$ in a nontrivial way, i.e. whether it has either fallen into a puncture or cut one of the components of $D_n \backslash \Gamma_{0\cup\cdots\cup i-1}$ into two, both of which contain at least one puncture. If yes, we let $\Gamma_i := \Gamma_i^p$, and have finished the induction step. If not, we rub out Γ_i^p, and start a new Γ_i^p at the next intersection point of γ_α with $D_n \backslash \Gamma_{0\cup\cdots\cup i-1}$. (This intersection point is just the end point of the previous Γ_i^p, unless this endpoint is in the interior of the previous Γ_i^p. Note that in this latter case not only Γ_i^p, but the entire segment of the geodesic γ_α up to its next intersection point with $\Gamma_{0\cup\cdots\cup i-1}$ cuts the disk in a trivial way.)

There is one special rule: if in the construction process we obtain an arc Γ_i^p which spirals *ad infinitum* towards a simple closed geodesic, then we define Γ_i to be the arc with end point in its own interior containing Γ_i^p in a regular neighbourhood, as shown in Figure 8 (this arc is unique up to loose isotopy).

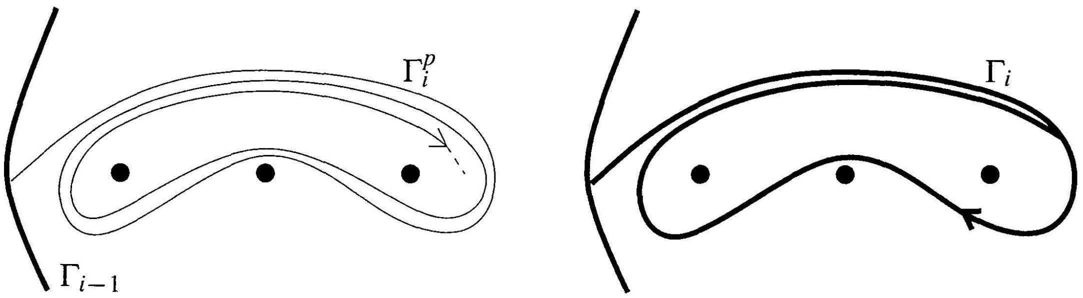

FIGURE 8
The curve diagram associated to a geodesic which spirals towards a closed geodesic

Since at most $n-1$ arcs can be constructed in this way, the process terminates after finitely many steps. We observe that the curve diagram $C(\gamma_\alpha)$ is total if and only if the geodesic γ_α fills D_n. More generally, two punctures are in the same path component of $D_n \backslash \gamma_\alpha$ if and only if they are in the same path component of $D_n \backslash C(\gamma_\alpha)$. We also note that for every geodesic γ_α and $\varphi \in B_n$ we have $C(\varphi(\gamma_\alpha)) = \varphi(C(\gamma_\alpha))$.

THEOREM 6.1. *For any* $\alpha \in (0, \pi)$ *and* $\varphi \in B_n$ *we have:*
(a) *if the curve diagrams* $\varphi(C(\gamma_\alpha))$ *and* $C(\gamma_\alpha)$ *are isotopic then* $\varphi(\alpha) = \alpha$*;*
(b) *if* $\varphi(C(\gamma_\alpha)) > C(\gamma_\alpha)$ *(in the curve diagram sense) then we have* $\varphi(\alpha) > \alpha$ *in* $\mathbf{R}$.

COROLLARY 6.2. *For every geodesic* γ_α *of finite type (where* $\alpha \in (0, \pi)$*), the ordering of* B_n *associated to* α *by Remark 1.2(1) coincides with the ordering associated to the curve diagram* $C(\gamma_\alpha)$ *by Definition 4.2.*

Proof of the theorem. We shall need a generalisation of the concept of relative "reduction" of two simple curves in D_n to the case where one of the two curves is authorised to have self-intersections, but no D-disks with itself. For instance, we shall be interested in the case where one of the two curves is a simple geodesic, and the other is a homeomorphic image of a non-simple geodesic.

Suppose that C is a disjoint collection of simple closed geodesics and properly embedded geodesic arcs connecting distinct punctures in D_n. Then we say that $\varphi(\gamma_\alpha)$ is *reducible* with respect to C if there are D-disks enclosed by $\varphi(\gamma_\alpha)$ and C, i.e. if there are finite segments of $\varphi(\gamma_\alpha)$ and of C with the same start and end points which are homotopic with fixed end points. If $\varphi(\gamma_\alpha)$ is not reducible then we say it is *reduced* with respect to C. Equivalently, any component of the preimage of $\varphi(\gamma_\alpha)$ in the universal cover $\tilde{D_n}$ intersects any component of the preimage of C at most once.

Lemma 6.3. *One can pull $\varphi(\gamma_\alpha)$ tight with respect to C, i.e. there exists an isotopy of φ which makes $\varphi(\gamma_\alpha)$ and C reduced with respect to each other.*

Proof. The proof is an easy exercise – it is in fact similar to the proof of the "triple reduction lemma" 2.1 of [9]. □

We need some more notation. We still write Γ for $C(\gamma_\alpha)$, denote by j the number of arcs of Γ, and consider the partial curve diagrams $\Gamma_{0\cup\cdots\cup i-1}$ for $i \in \{1,\ldots,j\}$; all their arcs are geodesics. Every path component of $D_n \backslash \Gamma_{0\cup\cdots\cup i-1}$ contains at least one puncture in its interior. The boundary curve of each component with at least two punctures is isotopic to a unique simple closed geodesic, which bounds a disk (with these punctures in its interior) in D_n. Removing all these disks from D_n yields a planar surface with a number of geodesic boundary components (one of them being ∂D_n, the others corresponding to the at least twice punctured components of $D_n \backslash \Gamma_{0\cup\cdots\cup i-1}$) and a number of punctures (corresponding to once-punctured components of $D_n \backslash \Gamma_{0\cup\cdots\cup i-1}$). We denote this surface by $N\Gamma_{0\cup\cdots\cup i-1}$; it is a regular neighbourhood of $\partial D_n \cup \Gamma_{0\cup\cdots\cup i-1}$ in D_n, and contains the complete initial segment of the geodesic γ_α up to the starting point of the arc $\Gamma_i \subset \gamma_\alpha$.

We are now ready to prove the theorem. For part (a) suppose that we are given $\alpha \in (0,\pi)$, and $\varphi \in B_n$, and that the curve diagrams Γ and $\varphi(\Gamma)$ are isotopic. Then we can modify the map φ by an isotopy which fixes ∂D_n such that the restriction $\varphi|_{N\Gamma}$ becomes the identity map. But by construction of $\Gamma = C(\gamma_\alpha)$, the geodesic γ_α is entirely contained in $N\Gamma$, and is thus mapped identically. This proves part (a) of the theorem.

For part (b) suppose that we are given $\alpha \in (0,\pi)$ and $\varphi \in B_n$, and that for some $i \in \{1,\ldots,j\}$ the curve diagrams $\Gamma_{0\cup\cdots\cup i-1}$ and $\varphi(\Gamma_{0\cup\cdots\cup i-1})$ are isotopic, whereas $\varphi(\Gamma_i)$ is "more to the left" than Γ_i. Our aim is to prove that $\varphi(\alpha) > \alpha$, i.e. that the end points of the liftings of $\varphi(\gamma_\alpha)$ and γ_α on $\partial \tilde{D}_n^{=} \backslash \Pi \cong (0,\pi)$ are different, with that of $\varphi(\gamma_\alpha)$ being "higher" in Figure 1.

Firstly, the map φ sends $\Gamma_{0\cup\cdots\cup i-1}$ to a curve diagram which is isotopic to $\Gamma_{0\cup\cdots\cup i-1}$; therefore we can assume, after an isotopy of φ which fixes ∂D_n, that the restriction $\varphi|_{N\Gamma_{0\cup\cdots\cup i-1}}$ is the identity map. Note that γ_α, being a geodesic, is already reduced with respect to the collection of geodesics $\partial N\Gamma_{0\cup\cdots\cup i-1}$, and therefore $\varphi(\gamma_\alpha)$ is also reduced with respect to $\partial N\Gamma_{0\cup\cdots\cup i-1} = \varphi(\partial N\Gamma_{0\cup\cdots\cup i-1})$.

Next, we note that the arc Γ_i will cut precisely one of the components of $D_n \backslash N\Gamma_{0\cup\cdots\cup i-1}$ in two, and leave the other components untouched. This critical component is an at least twice punctured disk, and we shall denote it by D_c. The preimage of D_c in the universal cover $\widetilde{D_n}$ has many path components, but we shall be interested in one particular component $\widetilde{D_c}$, namely the one which is cut in two by the segment corresponding to $\Gamma_i \subset \gamma_\alpha$ in the geodesic $\widetilde{\gamma}_\alpha$ in $\widetilde{D_n}$.

We now distinguish three cases: firstly, the arc Γ_i falls into a puncture inside D_c; secondly, the arc Γ_i has its end point in $N\Gamma_{0\cup\cdots\cup i-1}$ (either on $\Gamma_{0\cup\cdots\cup i-1}$ or in the initial segment $\Gamma_i \cap N\Gamma_{0\cup\cdots\cup i-1}$ of Γ_i); thirdly, the end point of the arc Γ_i lies in the interior of D_c (and then necessarily in the interior of Γ_i).

The first case is the easiest: by an isotopy of φ which is fixed outside D_c we can pull $\varphi(\Gamma_i) \cap D_c$ tight with respect to $\Gamma_i \cap D_c$. The effect of this isotopy is to make the images of the liftings $\widetilde{\varphi}(\widetilde{\gamma}_\alpha) \cap \widetilde{D}_c$ and $\widetilde{\gamma}_\alpha \cap \widetilde{D}_c$ disjoint, except for the common starting point. Moreover, both liftings run inside $\widetilde{D}_c$ all the way to the circle at infinity. By the hypothesis that $\varphi(\Gamma) > \Gamma$, we have that an initial segment of $\widetilde{\varphi}(\widetilde{\gamma}_\alpha)$ lies to the left of the corresponding segment $\widetilde{\gamma}_\alpha$, and we conclude that its end point on the circle at infinity also lies more to the left. This proves the theorem in the first case.

LEMMA 6.4. *If γ is a (finite or infinite) geodesic starting on the boundary of the punctured disk D_c, and if φ is an automorphism of D_c which acts nontrivially on γ, then two liftings of γ and $\varphi(\gamma)$ to the universal cover $\widetilde{D_c}$ of D_c with the same starting point in $\partial\widetilde{D_c}$ have end points either on different components of $\partial\widetilde{D_c}$ (if γ is finite) or on different points at infinity (if γ is infinite).* $\square$

In the second case, we can pull the arc $\varphi(\Gamma_i) \cap D_c$ tight with respect to $\Gamma_i \cap D_c$ by an isotopy of φ as in the first case, thus making their liftings disjoint (except for the common starting point). We now have by hypothesis that the point of intersection of $\widetilde{\varphi}(\widetilde{\Gamma_i})$ with $\partial\widetilde{D_c}$ where $\widetilde{\varphi}(\widetilde{\Gamma_i})$ exits $\widetilde{D_c}$ lies to the left of the one of $\widetilde{\Gamma_i}$. By the previous lemma, the two points will even lie on different boundary components of $\widetilde{D_c}$, and therefore there is a point of $\partial\widetilde{D_c}$ between these two boundary components which lies on the circle at infinity. For the liftings of our geodesic and its image this means the following: $\widetilde{\gamma}_\alpha$ and $\widetilde{\varphi}(\widetilde{\gamma}_\alpha)$ enter $\partial\widetilde{D_c}$ at the same point, but exit into different components of $\widetilde{D_n} \backslash \widetilde{D_c}$, with $\widetilde{\varphi}(\widetilde{\gamma}_\alpha)$ choosing the one that lies more to the left. Since $\widetilde{\gamma}_\alpha$ and $\widetilde{\varphi}(\widetilde{\gamma}_\alpha)$ do not intersect $\partial\widetilde{D_c}$ again, they stay inside their chosen component of

$D_n^{\sim} \backslash D_c^{\sim}$. Hence we have for their end points that $\varphi(\alpha) > \alpha$, and the theorem is proved in the second case.

We now turn to the third case, which includes the possibility that γ_α spirals towards a closed geodesic inside D_c. We consider the arc $\Sigma := \Gamma_i'$ as in Figure 3, and for simplicity we choose Σ to be a geodesic arc. We denote by $D_{cc} \subset D_c$ the subdisk cut off by Σ (so that $\Sigma = \partial D_{cc}$). Since Σ is geodesic, we have that $\gamma_\alpha \cap D_c$ is reduced with respect to Σ. After an isotopy of φ inside D_c we can assume by Lemma 6.3 that the first component of $\varphi(\gamma_\alpha) \cap D_c$ (the one that contains $\varphi(\Gamma_i) \cap D_c$) is also reduced with respect to Σ. By the hypothesis that $\varphi(\Gamma_i)$ sets off more to the left than Γ_i, we are now in one of the situations indicated in Figure 9.

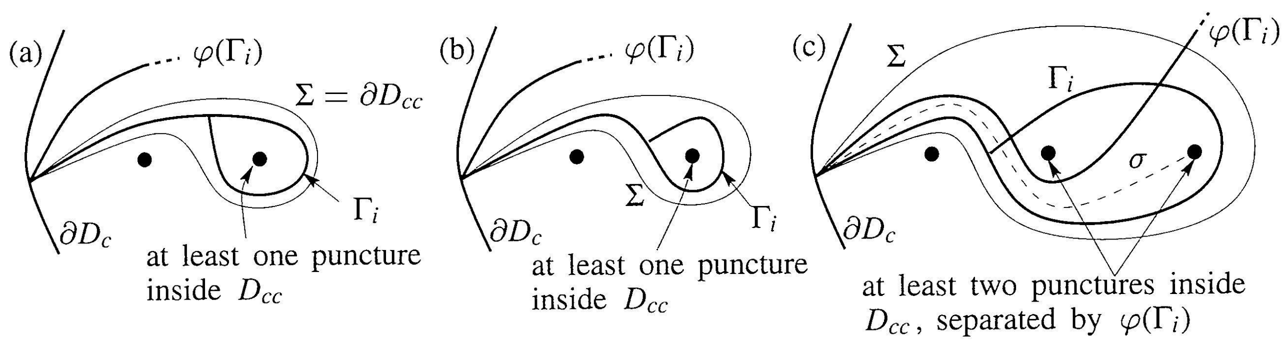

FIGURE 9

The critical disk D_c containing Γ_i and $\varphi(\Gamma_i)$

A first possiblity is that an initial segment of $\varphi(\Gamma_i) \cap D_c$ lies to the left of the tip of D_{cc} (Figures 9(a) and (b)); in the universal cover $D_c^{\sim}$ we now have three arcs, namely $\widetilde{\varphi}(\widetilde{\gamma}_\alpha) \cap D_c^{\sim}$, a lifting of Σ, and $\widetilde{\gamma}_\alpha \cap D_c^{\sim}$ (and, in fact, a fourth arc, another lifting of Σ) starting at the same point of $\partial D_c^{\sim}$, and setting off into different directions, namely in the given order from left to right. Moreover, the liftings of Σ are disjoint from the interiors of the other two arcs, by reducedness. Thus the end point of $\widetilde{\varphi}(\widetilde{\gamma}_\alpha) \cap D_c^{\sim}$ on $\partial D_c^{\sim}$ lies more to the left than that of $\widetilde{\gamma}_\alpha \cap D_c^{\sim}$. Even stronger, by Lemma 6.4 they lie either on different points at infinity (in which case we are done) or they leave $D_c^{\sim}$ through different components of $\partial D_c^{\sim}$ (in which case we argue as above that their remainders are trapped in different components of $D_n^{\sim} \backslash D_c^{\sim}$, so that $\widetilde{\varphi}(\widetilde{\gamma}_\alpha)$ stays to the left of $\widetilde{\gamma}_\alpha$).

The second possibility is that some initial segment of $\varphi(\Gamma_i) \cap D_c$ lies in D_{cc} (Figure 9(c)); then D_{cc}, cut along this initial segment, has precisely two path components, each of which contains at least one puncture. Since $\varphi(\Gamma_i)$ is oriented, we can refer to them as the "left" and the "right" half of D_{cc}. We now consider a geodesic arc σ which is embedded in the right half of D_{cc}, starts at the tip of D_{cc} (i.e. at the same point as $\Gamma_i \cap D_c$ and $\varphi(\Gamma_i) \cap D_c$), and falls into one of the punctures in the right half of D_{cc}. By construction, $\gamma_\alpha \cap D_{cc}$ is reduced with respect to σ, since both are geodesics, and the first component of $\varphi(\gamma_\alpha) \cap D_{cc}$ is even disjoint from σ. In the universal cover we now have that the lifting $\widetilde{\sigma}$ of σ ends on the circle at infinity, thus separating $\widetilde{D}_{cc}$ into two components, the left one containing the lift of $\varphi(\gamma_\alpha) \cap D_{cc}$, and the right one the lift of $\gamma_\alpha \cap D_{cc}$. Thus lifts of these two curves, not being allowed to intersect any component of $\partial D_{cc}^{\sim}$ and $\partial D_c^{\sim}$ more than once, go on to hit different points of $\partial D_n^{\sim}$, with $\widetilde{\varphi}(\widetilde{\gamma}_\alpha)$ staying more to the left than $\widetilde{\gamma}_\alpha$. This completes the proof of the third case, and thus of Theorem 6.1. □

Proof of Theorem 3.3 (b). If γ_α fills D_n, then $C(\gamma_\alpha)$ is a total curve diagram, and thus induces a *total* ordering of B_n. By Corollary 6.2, the ordering of B_n associated to the point $\alpha \in (0, \pi)$ agrees with this ordering. □

Proof of Theorem 3.4 (b). For any two geodesics γ_α and γ_β of finite type one can work out their associated curve diagrams $C(\gamma_\alpha)$ and $C(\gamma_\beta)$. By Corollary 6.2 it is sufficient to decide whether or not the orderings associated to the two curve diagrams coincide, which can be done by Theorem 5.2. □

Proof of Theorem 3.5. It only remains to be proved that $N_n = M_n$ (where M_n is given in Proposition 5.3), i.e. that every curve diagram is realized up to loose isotopy as $C(\gamma_\alpha)$ for some geodesic γ_α, $\alpha \in (0, \pi)$. This is left as an exercise to the reader. □

7. Orderings associated to geodesics of infinite type

In this section we prove the results concerning orderings of infinite type, and explain the essential differences between finite and infinite type orderings.

We start by describing in more detail than in Section 3 the structure of geodesics of infinite type. We define the *curve diagram* $C(\gamma_\alpha)$ *associated to a geodesic of infinite type* by precisely the same inductive construction procedure as in the finite type case. Except for a finite initial segment, the last arc Γ_j will lie in some path component D_c of $D_n \backslash N\Gamma_{0\cup\cdots\cup j-1}$, the only difference with the finite type case is that Γ_j goes on for ever, without falling into a puncture and without spiralling. The closure of Γ_j inside this critical component D_c is a geodesic lamination; the lamination has no closed leaves, for such a leaf would have to be the limit of an infinite spiral of Γ_j (see [17, Appendix]). All self-intersections of the geodesic γ_α occur inside the finite