4. Orderings of mapping class groups using curve diagrams

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We do not know if there exists a "closed" formula for N_n. The following list gives the first few values:

n	2	3	4	5	6	7	8
N_n	1	1	3	9	39	189	1197

Theorems 3.4 and 3.5 almost certainly generalise to mapping class groups of other negatively curved surfaces, but in order to keep our machinery simple, we stick to the special case of punctured disks.

4. Orderings of mapping class groups using curve diagrams

In this section we present another method for constructing left orderings on B_n, using certain diagrams on D_n, which we call *curve diagrams*. Both the definition of curve diagrams and the orderings associated to them are generalisations of similar concepts in [9].

Convention. Whenever we talk about geodesics in D_n, we think of the punctures as being holes in the disk, whose neighbourhoods on the disk have the geometry of cusps. By contrast, when we talk about curve diagrams, we think of the punctures as distinguished points on, and belonging to, the disk, and we ignore the geometric structure. This changing perspective should not cause confusion.

Definition 4.1. A (partial) curve diagram Γ is a diagram on D_n consisting of $j \leqslant n-1$ closed, oriented arcs which are labelled $\Gamma_1, \ldots, \Gamma_j$. Moreover, the boundary circle of D_n is labelled Γ_0, and by abuse of notation we shall refer to it as an "arc" of Γ. We require:

(1) every path component of $D_n \backslash \Gamma$ has at least one puncture in its interior,

(2) $\bigcup_{i=0}^{j} int(\Gamma_i)$ is embedded and disjoint from the punctures (where *int* denotes the interior),

(3) the starting point of the i^{th} arc lies in $\bigcup_{k=0}^{i-1} \Gamma_k$, i.e. on one of the previous arcs,

(4) the end point of the i^{th} arc lies in one of the previous arcs, or on an earlier point of the i^{th} arc, or in a puncture.

In the special case that $j = n-1$, so that in (1) every path component contains precisely one puncture, we say Γ is a total curve diagram.

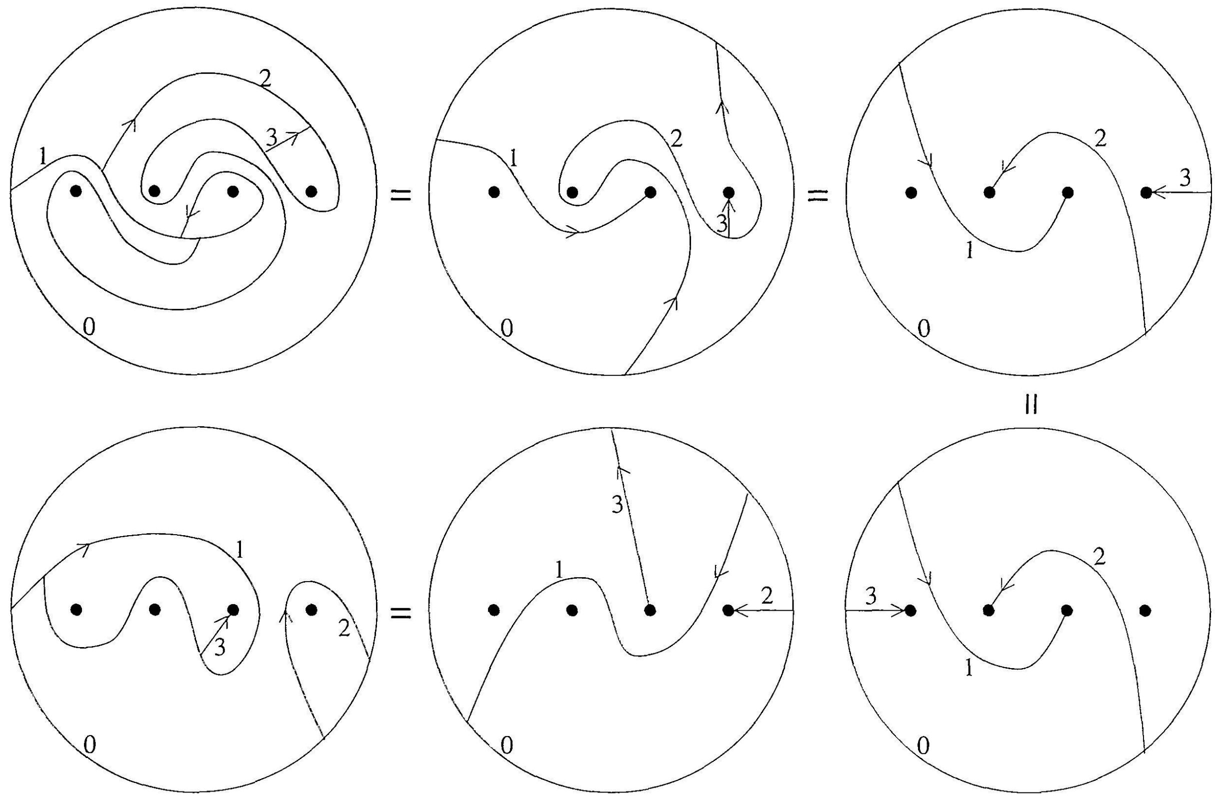

FIGURE 2

Examples of total curve diagrams on D_4
(The meaning of the equality signs will be explained in § 5.)

REMARKS. For simplicity we shall sometimes label arcs $0, \ldots, j$, instead of $\Gamma_0, \ldots, \Gamma_j$. Moreover, we shall use the abbreviated notation $\Gamma_{0\cup\cdots\cup i} := \bigcup_{k=0}^{i} \Gamma_k$. Note for (1) that the number of path components of $D_n \backslash \Gamma$ equals 1 plus the number of arcs of Γ not ending in a puncture, so it can be anything between 1 and n. Note for (3) that the start point of the i^{th} arc can lie in a puncture, if this puncture was the end point of one of the previous arcs. Finally note that if $i < j$ then Γ_i is disjoint from the interior of Γ_j.

We now explain how to associate a partial left ordering of $\mathcal{MCG}(D_n) = B_n$ to a partial curve diagram (with total curve diagrams giving rise to total orderings). The essential ingredient in this definition is the well-known procedure of "pulling tight" or "reducing" two properly embedded curves in a surface. In brief, two simple closed or properly embedded curves in a surface can be isotoped into a relative position in which they have minimal possible intersection number, and this relative position is unique. Moreover, it can be found in a very naive way: whenever one sees a D-disk (or "bigon") enclosed by a pair of segments of the curves, one "squashes" it, i.e. one reduces the intersection number of the two curves, by isotoping the arcs across the disk. A systematic exposition of these ideas can for instance be found in Section 2 of [9].

Our definition of the ordering of $\mathcal{MCG}(S)$ associated to a curve diagram will be a variation of the definition in [9]. We briefly remind the reader of this comparison method. Let Γ be a partial curve diagram in which all j arcs are embedded (no curve Γ_i has end point in its own interior), and let φ and ψ be two homeomorphisms of D_n. If $\varphi(\Gamma_k) \neq \psi(\Gamma_k)$, then we will define either $\varphi < \psi$ or $\psi < \varphi$, according to the following rule. There is an $i \leq j$ such that $\varphi(\Gamma_{0\cup\cdots\cup i-1})$ and $\psi(\Gamma_{0\cup\cdots\cup i-1})$ are isotopic, whereas $\varphi(\Gamma_{0\cup\cdots\cup i})$ and $\psi(\Gamma_{0\cup\cdots\cup i})$ are not. Then we replace φ by an isotopic map, also denoted φ, such that the restrictions of φ and ψ to $\Gamma_{0\cup\cdots\cup i-1}$ are exactly the same maps. At this point, $\varphi(\Gamma_i)$ and $\psi(\Gamma_i)$ have the same starting point and lie in the same path component of $D_n \backslash \varphi(\Gamma_{0\cup\cdots\cup i-1})$. Next we "pull $\varphi(\Gamma_i)$ tight" with respect to $\psi(\Gamma_i)$, i.e. we isotope φ so as to minimise the number of intersections of $\varphi(\Gamma_i)$ and $\psi(\Gamma_i)$, as described above. This can be done by an isotopy which fixes $\varphi(\Gamma_{0\cup\cdots\cup i-1})$. Restricting finally our attention to small initial segments of $\varphi(\Gamma_i)$ and $\psi(\Gamma_i)$, we see that the two curves set off from their common starting point into the interior of a component of $D_n \backslash \varphi(\Gamma_{0\cup\cdots\cup i-1})$ in different directions, one of them "going more to the left"; if it is $\varphi(\Gamma_i)$ say, then we define $\varphi > \psi$, otherwise $\psi < \varphi$. The resulting (possibly parital) ordering is left invariant, because the *relative* position of $\chi \circ \varphi(\Gamma)$ and $\chi \circ \psi(\Gamma)$ is the same as that of $\varphi(\Gamma)$ and $\psi(\Gamma)$ for all $\chi \in \mathcal{MCG}(S)$.

We shall use the following variant of this comparison method: first we make $\varphi(\Gamma_{0\cup\cdots\cup i-1})$ and $\psi(\Gamma_{0\cup\cdots\cup i-1})$ agree for maximal possible i, as before. If the arc Γ_i is embedded, then we proceed as before to compare $\varphi(\Gamma_i)$ and $\psi(\Gamma_i)$. If the arc Γ_i has end point in the interior of Γ_i itself, then we consider the embedded arc Γ_i' which, by definition, is obtained from Γ_i by sliding the end point back along Γ_i so as to make start and end point coincide, as illustrated in Figure 3. We then ignore the original arc Γ_i, and compare $\varphi(\Gamma_i')$ and $\psi(\Gamma_i')$ as before.

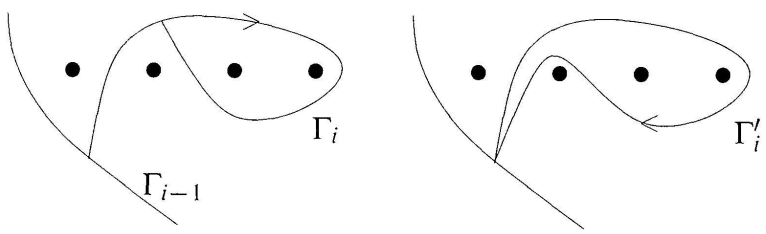

FIGURE 3

The embedded arc Γ_i' obtained from Γ_i by sliding the end point

DEFINITION 4.2. The ordering defined in this way is *the ordering associated to* the curve diagram Γ.

LEMMA 4.3. *The ordering associated to a curve diagram* Γ *is total if and only if* Γ *is a total curve diagram.*

Proof. If Γ is total, i.e. if all components of $D_n \backslash \Gamma$ are once-punctured disks, then any homeomorphism of D_n which fixes Γ is isotopic to the identity; this follows from the Alexander trick (see e.g. [21]). Conversely, if $D_n \backslash \Gamma$ has a path component which contains at least two holes, then we can push the boundary curve of this path component slightly into its interior, to make it disjoint from Γ. A Dehn twist along such a curve is a nontrivial element of B_n, and acts trivially on Γ. $\square$

EXAMPLE. For any n, the Dehornoy ordering [6] is defined by the diagram consisting of $n-1$ horizontal line segments, connecting ∂D_n to the first (leftmost) hole, the first to the second hole, and so on. The arcs are oriented from left to right, and labelled $1, \ldots, n-1$ in this order (see [9]).

DEFINITION 4.4. A (possibly partial) order on a group G is *discrete* if the positive cone $P = \{g \in G \mid g > 1\}$ has a minimal element. (If the ordering is total then this element is necessarily unique.)

In a group with a discrete total left-invariant order every element has a unique predecessor and successor. We note that an ordering is non-discrete if and only if for all $a, c \in G$ there exists a $b \in G$ such that $a < b < c$.

LEMMA 4.5. *The total ordering associated to a total curve diagram* Γ *is discrete.*

Proof. The curve diagram $\Gamma_{0 \cup \cdots \cup n-2}$ (which is obtained from Γ by removing the arc of maximal index) cuts D_n into a number of once-punctured disks and one twice-punctured disk. We observe that the unique smallest element is the positive half-twist interchanging the two punctures inside this disk. $\square$

REMARK. It is an easy exercise to prove that the partial orderings associated to partial curve diagrams are in general not discrete. However, we shall see in the proof of Theorem 3.4(a) that even such orderings have a certain discreteness property.