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The Novikov conjecture is that

$$\langle \mathbf{L}(M) \cup f^*(a), [M] \rangle$$

is an invariant of oriented homotopy type, where  $\mathbf{L}(M)$  is the total  $\mathbf{L}$  class of  $TM$  and  $a$  is any element in  $H^*(BG; \mathbf{Q})$ .

Kasparov [19] and Mischenko-Fomenko [21] [22] define a map

$$K_0(BG) \rightarrow K_0 C^*G$$

and prove that the Novikov conjecture is implied by its rational injectivity. This enabled them to prove the Novikov conjecture for any discrete subgroup of a linear Lie group. The relation with our conjecture is clear from the following commutative diagram

$$\begin{array}{ccc} K_0(BG) & \longrightarrow & K_0 C^*G \\ & \searrow & \swarrow \\ & K^0(\cdot, G) & \end{array}$$

and the Proposition of § 6 above. (In this factorization, the topological definition of  $K$ -homology given in [9] is being used.)  $\square$

**COROLLARY 5.** *(Stable) Riemannian geometry conjectures of Gromov-Lawson-Rosenberg [30].*

For the same reason our conjecture implies the stable<sup>1)</sup> form of the Riemannian geometry conjectures of Gromov-Lawson-Rosenberg [30] on topological obstructions to the existence of metrics of positive scalar curvature.

## 8. TWISTING BY A 2-COCYCLE

This section is motivated by the papers [16], [26], [29], on the range of the trace for the  $C^*$ -algebra of the projective regular representation of a discrete group.

All of §2 adapts to the projective situation where together with the  $G$ -manifold  $X$  one is given a 2-cocycle  $\gamma \in Z^2(X \rtimes G, S^1)$ . For simplicity we

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<sup>1)</sup> Paul Baum comments: It is important to emphasize "stable" because Thomas Schick has shown that the original unstable Gromov-Lawson-Rosenberg conjecture is false. On the other hand, Stephan Stolz (with contributions from J. Rosenberg and others) has proved that the real form of Baum-Connes implies the stable Gromov-Lawson-Rosenberg conjecture. Also, Max Karoubi and I have proved that the usual (i.e. complex  $K$ -theory) form of Baum-Connes implies the real form of Baum-Connes.

shall stick to the case  $X = \text{pt} = \cdot$  and  $G \text{ discrete} = \Gamma$ ; then  $\gamma \in Z^2(\Gamma, S^1)$  is a map:  $\Gamma \times \Gamma \rightarrow S^1$  such that:

$$\gamma(g_2, g_3) \gamma(g_1 g_2, g_3)^{-1} \gamma(g_1, g_2 g_3) \gamma(g_1, g_2)^{-1} = 1 \quad \text{for every } g_1, g_2, g_3 \in \Gamma.$$

Given a *proper*  $\Gamma$ -manifold  $Z$ , a  $(\Gamma, \gamma)$ -vector bundle on  $Z$  is a smooth (complex) vector bundle  $E$  on  $Z$  together with a smooth map  $E \times \Gamma \rightarrow E$  such that (with  $\pi: E \rightarrow Z$  the projection):

- a)  $\pi(\xi g) = \pi(\xi)g$  for each  $\xi \in E$ ,  $g \in \Gamma$ ;
- b)  $\xi(g_1 g_2) = \gamma(g_1, g_2) (\xi g_1) g_2$  for each  $g_1, g_2 \in \Gamma$ .

In b),  $\gamma(g_1, g_2) \in S^1$  is viewed as a complex number of modulus 1. As in §2, we let  $V_{(\Gamma, \gamma)}^0(Z)$  be the collection of triples  $(E_0, E_1, \sigma)$  where  $E_0, E_1$  are  $(\Gamma, \gamma)$ -vector bundles over  $Z$  and  $\sigma$  is a smooth morphism of vector bundles such that:

- 1)  $\sigma(\xi g) = \sigma(\xi)g$  for each  $\xi \in E_0$ ,  $g \in \Gamma$ ;
- 2)  $\text{Support}(\sigma)$  is  $\Gamma$ -compact.

The groups  $K_{(\Gamma, \gamma)}^i(Z)$  are then defined as in [5], [31]. The Thom isomorphism as formulated in §2 still holds in this context, and this allows us to define Gysin maps:

$$h!: K_{(\Gamma, \gamma)}^i(T^*Z_1) \rightarrow K_{(\Gamma, \gamma)}^i(T^*Z_2)$$

for a  $\Gamma$ -map  $h$  of the proper  $\Gamma$ -manifold  $Z_1$  to the proper  $\Gamma$ -manifold  $Z_2$ .

Thus as in §2 we can define the geometric group also in this twisted situation, we denote it by  $K_\gamma^*(X, G)$  in general, and  $= K_\gamma^*(\cdot, \Gamma)$  in our special case.

Let then  $C_r^*(\Gamma, \gamma)$  be the reduced  $C^*$ -algebra of the pair  $(\Gamma, \gamma)$ , i.e. the  $C^*$ -algebra generated in  $\ell^2(\Gamma)$  by the projective regular representation  $\lambda$  of  $\Gamma$ :

$$(\lambda(g) \xi)(g') = \gamma(g, g^{-1} g') \xi(g^{-1} g').$$

As in §2 we get a map  $\mu$  from  $K_\gamma^*(\text{pt}, \Gamma)$  to  $K_*(C_r^*(\Gamma, \gamma))$ , where  $\mu(Z, \xi)$  is the analytical index of the  $K$ -cocycle  $(Z, \xi) \in V_{(\Gamma, \gamma)}^*(T^*Z)$ . The only part of the construction which is modified by the presence of  $\gamma$  is that of the  $C^*$ -module over  $C_r^*(\Gamma, \gamma)$  attached to a  $(\Gamma, \gamma)$ -bundle  $E$  on the proper  $\Gamma$ -manifold  $Z$ . More precisely, one starts with the space  $C_c(Z, E \otimes \Omega^{1/2})$  of compactly supported continuous  $\frac{1}{2}$ -density sections of  $E$  and, after choosing a  $\Gamma$ -invariant metric on  $E$ , one defines:

$$\langle \xi, \eta \rangle(g) = \int_X \langle \xi_x, (\eta_{xg}) g^{-1} \rangle \quad \text{for each } g \in \Gamma,$$

which gives a  $C_c(\Gamma)$ -valued sesquilinear form on  $C_c(Z, E \otimes \Omega^{1/2})$ . One checks that for any  $\xi \in C_c(Z, E \otimes \Omega^{1/2})$ ,  $\langle \xi, \xi \rangle$  is a *positive* element of  $C_r^*(\Gamma)$ , since for any  $\eta \in \ell^2(\Gamma)$  one has:

$$\begin{aligned} \langle \eta, \lambda(\langle \xi, \xi \rangle) \eta \rangle &= \sum \bar{\eta}(g) \langle \xi, \xi \rangle(h) (\lambda(h) \eta)(g) \\ &= \sum \gamma(h, h^{-1}g) \bar{\eta}(g) \eta(h^{-1}g) \int_X \langle \xi_x, (\xi_x h) h^{-1} \rangle \\ &= \sum \bar{\eta}(g) \eta(h^{-1}g) \int_X \langle (\xi_{xg^{-1}})g, (\xi_{xg^{-1}h})h^{-1}g \rangle \geq 0. \end{aligned}$$

Then, by completion with respect to the norm  $\|\langle \xi, \xi \rangle\|^{1/2}$ , one gets a  $C^*$ -module over  $C_r^*(\Gamma, \gamma)$ , which we denote by  $L^2(Z, E)$ . The right action is given by:

$$(\xi f)(x) = \sum_{\Gamma} (\xi_{xg^{-1}})g f(g) \text{ for each } \xi \in C_c(Z, E \otimes \Omega^{1/2}), f \in C_c(\Gamma).$$

Next, we can choose a  $\Gamma$ -invariant Riemannian metric on  $Z$ , represent every class in  $K_{(\Gamma, \gamma)}^0(T^*Z)$  by a pair  $E_0, E_1$  of  $(\Gamma, \gamma)$ -hermitian bundles on  $Z$  and a symbol  $\sigma$  which is an isomorphism of the pull back of  $E_0$  to  $S^*Z$  to that of  $E_1$ , and is independent of  $\xi$ ,  $\pi(\xi) = z$ , outside a  $\Gamma$ -compact subset of  $Z$ . Letting  $P_\sigma$  be the corresponding order 0 pseudo-differential operator, one gets a Kasparov  $(\mathbb{C}, C_r^*(\Gamma, \gamma))$ -bimodule: the triple  $(L^2(Z, E_0), L^2(Z, E_1), P_\sigma)$  which gives an element of  $K_0(C_r^*(\Gamma, \gamma))$ . It is important to give another description of the map  $\mu: K_{(\Gamma, \gamma)}^0(T^*Z) \rightarrow K_0(C_r^*(\Gamma, \gamma))$ , using Kasparov products.

**PROPOSITION 1.** a) *Let  $X$  be a proper  $\Gamma$ -manifold, then  $K_{(\Gamma, \gamma)}^i(X)$  is canonically isomorphic to  $K_i(C_0(X) \rtimes_\gamma \Gamma)$ , where  $C_0(X) \rtimes_\gamma \Gamma$  is the twisted crossed product of  $C_0(X)$  by  $\Gamma$ .*

b) (Compare [19]). *For any  $C^*$ -algebras  $A, B$  on which  $\Gamma$  acts by automorphisms, one has a natural map from  $KK_\Gamma(A, B)$  to  $KK(A \rtimes_\gamma \Gamma, B \rtimes_\gamma \Gamma)$ .*

*Proof.* a) One can consider  $A = C_0(X) \rtimes_\gamma \Gamma$  as the  $C^*$ -algebra of the groupoid  $X \rtimes \Gamma = G$  with units  $G^{(0)} = X$ , source and range maps  $s(x, g) = xg$ ,  $r(x, g) = x$  and composition  $(x, g) \cdot (x', g') = (x, gg')$  with the 2-cocycle  $\gamma \circ \pi$  where  $\pi$  is the natural homomorphism  $G \rightarrow \Gamma: \pi(x, g) = g$ .

Thus  $A$  is the completion of this convolution algebra  $C_c(G)$ :

$$\begin{aligned} (f_1 * f_2)(x, g) &= \sum_{\Gamma} f_1(x, h) f_2(xh, h^{-1}g) \gamma(h, h^{-1}g) \\ f^*(x, g) &= \bar{f}(xg, g^{-1}) \end{aligned}$$



with the norm  $\|f\| = \sup \|\pi_x(f)\|$ , where for each  $x \in X$  the representation  $\pi_x$  of  $C_c(G)$  in  $\ell^2(\Gamma)$  is given by:

$$(\pi_x(f)\xi)(g) = \sum_{\Gamma} f(xg^{-1}, h) \xi(h^{-1}g) \gamma(h, h^{-1}g) \text{ for each } \xi \in \ell^2(\Gamma).$$

Now, given a  $(\Gamma, \gamma)$ -vector bundle  $E$  on  $X$ , one can endow  $E$  with a  $\Gamma$ -invariant hermitian metric and define a  $C^*$ -module  $\mathcal{E}$  over  $A = C_0(X) \rtimes_{\gamma} \Gamma$  as follows. For any  $\xi, \eta \in C_c(X, E)$  let  $\langle \xi, \eta \rangle \in C_c(X \rtimes \Gamma)$  be given by  $\langle \xi, \eta \rangle(x, g) = \langle \xi_x g, \eta_{xg} \rangle$ ; then  $\langle \xi, \xi \rangle$  is a positive element of  $A = C_0(X) \rtimes_{\gamma} \Gamma$ , since for any  $\eta \in \ell^2(\Gamma)$  and  $x \in X$  one has:

$$\begin{aligned} \langle \eta, \pi_x(\langle \xi, \xi \rangle) \eta \rangle &= \\ \sum \sum \langle \xi_{xg^{-1}} h, \xi_{xg^{-1}} h \rangle \eta(h^{-1}g) \bar{\eta}(g) \gamma(h, h^{-1}g) &= \langle \alpha, \alpha \rangle \geq 0, \end{aligned}$$

where  $\alpha = \sum (\xi_{xg^{-1}})g \eta(g) \in E_x$ .

Let  $\mathcal{E}$  be the completion of  $C_c(X, E)$  with the norm  $\|\xi\| = \|\langle \xi, \xi \rangle\|$ ; then  $\mathcal{E}$  is a  $C^*$ -module over  $A$ , with:

$$(\xi f)(x) = \sum f(xg^{-1}, g) \xi(xg')g \text{ for every } f \in C_c(X \rtimes \Gamma), \xi \in C_0(X, E).$$

(One easily checks that  $\langle \xi, \eta f \rangle = \langle \xi, \eta \rangle * f$  and that this right action of  $C_c(X \rtimes \Gamma)$  extends to an action of  $A$ .)

The equality  $(\eta \langle \eta, \xi \rangle)(x) = \sum \langle (\eta_{xg^{-1}})g, \xi_x \rangle (\eta_{xg^{-1}})g$  shows that any endomorphism  $\sigma$  of the vector bundle  $E$  which commutes with  $\Gamma$  and has  $\Gamma$ -compact support defines an  $A$ -compact endomorphism of  $\mathcal{E}$  by the equality:  $(T\xi)(x) = \sigma(x) \xi(x)$  for every  $x \in X$ . Thus, to any triple  $(E_0, E_1, \sigma) \in V_{(\Gamma, \gamma)}^0(X)$  corresponds an element of  $KK(\mathbf{C}, A)$ ,  $A = C_0(X) \rtimes_{\gamma} \Gamma$ , which obviously depends only upon the class of the triple in  $K_{(\Gamma, \gamma)}^0(X)$ . Let us prove that this map is an isomorphism assuming that  $\Gamma$  is *torsion free*. We may then assume that  $X$  is  $\Gamma$ -compact. We claim first that  $A = C_0(X) \rtimes_{\gamma} \Gamma$  is Morita equivalent to a  $C^*$ -algebra with unit. Indeed, with  $V = X/\Gamma$ ,  $A$  is the  $C^*$ -algebra of the continuous field of elementary  $C^*$ -algebras  $A_t = C_0(\pi^{-1}(t)) \rtimes_{\gamma} \Gamma$ , where  $\pi: X \rightarrow X/\Gamma = V$  is the projection. By a simple computation, one gets that the Dixmier-Douady obstruction  $\delta(A) \in H^3(V, \mathbf{Z})$  is given by  $\delta(A) = \phi^*(\partial\gamma)$  where  $\phi: V \rightarrow B\Gamma$  is the classifying map, and  $\partial\gamma \in H^3(B\Gamma, \mathbf{Z})$  is the boundary of  $\gamma \in H^2(B\Gamma, S^1) = H^2(\Gamma, S^1)$  in the exact sequence:

$$H^2(\Gamma, \mathbf{Z}) \rightarrow H^2(\Gamma, \mathbf{R}) \rightarrow H^2(\Gamma, S^1) \xrightarrow{\partial} H^3(\Gamma, \mathbf{Z}) \rightarrow H^3(\Gamma, \mathbf{R}) \rightarrow \dots$$

In particular  $\delta(A)$  is a torsion element in  $H^3(V, \mathbf{Z})$  so that there exists a bundle of matrix algebras over  $V$  with the same Dixmier-Douady obstruction and  $A$  is Morita equivalent to a unital  $C^*$ -algebra. It follows then that  $K_0(A)$

is obtained from  $C^*$ -modules  $\mathcal{E}$  over  $A$  with the property  $\text{id}_{\mathcal{E}} \in \text{End}_A^0(\mathcal{E})$ , i.e. all endomorphisms of  $\mathcal{E}$  are  $A$ -compact. Finally, the above construction sets up a surjective map from  $(\Gamma, \gamma)$ -vector bundles on  $X$  to  $C^*$ -modules over  $A$  with the above property. Given  $\mathcal{E}$ , the fiber  $E_x$  of the corresponding vector bundle is:

$$E_x = \mathcal{E} \widehat{\otimes}_A \ell^2(\Gamma)$$

where  $A = C_0(X) \rtimes_{\gamma} \Gamma$  acts in  $\ell^2(\Gamma)$  by the representation  $\pi_x$ . Since  $\pi_x(A) \subset \text{Compacts}$ , one gets that  $E_x$  is a finite dimensional Hilbert space.

b) The proof is the same as in [19], one defines for any  $\Gamma$ -equivariant  $C^*$ -module  $\mathcal{E}$  over  $B$  the crossed product  $\mathcal{E} \rtimes_{\gamma} \Gamma$  twisted by the 2-cocycle  $\gamma$ .  $\square$

We can now state:

**THEOREM 2.** *For any element  $x$  of  $K_{(\Gamma, \gamma)}^0(T^*Z) = K_0(A)$  (where  $A = C_0(T^*Z) \rtimes_{\gamma} \Gamma$ , and  $Z$  a proper  $\Gamma$ -manifold), one has:*

$$\mu(x) = x \otimes j_{(\Gamma, \gamma)}(D),$$

where  $D \in KK_{\Gamma}(C_0(T^*Z), \mathbf{C})$  is the class of the Dirac operator.

Note that  $x \in KK(\mathbf{C}, C_0(T^*Z) \rtimes_{\gamma} \Gamma)$  and that

$$j_{(\Gamma, \gamma)}(D) \in KK(C_0(T^*Z) \rtimes_{\gamma} \Gamma, C_r^*(\Gamma, \gamma)),$$

so that the above equality is meaningful. The proof is straightforward.

To show how to use this theorem, we shall combine it with the recent result of G. G. Kasparov ([19]) to compute  $K_i(C_r^*(\Gamma, \gamma))$  in the following example: we let  $\Gamma = \pi_1(M)$  be the fundamental group of a Riemann surface  $M$  with genus  $> 1$ . From the exact sequence  $0 \rightarrow H^2(\Gamma, \mathbf{Z}) \rightarrow H^2(\Gamma, \mathbf{R}) \rightarrow H^2(\Gamma, S^1) \rightarrow 0$  one gets  $H^2(\Gamma, S^1) = \mathbf{R}/\mathbf{Z}$ , so that there are many non trivial cocycles in this example. The geometric group  $K_{\gamma}^i(\text{pt}, \Gamma)$  is easily determined: since the universal cover  $\tilde{M}$  of  $M$  (the Poincaré disc) is a final object in the category of proper  $\Gamma$ -manifolds, and homotopy classes of  $\Gamma$ -maps, it is enough to compute  $K_{(\Gamma, \gamma)}^i(T^*\tilde{M})$ . Since  $\tilde{M}$  has a  $\Gamma$ -invariant  $\text{Spin}^c$ -structure, the Thom isomorphism hence gives:  $K_{\gamma}^i(\text{pt}, \Gamma) = K_{(\Gamma, \gamma)}^i(\tilde{M})$ . By Proposition 1, one has  $K_{(\Gamma, \gamma)}^i(\tilde{M}) = K_i(C_0(\tilde{M}) \rtimes_{\gamma} \Gamma)$  and the latter  $C^*$ -algebra is Morita equivalent to  $C(M)$  (see the proof of a) in Proposition 1). Thus we get:  $K_{\gamma}^0(\text{pt}, \Gamma) = \mathbf{Z}^2$ ,  $K_{\gamma}^1(\text{pt}, \Gamma) = \mathbf{Z}^{2g}$ .

**THEOREM 3.** *Let  $\Gamma$  be the fundamental group of a Riemann surface of genus  $> 1$ , and  $\gamma \in H^2(\Gamma, S^1)$ , then the map  $\mu: K_\gamma^*(\text{pt}, \Gamma) \rightarrow K_*(C_r^*(\Gamma, \gamma))$  is an isomorphism.*

*Proof.* Let  $D \in KK_G(C_0(U), \mathbf{C})$  be the  $G = \text{PSL}(2, \mathbf{R})$  equivariant Dirac operator on the Poincaré disc  $U = G/G_c$  (cf. [19]). Identify  $\tilde{M}$  with  $U$  and  $\Gamma$  with a subgroup of  $G$ . Then by Proposition 1 b) and Theorem 2 it is enough to show that the restriction of  $D$  to an element of  $KK_\Gamma(C_0(U), \mathbf{C})$  is an invertible element. This follows from [19] which shows that  $D$  is an invertible element of  $KK_G(C_0(U), \mathbf{C})$ , and from the multiplicative property of the restriction to subgroups.

We shall now show how to prove that the  $C^*$ -algebras  $C_r^*(\Gamma, \gamma)$  are pairwise non-isomorphic when  $\gamma$  varies in  $H^2(\Gamma, S^1)$ . In fact we shall compute in full generality the composition  $\zeta \circ \mu$  of the canonical trace  $\zeta$  on  $C_r^*(\Gamma, \gamma)$  (viewed as a map from  $K_0$  to  $\mathbf{C}$ ) with the above map  $\mu: K_\gamma^0(\text{pt}, \Gamma) \rightarrow K_0(C_r^*(\Gamma, \gamma))$ .

The computation is a generalization of the index theorem for covering spaces of Atiyah ([3]).

**LEMMA 4.** *Let  $Z$  be a proper  $\Gamma$ -manifold and  $E$  a  $(\Gamma, \gamma)$  vector bundle on  $Z$ . There exists a  $\Gamma$ -invariant connection  $\nabla$  on  $E$ .*

*Proof.* For any  $(\Gamma, \gamma)$ -vector bundle  $F$  on  $Z$  and section  $\xi \in C_c^\infty(Z, F)$  let, for  $g \in \Gamma$ ,  $g\xi \in C_c^\infty(Z, F)$  be given by:  $(g\xi)(x) = (\xi(xg))g^{-1} \in F_x$  for every  $x \in Z$ .

In this way one gets a natural  $\gamma$ -action of  $\Gamma$  on both  $C_c^\infty(Z, E)$  and  $C_c^\infty(Z, E \otimes T^*Z)$ , and one looks for a connection

$$\nabla: C_c^\infty(Z, E) \rightarrow C_c^\infty(Z, E \otimes T^*Z)$$

such that  $\nabla(g\xi) = g(\nabla\xi)$  for every  $\xi$ . Let  $f \in C^\infty(Z)$ ,  $0 \leq f \leq 1$ , be such that  $\sum_\Gamma f(xg) = 1$  for every  $x \in Z$  and  $\nabla_0$  be a connection on  $E$ . Put  $\nabla = \sum_\Gamma g^{-1}(f\nabla_0)g$ . By construction  $\nabla$  is  $\Gamma$ -invariant, moreover each  $g^{-1}\nabla_0 g$  is a connection on  $E$  thus  $\nabla$  is a connection on  $E$ .  $\square$

*Proof of Theorem 3, continued.* Assuming now that  $Z$  is  $\Gamma$ -compact, let for a  $\Gamma$ -invariant connection  $\nabla$  on  $E$ ,  $\omega_\nabla$  be the canonical differential form on  $Z$  which represents locally the Chern character  $\text{ch}(E)$ . By construction  $\omega_\nabla$  is  $\Gamma$ -invariant and hence determines a cohomology class in  $Z/\Gamma$ . One checks as usual that this class does not depend upon the choice of  $\nabla$  and

we shall denote it by  $[E] \in H^*(Z/\Gamma, \mathbf{R})$ . This construction easily extends to give a map  $\text{ch}$  from  $K_{(\Gamma, \gamma)}^0(Z)$  to  $H^*(Z/\Gamma, \mathbf{R})$  for any proper  $\Gamma$ -manifold  $Z$ . However, in the presence of the 2-cocycle  $\gamma$  the range of this map is *no longer necessarily contained* in  $H^*(Z/\Gamma, \mathbf{Q})$ .

To be more precise, let us make a few simplifying assumptions and compute exactly the range of this Chern character:

$$\text{ch}: K_{(\Gamma, \gamma)}^0(Z) \rightarrow H^*(Z/\Gamma, \mathbf{R}).$$

Thus let us assume that  $\Gamma$  is torsion free and that the image of  $\gamma \in H^2(\Gamma, S^1)$  in  $H^3(\Gamma, \mathbf{Z})$  under the connecting map of the long exact sequence:

$$\dots \rightarrow H^2(\Gamma, \mathbf{Z}) \rightarrow H^2(\Gamma, \mathbf{R}) \rightarrow H^2(\Gamma, S^1) \rightarrow H^3(\Gamma, \mathbf{Z}) \rightarrow \dots$$

is equal to 0 (it is always a torsion element).

Let then  $\rho \in H^2(\Gamma, \mathbf{R})$  be such that  $e(\rho) = \gamma$  where  $e: \mathbf{R} \rightarrow S^1$  is given by  $e(s) = \exp(2\pi i s)$ , for each  $s \in \mathbf{R}$ .

LEMMA 5. a) Let  $\rho \in Z^2(\Gamma, \mathbf{R})$  and  $Z$  be a proper  $\Gamma$ -manifold, then there exists a smooth function  $c \in C^\infty(Z \rtimes \Gamma)$  such that:

$$c(x, g_1) + c(xg_1, g_2) = c(x, g_1g_2) - \rho(g_1, g_2)$$

for every  $x \in Z$ ,  $g_1, g_2 \in \Gamma$ .

b) If  $\gamma = e(\rho)$  there exists an isomorphism  $r: K_\Gamma^0(Z) \rightarrow K_{(\Gamma, \gamma)}^0(Z)$  making the following diagram commutative:

$$\begin{array}{ccc} K_\Gamma^0(Z) & \xrightarrow{r} & K_{(\Gamma, \gamma)}^0(Z) \\ \downarrow \text{ch} & & \downarrow \text{ch} \\ H^*(Z/\Gamma) & \xrightarrow{m} & H^*(Z/\Gamma), \end{array}$$

where  $m$  is multiplication by the cohomology class  $\exp(\phi^*\rho)$  and where  $\phi: Z/\Gamma \rightarrow B\Gamma$  is the classifying map.

*Proof.* a) Let  $M = Z/\Gamma$ ,  $\pi: Z \rightarrow M$  the projection. Since  $Z$  is a locally trivial  $\Gamma$ -principal bundle, it is easy to construct  $c$  on the open set  $\pi^{-1}(U)$  for  $U$  small enough. Then one combines such  $c_U$  by a smooth partition of unity on  $M$ :

$$c(x, g) = \sum \phi_U(\pi(x)) c_U(x, g).$$

b) Let  $c \in C^\infty(Z \rtimes \Gamma)$  be as in a) and let us endow the trivial line bundle on  $Z$  (with total space  $Z \times \mathbf{C}$ ) with a structure of  $(\Gamma, \gamma)$ -bundle. We take:

$$(x, \lambda)g = (xg, e(c(x, g))\lambda).$$

(One has  $((x, \lambda)g_1)g_2 = (xg_1g_2, e(c(x, g_1) + c(xg_1, g_2))\lambda) = \gamma^{-1}(g_1, g_2)(x\lambda)(g_1g_2).$ )

Let  $L$  be the  $(\Gamma, \gamma)$ -line bundle on  $Z$  thus obtained. It is obvious that tensoring by  $L$  gives an isomorphism of  $V_{(\Gamma)}^0(Z)$  with  $V_{(\Gamma, \gamma)}^0 Z$  and hence of  $K_{\Gamma}^0(Z)$  with  $K_{(\Gamma, \gamma)}^0(Z)$ .  $\square$

*End of proof of Theorem 3.* To conclude, it is enough to compute  $\text{ch}(L)$ . Let  $\xi \in C^\infty(Z, L)$  be the section  $\xi(x) = 1$  for every  $x \in Z$ . Let  $\nabla$  be a  $\Gamma$ -invariant connection on  $L$ , one has  $\text{ch}(L) = \exp(\omega)$  where  $\omega \in H^2(Z/\Gamma, \mathbf{R})$  corresponds to the  $\Gamma$ -invariant 2-form  $\theta = \frac{1}{2\pi i} d(\nabla \xi / \xi)$  on  $Z$ . Let  $\alpha = \frac{1}{2\pi i} \nabla \xi / \xi$ , then  $\alpha$  is a 1-form on  $Z$ , and let us compute for any  $g \in \Gamma$  the difference  $\alpha - \phi^* \alpha$  where  $\phi(x) = xg$  for every  $x \in Z$ . Since  $\nabla$  is  $\Gamma$ -invariant, one has  $\phi^* \alpha = \frac{1}{2\pi i} \nabla g(\xi) / g(\xi)$ , and as  $g(\xi)(x) = e(c(xg, g^{-1})) \xi(x)$  one gets  $\phi^* \alpha - \alpha = d\psi_g$ , where  $\psi_g(x) = c(xg, g^{-1})$  for every  $x \in Z$ . One has  $\psi_{g_1g_2} - g_1\psi_{g_2} - \psi_{g_1} = \rho(g_2^{-1}, g_1^{-1})$ . This shows that the class of  $\theta$  in  $H^2(Z/\Gamma, \mathbf{R})$  is the pull back of the class of  $-\rho$  in  $H^2(B\Gamma, \mathbf{R})$ , by the classifying map:  $Z/\Gamma \rightarrow B\Gamma$ .  $\square$

Using this map  $\text{ch}: K_{(\Gamma, \gamma)}^*(Z) \rightarrow H^*(Z/\Gamma, \mathbf{R})$  we get, by the same five steps as in §6, a map

$$K_{\gamma}^*(\text{pt}, \Gamma) \xrightarrow{\text{ch}} H_*(B\Gamma, \mathbf{R}).$$

Again as in §6, let  $\epsilon$  be the map from  $B\Gamma$  to a point, and  $\text{tr}_{\Gamma}$  be the canonical trace on  $C_r^*(\Gamma, \gamma)$ .

**THEOREM 6.** *For any discrete group  $\Gamma$  and 2-cocycle  $\gamma$  the following diagram is commutative:*

$$\begin{array}{ccc} K_{\gamma}^0(\text{pt}, \Gamma) & \xrightarrow{\mu} & K_0(C_r^*(\Gamma, \gamma)) \\ \downarrow \text{ch} & & \downarrow \text{tr}_{\Gamma} \\ H_*(B\Gamma, \mathbf{R}) & \xrightarrow{\epsilon^*} & \mathbf{C}. \end{array}$$

The proof is a simple adaptation of the heat equation method to compute the  $\Gamma$ -index of the  $(\Gamma, \gamma)$ -Dirac operator on a  $\Gamma$ -manifold  $Z$ .

COROLLARY 7. *If  $\gamma = e(\rho)$ , for some  $\rho \in H^2(\Gamma, \mathbf{R})$ , then the subgroup of  $\mathbf{R}$ ,  $\Delta = \text{tr}_\Gamma(K_0(C_r^*(\Gamma, \gamma)))$  contains the group:*

$$\langle \text{ch } K_*(B\Gamma), \exp(\rho) \rangle.$$

This follows from Theorem 6 and Lemma 5 b).

Moreover, when the map  $\mu$  is an isomorphism, one can conclude that  $\Delta = \langle \text{ch } K_*(B\Gamma), \exp(\rho) \rangle$ . Thus using Theorem 3 we get:

COROLLARY 8. *Let  $\Gamma$  be the fundamental group of a compact Riemann surface of positive genus,  $\gamma \in H^2(\Gamma, S^1)$  be a 2-cocycle and  $\theta \in \mathbf{R}/\mathbf{Z}$  the class of  $\gamma$  in  $H^2(\Gamma, \mathbf{R})/H^2(\Gamma, \mathbf{Z}) = \mathbf{R}/\mathbf{Z}$ . Then the image of  $K_0(C_r^*(\Gamma, \gamma))$  by the canonical trace  $\zeta = \text{Tr}_\Gamma$  is equal to the subgroup  $\mathbf{Z} + \theta\mathbf{Z} \subset \mathbf{R}$ .*

Since, for  $g > 1$ , the trace  $\text{tr}_\Gamma$  is the unique normalized trace on  $C_r^*(\Gamma, \gamma)$  (for any value of  $\gamma$ ), one gets that the corresponding  $C^*$ -algebras are isomorphic only when the  $\Gamma$ 's are the same (using  $K_1$ ) and when the  $\gamma$ 's are equal or opposite (in  $H^2(\Gamma, S^1)$ ).

## 9. FOLIATIONS

Let  $V$  be a  $C^\infty$ -manifold, and let  $F$  be a  $C^\infty$ -foliation of  $V$ . Thus  $F$  is a  $C^\infty$ -integrable sub-vector bundle of  $TV$ . As in [33] let  $G$  be the holonomy groupoid (graph) of  $(V, F)$ . The manifold  $V$  is assumed to be Hausdorff and second countable.  $G$ , however, is a  $C^\infty$ -manifold which might not be Hausdorff. A point in  $G$  is an equivalence class of  $C^\infty$ -paths

$$\gamma: [0, 1] \rightarrow V$$

such that  $\gamma(t)$  remains within one leaf of the foliation for all  $t \in [0, 1]$ . Set  $s(\gamma) = \gamma(0)$ ,  $r(\gamma) = \gamma(1)$ . The equivalence relation on the  $\gamma$  preserves  $s(\gamma)$  and  $r(\gamma)$  so  $G$  comes equipped with two maps  $G \begin{smallmatrix} \xrightarrow{s} \\ \xrightarrow{r} \end{smallmatrix} V$ .

Let  $Z$  be a possibly non-Hausdorff  $C^\infty$ -manifold. Assume given a  $C^\infty$ -map  $\rho: Z \rightarrow V$ , set

$$Z \circ G = \{(z, \gamma) \in Z \times G \mid \rho(z) = s(\gamma)\}.$$

A  $C^\infty$  right action of  $G$  on  $Z$  is a  $C^\infty$ -map