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Autor: Fox, Glenn J.
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A p-ADIC L-FUNCTION OF TWO VARIABLES

by Glenn J. FOX *)

ABSTRACT. For p prime and χ a primitive Dirichlet character, we derive a p-adic function $L_p(s,t;\chi)$, where $t \in \mathbf{C}_p$, $|t|_p \leq 1$, and $s \in \mathbf{C}_p$, $|s-1|_p < |p|_p^{1/(p-1)}|q|_p^{-1}$, $s \neq 1$ if $\chi = 1$, with $q = 4$ if $p = 2$ and $q = p$ if $p > 2$, that interpolates the values

$$L_p(1-n,t;\chi) = -\frac{1}{n}\Big(B_{n,\chi_n}(qt) - \chi_n(p)p^{n-1}B_{n,\chi_n}(p^{-1}qt)\Big),$$

for $n \in \mathbf{Z}$, $n \geq 1$. Here $B_{n,\chi}(t)$ is the n^{th} generalized Bernoulli polynomial associated with the character χ, and $\chi_n = \chi\omega^{-n}$, where ω is the Teichmüller character. This function is then a two-variable analogue of the p-adic L-function $L_p(s;\chi)$, where $s \in \mathbf{C}_p$, $|s-1|_p < |p|_p^{1/(p-1)}|q|_p^{-1}$, $s \neq 1$ if $\chi = 1$, in that this function satisfies $L_p(s,0;\chi) = L_p(s;\chi)$. In addition to deriving this function, we establish several properties and applications of $L_p(s,t;\chi)$.

1. INTRODUCTION

Given a primitive Dirichlet character χ, having conductor f_χ (see Section 2 for definitions), the Dirichlet L-function associated with χ is defined by

$$L(s;\chi) = \sum_{b=1}^{\infty} \frac{\chi(b)}{b^s},$$

where $s \in \mathbf{C}$, $\Re(s) > 1$. This function can be continued analytically to the entire complex plane, except for a simple pole at $s = 1$ when $\chi = 1$, in which case we have the Riemann zeta function, $\zeta(s) = L(s;1)$. It is believed that the analysis of Dirichlet L-functions began with Euler's study of $\zeta(s)$, in which he considered the function only for real values of s. It was Riemann

*) A majority of these results were obtained while the author was a graduate student at the University of Georgia, Athens, under the direction of Andrew Granville.

who extended this study to a complex variable [17]. Of notable interest are the values of $L(s;\chi)$ at $s = n$, $n \in \mathbf{Z}$. Euler was able to evaluate $\zeta(s)$ at the positive even integers. However, the determination of the values of this function at odd $s \geq 3$ remains an open problem. Similarly, the values of $L(s;\chi)$ can be determined at either the positive even or odd integers depending on the sign of $\chi(-1)$. Furthermore, these functions can be readily evaluated at all integer values of $s \leq 0$. Because of a functional equation (7) that the Dirichlet L-functions satisfy (discovered by Riemann [17] for $\zeta(s)$), we can obtain a relationship between the values of $L(s;\chi)$ at positive and negative $s \in \mathbf{Z}$.

Jakob Bernoulli was the first to consider a particular sequence of rational numbers in the study of finite sums of a given power of consecutive integers [4]. In this study, he gave a defining relationship that enables the generation of this sequence. This sequence of numbers has, since that time, come to be known as the Bernoulli numbers, B_n, $n \in \mathbf{Z}$, $n \geq 0$. They are given by $B_0 = 1, B_1 = -1/2, B_2 = 1/6, B_3 = 0, B_4 = -1/30, \ldots$, where $B_n = 0$ for odd $n \geq 3$, and for all $n \geq 1$,

$$B_n = -\frac{1}{n+1}\sum_{m=0}^{n-1}\binom{n+1}{m}B_m.$$

The Bernoulli polynomials were first introduced by Raabe in [16]. They can be expressed in the form

$$B_n(t) = \sum_{m=0}^{n}\binom{n}{m}B_{n-m}t^m,$$

where $n \in \mathbf{Z}$, $n \geq 0$. The form in which they are currently defined has been somewhat modified from Raabe's original construction, but the results that he obtained set the framework for a continuing history of analysis on these polynomials.

The generalized Bernoulli numbers associated with the Dirichlet character χ, $B_{n,\chi}$, $n \in \mathbf{Z}$, $n \geq 0$, were defined in [12], [3], [1], and [15]. We obtain the standard Bernoulli numbers when $\chi = 1$, in that $B_{n,1} = B_n$ if $n \neq 1$, and $B_{1,1} = -B_1$. The generalized Bernoulli numbers share a particular relationship with the Dirichlet L-function, $L(s;\chi)$, in that

$$L(1-n;\chi) = -\frac{1}{n}B_{n,\chi},$$

for $n \in \mathbf{Z}$, $n \geq 1$. The generalized Bernoulli polynomials, $B_{n,\chi}(t)$, are given by

$$B_{n,\chi}(t) = \sum_{m=0}^{n} \binom{n}{m} B_{n-m,\chi} t^m,$$

where $n \in \mathbf{Z}$, $n \geq 0$.

During the development of p-adic analysis, effort was made to derive a meromorphic function, defined over the p-adic number field, that would interpolate the same, or at least similar, values as the Dirichlet L-function at nonpositive integers. In [14] Kubota and Leopoldt proved the existence of such a function, considered the p-adic equivalent of the Dirichlet L-function. This function, $L_p(s;\chi)$, yields the values

$$L_p(1-n;\chi) = -\frac{1}{n}\left(1-\chi_n(p)p^{n-1}\right) B_{n,\chi_n},$$

for $n \in \mathbf{Z}$, $n \geq 1$, where $\chi_n = \chi\omega^{-n}$, with ω the Teichmüller character. The function $L_p(s;\chi)$ can be expressed in the form

$$L_p(s;\chi) = \frac{a_{-1}}{s-1} + \sum_{n=0}^{\infty} a_n (s-1)^n,$$

where

$$a_{-1} = \begin{cases} 1-\frac{1}{p}, & \text{if } \chi = 1 \\ 0, & \text{if } \chi \neq 1, \end{cases}$$

and $a_n \in \mathbf{Q}_p(\chi)$, a finite extension of $\mathbf{Q}_p$, for $n \geq 0$. The power series given in the above expression converges in $\mathfrak{D} = \{s \in \mathbf{C}_p : |s-1|_p < r\}$, for $r = |p|_p^{1/(p-1)}|q|_p^{-1}$, where $q = 4$ if $p = 2$, and $q = p$ otherwise. Much additional information about these functions can be found in [19].

We have found a more general form for the p-adic L-function $L_p(s;\chi)$. Instead of generating a function of one variable that interpolates an expression involving generalized Bernoulli numbers, we have sought out a function of two variables that in one variable interpolates an expression that involves generalized Bernoulli polynomials in the other variable, such that when this second variable is 0, we obtain the familiar function $L_p(s;\chi)$. We have constructed such a function for all primes p, and so we have been able to prove the existence of a p-adic L-function, $L_p(s,t;\chi)$, where $s \in \mathbf{C}_p$ such that $|s-1|_p < r$, except $s \neq 1$ when $\chi = 1$, and $t \in \mathbf{C}_p$ such that $|t|_p \leq 1$, which interpolates the polynomials

$$L_p(1-n,t;\chi) = -\frac{1}{n}\left(B_{n,\chi_n}(qt) - \chi_n(p)p^{n-1}B_{n,\chi_n}\left(p^{-1}qt\right)\right),$$

for $n \in \mathbf{Z}$, $n \geq 1$. This function also has an expansion

$$L_p(s,t;\chi) = \frac{a_{-1}(t)}{s-1} + \sum_{n=0}^{\infty} a_n(t)(s-1)^n,$$

where

$$a_{-1}(t) = \begin{cases} 1 - \frac{1}{p}, & \text{if } \chi = 1 \\ 0, & \text{if } \chi \neq 1 . \end{cases}$$

If $\chi(-1) = -1$, then $B_{n,\chi_n} = 0$ for each $n \geq 0$. Thus the corresponding p-adic L-function, $L_p(s;\chi)$, vanishes on a set that has a limit point in $\mathbf{Z}_p$. This implies that $L_p(s;\chi)$ must vanish identically for all $s \in \mathfrak{D}$. Because of this, proofs of the existence of this function need only deal with the case of those χ such that $\chi(-1) = 1$, and properties associated with these χ can then be utilized to enhance the efficiency of the proof. In the more generalized form, the p-adic L-function $L_p(s,t;\chi)$ must satisfy $L_p(s,0;\chi) = L_p(s;\chi)$, and so $L_p(s,0;\chi)$ vanishes for all $s \in \mathfrak{D}$ when $\chi(-1) = -1$, but this property does not hold for all t for any given χ. Thus we cannot focus the proof of the existence of $L_p(s,t;\chi)$ solely on those χ such that $\chi(-1) = 1$.

In Section 3, we derive $L_p(s,t;\chi)$ according to the method given in [13], Chapter 3. In this method, if a sequence $\{b_n\}_{n=0}^{\infty}$, in a finite extension of $\mathbf{Q}_p$, is given such that

$$c_n = \sum_{m=0}^{n} \binom{n}{m} (-1)^{n-m} b_m$$

satisfies $|c_n|_p \leq C\rho^n$, for all $n \geq 0$, where $C, \rho \in \mathbf{R}$, with $C > 0$ and $0 < \rho < |p|_p^{1/(p-1)}$, then a power series $A(s)$ can be generated such that $A(n) = b_n$, for each n, and such that $A(s)$ converges on $\{s \in \mathbf{C}_p : |s|_p < |p|_p^{1/(p-1)}\rho^{-1}\}$. It is then shown that, given a Dirichlet character χ, the values $b_n = (1 - \chi_n(p)p^{n-1})B_{n,\chi_n}$, $n \geq 0$, form such a sequence, and thus we have a power series $A_\chi(s)$ which interpolates the b_n and which converges in the domain $\mathfrak{D}$. The p-adic L-function, $L_p(s;\chi)$, is generated by taking $L_p(s;\chi) = (s-1)^{-1}A_\chi(1-s)$.

In our work we first let τ be an element of a finite field extension of $\mathbf{Q}_p$, contained in the algebraic closure, $\overline{\mathbf{Q}}_p$, of $\mathbf{Q}_p$, with $|\tau|_p \leq 1$. We then define the sequence $\{b_n(\tau)\}_{n=0}^{\infty}$ by

$$b_n(\tau) = B_{n,\chi_n}(q\tau) - \chi_n(p)p^{n-1}B_{n,\chi_n}\left(p^{-1}q\tau\right).$$

The sequence $\{c_n(\tau)\}_{n=0}^{\infty}$ is defined as above, and we prove

PROPOSITION 3.3. *For all* $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$, *and for* $n \in \mathbf{Z}$, $n \geq 0$, *we have* $|c_n(\tau)|_p \leq |pqf_\chi|_p^{-1}|q|_p^n$.

At this point it follows that a p-adic power series $A_\chi(s,\tau)$ exists, satisfying $A_\chi(n,\tau) = b_n(\tau)$, and converging in $\mathfrak{D}$. We can then form the

p-adic function $L_p(s, \tau; \chi)$, satisfying $L_p(1-n, \tau; \chi) = -b_n(\tau)/n$, by taking $L_p(s, \tau; \chi) = (s-1)^{-1}A_\chi(1-s, \tau)$. However, this is only for $\tau \in \overline{\mathbf{Q}}_p$, $|\tau|_p \leq 1$. In order to prove this for all $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$, we derive a means of defining $L_p(s, \tau; \chi)$ for each such τ, and then prove the following:

LEMMA 3.12. *Let $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$, and let $\{\tau_i\}_{i=1}^{\infty}$ be a sequence in $\overline{\mathbf{Q}}_p$, with $|\tau_i|_p \leq 1$, such that $\tau_i \to \tau$. Then for each $n \in \mathbf{Z}$, $n \geq 1$,*

$$\lim_{i\to\infty} L_p(1-n, \tau_i; \chi) = L_p(1-n, \tau; \chi).$$

Therefore, as a consequence of this, we deduce

THEOREM 3.13. *For each $\tau \in \mathbf{C}_p$, with $|\tau|_p \leq 1$, there exists a unique p-adic, meromorphic function $L_p(s, \tau; \chi)$ that satisfies*

$$L_p(1-n, \tau; \chi) = -\frac{1}{n}\left(B_{n,\chi_n}(q\tau) - \chi_n(p)p^{n-1}B_{n,\chi_n}\left(p^{-1}q\tau\right)\right),$$

for each $n \in \mathbf{Z}$, $n \geq 1$. Furthermore, this function can be expressed in the form

$$L_p(s, \tau; \chi) = \frac{a_{-1}(\tau)}{s-1} + \sum_{n=0}^{\infty} a_n(\tau)(s-1)^n,$$

where the power series converges in the domain $\mathfrak{D}$, and

$$a_{-1}(\tau) = \begin{cases} 1-\frac{1}{p}, & \text{if } \chi = 1 \\ 0, & \text{if } \chi \neq 1. \end{cases}$$

Once we have established the existence of $L_p(s, \tau; \chi)$ for all $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$, we proceed to investigate the properties of the two variable function $L_p(s, t; \chi)$, where $s \in \mathfrak{D}$, $s \neq 1$ if $\chi = 1$, and $t \in \mathbf{C}_p$ with $|t|_p \leq 1$. In Section 4 we derive the following for all primes p:

THEOREM 4.3. *Let $t \in \mathbf{C}_p$, $|t|_p \leq 1$, and $s \in \mathfrak{D}$, except $s \neq 1$ if $\chi = 1$. Then $L_p(s, -t; \chi) = \chi(-1)L_p(s, t; \chi)$.*

This property follows from a similar property for the generalized Bernoulli polynomials. An immediate consequence of this is that $L_p(s; \chi) = 0$ when χ is odd. Another property of $L_p(s, t; \chi)$ is given by

LEMMA 4.6. *Let* $t \in \mathbf{C}_p$, $|t|_p \leq 1$, *and* $s \in \mathfrak{D}$, *except* $s \neq 1$ *if* $\chi = 1$. *Then*

$$\frac{\partial^n}{\partial t^n} L_p(s,t;\chi) = n! q^n \binom{-s}{n} L_p(s+n,t;\chi_n),$$

for $n \in \mathbf{Z}$, $n \geq 0$.

Here we are taking

$$\left.\binom{-s}{n} L_p(s+n,t;\chi)\right|_{s=1-n} = -\frac{1}{n}\left(1-\chi(p)p^{-1}\right) B_{0,\chi},$$

for $n \in \mathbf{Z}$, $n \geq 1$. Note that this result implies that

$$\frac{\partial^{p-1}}{\partial t^{p-1}} L_p(s,t;\chi) = (p-1)! q^{p-1} \binom{-s}{p-1} L_p(s+p-1,t;\chi).$$

Because of this lemma we can find a power series expansion of $L_p(s,t;\chi)$ in the variable t about any $\alpha \in \mathbf{C}_p$, $|\alpha|_p \leq 1$.

THEOREM 4.7. *Let* $t \in \mathbf{C}_p$, $|t|_p \leq 1$, *and* $s \in \mathfrak{D}$, *except* $s \neq 1$ *if* $\chi = 1$. *Then for* $\alpha \in \mathbf{C}_p$, $|\alpha|_p \leq 1$,

$$L_p(s,t;\chi) = \sum_{m=0}^{\infty} \binom{-s}{m} q^m (t-\alpha)^m L_p(s+m,\alpha;\chi_m).$$

When $\alpha = 0$, this theorem yields an expansion of $L_p(s,t;\chi)$ in terms of $L_p(s;\chi_m)$ for $m \in \mathbf{Z}$, and thus yields an additional method of derivation of $L_p(s,t;\chi)$.

Let $F_0 = \operatorname{lcm}(f_\chi, q)$, and let F be a positive multiple of $pq^{-1}F_0$. If we define $\langle a+qt\rangle = \omega^{-1}(a)(a+qt)$ for $a \in \mathbf{Z}$, $(a,p)=1$, and $t \in \mathbf{C}_p$, $|t|_p \leq 1$, where ω is the Teichmüller character, then we have the following:

THEOREM 4.8. *Let* $t \in \mathbf{C}_p$, $|t|_p \leq 1$, *and* $s \in \mathfrak{D}$, *except* $s \neq 1$ *if* $\chi = 1$. *Then*

$$L_p(s,t+F;\chi) - L_p(s,t;\chi) = -\sum_{\substack{a=1\\(a,p)=1}}^{qF} \chi_1(a)\langle a+qt\rangle^{-s}.$$

We then have a connection between certain finite sums and the function $L_p(s,t;\chi)$. As a result of this, we obtain

COROLLARY 4.9. *Let $s \in \mathfrak{D}$, except $s \neq 1$ if $\chi = 1$. Then*

$$L_p(s, F; \chi) = L_p(s; \chi) - \sum_{\substack{a=1 \\ (a,p)=1}}^{qF} \chi_1(a)\langle a\rangle^{-s}.$$

Thus, when t takes on certain values, we have a finite expression for $L_p(s, t; \chi)$ in terms of previously known functions.

By combining the previous two theorems, we can obtain the relation

$$\sum_{\substack{a=1 \\ (a,p)=1}}^{qF} \chi_1(a)\langle a\rangle^{-s} = -\sum_{m=1}^{\infty} \binom{-s}{m} q^m F^m L_p(s+m; \chi_m),$$

where F is a positive multiple of $pq^{-1}F_0$, $F_0 = \operatorname{lcm}(f_\chi, q)$, and $s \in \mathfrak{D}$, except $s \neq 1$ if $\chi = 1$. This is a generalization of a result of Barsky found in [2] (see also [20]).

A number of congruences relating to the ordinary and the generalized Bernoulli numbers have found a considerable amount of interest. One of the more notable examples is the Kummer congruence for the ordinary Bernoulli numbers, which states that $p^{-1}\Delta_c \frac{1}{n} B_n \in \mathbf{Z}_p$, where $c \in \mathbf{Z}$ is positive with $c \equiv 0 \pmod{p-1}$, and $n \in \mathbf{Z}$ is positive, even, and $n \not\equiv 0 \pmod{p-1}$ (see [19], p. 61). Note that we are using Δ_c to denote the forward difference operator, $\Delta_c x_n = x_{n+c} - x_n$, so that

$$\Delta_c^k x_n = \sum_{m=0}^{k} \binom{k}{m} (-1)^{k-m} x_{n+mc}.$$

More generally, it can be shown that $p^{-k}\Delta_c^k \frac{1}{n} B_n \in \mathbf{Z}_p$, where $k \in \mathbf{Z}$, with $k \geq 1$, and c and n are as above, but with $n > k$.

The application of Kummer's congruence to generalized Bernoulli numbers was first treated by Carlitz in [5], with the result that $p^{-k}\Delta_c^k \frac{1}{n} B_{n,\chi} \in \mathbf{Z}_p[\chi]$, for positive $c \in \mathbf{Z}$ with $c \equiv 0 \pmod{p-1}$, $n, k \in \mathbf{Z}$ with $n > k \geq 1$, and χ such that $f_\chi \neq p^\mu$, where $\mu \in \mathbf{Z}$, $\mu \geq 0$. From [7] (see also [18]) we see that if the operator Δ_c^k is applied to the quantity $-(1 - \chi_n(p)p^{n-1})B_{n,\chi_n}/n$, the value of $L_p(1-n; \chi)$, for similar c and characters χ, then the congruence will still hold if the restriction $n > k$ is dropped, requiring only that $n \geq 1$. In addition to this, the divisibility requirements on c can be removed, yielding a congruence of the form

$$q^{-k}\Delta_c^k \frac{1}{n} \left(1 - \chi_n(p)p^{n-1}\right) B_{n,\chi_n} \in \mathbf{Z}_p[\chi],$$

for $c, n, k \in \mathbf{Z}$, each positive, and χ such that $f_\chi \neq p^\mu$, $\mu \in \mathbf{Z}$, $\mu \geq 0$. Recall that we are taking $q = 4$ if $p = 2$, and $q = p$ otherwise. If we denote

$$\beta_{n,\chi} = -\frac{1}{n}\left(1 - \chi_n(p)p^{n-1}\right) B_{n,\chi_n},$$

then this congruence can be expressed as $q^{-k}\Delta_c^k\beta_{n,\chi} \in \mathbf{Z}_p[\chi]$.

As an extension of the Kummer congruence, Gunaratne (see [10], [11]) has shown that if $p > 3$, $c, n, k \in \mathbf{Z}$ are positive, and $\chi = \omega^h$, where $h \in \mathbf{Z}$ and $h \not\equiv 0 \pmod{p-1}$, then the value of $p^{-k}\Delta_c^k\beta_{n,\chi}$ modulo $p\mathbf{Z}_p$ is independent of n, and further satisfies

$$p^{-k}\Delta_c^k\beta_{n,\chi} \equiv p^{-k'}\Delta_c^{k'}\beta_{n',\chi} \pmod{p\mathbf{Z}_p}$$

for positive $n', k' \in \mathbf{Z}$ with $k \equiv k' \pmod{p-1}$. Additionally, by means of the binomial coefficient operator

$$\binom{p^{-1}\Delta_c}{k} x_n = \frac{1}{k!}\left(\prod_{j=0}^{k-1}(p^{-1}\Delta_c - j)\right) x_n,$$

for these χ we have $\binom{p^{-1}\Delta_c}{k}\beta_{n,\chi} \in \mathbf{Z}_p$, with a value modulo $p\mathbf{Z}_p$ that is independent of n.

By utilizing Corollary 4.9, we can derive a collection of congruences, similar to the results of Gunaratne, relating to the generalized Bernoulli polynomials, but without a restriction on either p or χ.

THEOREM 4.10. *Let n, c, and k be positive integers, and let $\tau \in \mathbf{Z}_p$ such that $|\tau|_p \leq |pq^{-1}F_0|_p$. Then the quantity $q^{-k}\Delta_c^k\beta_{n,\chi}(\tau) - q^{-k}\Delta_c^k\beta_{n,\chi}(0) \in \mathbf{Z}_p[\chi]$, and, modulo $q\mathbf{Z}_p[\chi]$, is independent of n.*

Here we denote

$$\beta_{n,\chi}(t) = -\frac{1}{n}\left(B_{n,\chi_n}(qt) - \chi_n(p)p^{n-1}B_{n,\chi_n}\left(p^{-1}qt\right)\right),$$

the value of $L_p(1-n, t; \chi)$. In addition to this result, we have each of the following:

THEOREM 4.11. *Let n, c, k, and k' be positive integers with $k \equiv k' \pmod{p-1}$, and let $\tau \in \mathbf{Z}_p$ such that $|\tau|_p \leq |pq^{-1}F_0|_p$. Then*

$$\begin{aligned} &q^{-k}\Delta_c^k\beta_{n,\chi}(\tau) - q^{-k}\Delta_c^k\beta_{n,\chi}(0) \\ &\qquad \equiv q^{-k'}\Delta_c^{k'}\beta_{n,\chi}(\tau) - q^{-k'}\Delta_c^{k'}\beta_{n,\chi}(0) \pmod{p\mathbf{Z}_p[\chi]}. \end{aligned}$$

THEOREM 4.12. *Let n, c, and k be positive integers, and let $\tau \in \mathbf{Z}_p$ such that $|\tau|_p \leq |pq^{-1}F_0|_p$. Then the quantity*

$$\binom{q^{-1}\Delta_c}{k}\beta_{n,\chi}(\tau) - \binom{q^{-1}\Delta_c}{k}\beta_{n,\chi}(0) \in \mathbf{Z}_p[\chi],$$

and, modulo $q\mathbf{Z}_p[\chi]$, is independent of n.

These results show that if related congruences hold for

$$\beta_{n,\chi}(0) = -\frac{1}{n}(1 - \chi_n(p)p^{n-1})B_{n,\chi_n},$$

then they must also hold for $\beta_{n,\chi}(\tau)$, where τ is any element of $\mathbf{Z}_p$ such that $|\tau|_p \leq |pq^{-1}F_0|_p$.

In [9] Granville defined ordinary Bernoulli numbers of negative index, B_{-n}, where $n \in \mathbf{Z}$, $n \geq 1$, in the field $\mathbf{Q}_p$ according to

$$B_{-n} = \lim_{k\to\infty} B_{\phi(p^k)-n},$$

where the limit is taken in the p-adic sense. In a similar manner we define generalized Bernoulli numbers of negative index, $B_{-n,\chi}$, $n \in \mathbf{Z}$, $n \geq 1$, and a collection of functions that correspond to generalized Bernoulli polynomials of negative index, $B_{-n,\chi}(t)$, $n \in \mathbf{Z}$, $n \geq 1$. As a result of our definitions, we show that the $B_{-n,\chi}(t)$ are actually power series that can be written in the form

$$B_{-n,\chi}(t) = \sum_{m=0}^{\infty}\binom{-n}{m}B_{-n-m,\chi}t^m,$$

converging for $t \in \mathbf{C}_p$, $|t|_p < 1$. We close out by considering some properties of these functions.

2. PRELIMINARIES

The p-adic L-functions, $L_p(s;\chi)$, were first generated by Kubota and Leopoldt for the purpose of finding functions that would serve as analogues of the Dirichlet L-functions in the p-adic number field [14]. They are characterized by the fact that they interpolate a specific expression involving generalized Bernoulli numbers when the variable s is a nonpositive integer. In the following, for each $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$, we derive a p-adic function $L_p(s,\tau;\chi)$ that interpolates a specific expression involving generalized

Bernoulli polynomials in τ for similar values of the variable s. These functions are designed so that $L_p(s,0;\chi) = L_p(s;\chi)$. The method of derivation follows that found in [13], Chapter 3. However, this method will only account for those $\tau \in \overline{\mathbf{Q}}_p$ with $|\tau|_p \leq 1$. To complete the derivation we show that there exist functions $L_p(s,\tau;\chi)$ for all $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$, such that for every sequence $\{\tau_i\}_{i=0}^{\infty}$ in $\overline{\mathbf{Q}}_p$, with $|\tau_i|_p \leq 1$, converging to some $\tau \in \mathbf{C}_p$, the sequence $\{L_p(1-n,\tau_i;\chi)\}_{i=0}^{\infty}$, with $n \in \mathbf{Z}$, $n \geq 1$, converges to $L_p(1-n,\tau;\chi)$. Thus for each $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$, the function $L_p(s,\tau;\chi)$ must interpolate the appropriate expressions involving generalized Bernoulli polynomials for $s = 1-n$, $n \in \mathbf{Z}$, $n \geq 1$.

Before we begin the derivation, we must first define the concepts that we shall need and review some of their resulting properties.

2.1 Dirichlet characters

For $n \in \mathbf{Z}$, $n \geq 1$, a Dirichlet character to the modulus n is a multiplicative map $\chi : \mathbf{Z} \to \mathbf{C}$ such that $\chi(a+n) = \chi(a)$ for all $a \in \mathbf{Z}$, and $\chi(a) = 0$ if and only if $(a,n) \neq 1$. Since $a^{\phi(n)} \equiv 1 \pmod{n}$ for all a such that $(a,n) = 1$, $\chi(a)$ must be a root of unity for such a.

If χ is a Dirichlet character to the modulus n, then for any positive multiple m of n we can induce a Dirichlet character ψ to the modulus m according to

$$\psi(a) = \begin{cases} \chi(a), & \text{if } (a,m) = 1 \\ 0, & \text{if } (a,m) \neq 1. \end{cases}$$

The minimum modulus n for which a character χ cannot be induced from some character to the modulus m, $m < n$, is called the conductor of χ, denoted f_χ. We shall assume that each χ is defined modulo its conductor. Such a character is said to be primitive.

For primitive Dirichlet characters χ and ψ having conductors f_χ and f_ψ, respectively, we define the product, $\chi\psi$, to be the primitive character with $\chi\psi(a) = \chi(a)\psi(a)$ for all $a \in \mathbf{Z}$ such that $(a, f_\chi f_\psi) = 1$. Note that there may exist some values of a such that $\chi\psi(a) \neq \chi(a)\psi(a)$, due to the fact that our definition requires $\chi\psi$ to be a primitive character. The conductor $f_{\chi\psi}$ then divides $\operatorname{lcm}(f_\chi, f_\psi)$. With this operation defined, we can then consider the set of primitive Dirichlet characters to form a group under multiplication. The identity of the group is the principal character $\chi = 1$, having conductor $f_1 = 1$. The inverse of the character χ is the character $\chi^{-1} = \overline{\chi}$, the map of complex conjugates of the values of χ.

Since any Dirichlet character χ is multiplicative, we must have $\chi(-1) = \pm 1$. A character χ is said to be odd if $\chi(-1) = -1$, and even if $\chi(-1) = 1$.

2.2 Generalized Bernoulli polynomials

Let χ be a Dirichlet character with conductor f_χ. Then we define the functions, $B_{n,\chi}(t)$, $n \in \mathbf{Z}$, $n \geq 0$, by the generating function

$$(1) \qquad \sum_{a=1}^{f_\chi} \frac{\chi(a) x e^{(a+t)x}}{e^{f_\chi x} - 1} = \sum_{n=0}^{\infty} B_{n,\chi}(t) \frac{x^n}{n!}, \quad |x| < \frac{2\pi}{f_\chi}.$$

We define the generalized Bernoulli numbers associated with χ, $B_{n,\chi}$, $n \in \mathbf{Z}$, $n \geq 0$, by

$$\sum_{a=1}^{f_\chi} \frac{\chi(a) x e^{ax}}{e^{f_\chi x} - 1} = \sum_{n=0}^{\infty} B_{n,\chi} \frac{x^n}{n!}, \quad |x| < \frac{2\pi}{f_\chi},$$

so that $B_{n,\chi}(0) = B_{n,\chi}$. Note that

$$\sum_{a=1}^{f_\chi} \frac{\chi(a) x e^{(a+t)x}}{e^{f_\chi x} - 1} = e^{tx} \sum_{a=1}^{f_\chi} \frac{\chi(a) x e^{ax}}{e^{f_\chi x} - 1},$$

which implies that

$$\sum_{n=0}^{\infty} B_{n,\chi}(t) \frac{x^n}{n!} = e^{tx} \sum_{n=0}^{\infty} B_{n,\chi} \frac{x^n}{n!},$$

and from this we obtain

$$(2) \qquad B_{n,\chi}(t) = \sum_{m=0}^{n} \binom{n}{m} B_{n-m,\chi} t^m.$$

Thus the functions $B_{n,\chi}(t)$, defined in (1), are actually polynomials, called the generalized Bernoulli polynomials associated with χ. Let $\mathbf{Z}[\chi]$ denote the ring generated over $\mathbf{Z}$ by all the values $\chi(a)$, $a \in \mathbf{Z}$, and $\mathbf{Q}(\chi)$ the field generated over $\mathbf{Q}$ by all such values. Then it can be shown that $f_\chi B_{n,\chi}$ must be in $\mathbf{Z}[\chi]$ for each $n \geq 0$ whenever $\chi \neq 1$. In general, we have $B_{n,\chi} \in \mathbf{Q}(\chi)$ for each $n \geq 0$, and so $B_{n,\chi}(t) \in \mathbf{Q}(\chi)[t]$. The polynomials $B_{n,\chi}(t)$ exhibit the property that, for all $n \geq 0$,

$$(3) \qquad B_{n,\chi}(-t) = (-1)^n \chi(-1) B_{n,\chi}(t),$$

whenever $\chi \neq 1$. Thus $B_{n,\chi}(t)$, for $\chi \neq 1$, is either an even function or an odd function according to whether $(-1)^n \chi(-1)$ is 1 or -1. From (3) we obtain

$$B_{n,\chi} = (-1)^n \chi(-1) B_{n,\chi} ,$$

and so $B_{n,\chi} = 0$ whenever n is even and χ is odd, or whenever n is odd and χ is even, $\chi \neq 1$. Another property that the polynomials satisfy is that for $m \in \mathbf{Z}$, $m \geq 1$,

$$B_{n,\chi}(mf_\chi + t) - B_{n,\chi}(t) = n \sum_{a=1}^{mf_\chi} \chi(a)(a+t)^{n-1}, \tag{4}$$

for all $n \geq 0$. This can be derived from (1). Note that for $\chi = 1$ and $t = 0$ this becomes

$$\frac{1}{n}\left(B_{n,1}(m) - B_{n,1}\right) = \sum_{a=1}^{m} a^{n-1}.$$

If $\chi \neq 1$, then it can be shown that $\sum_{a=1}^{f_\chi} \chi(a) = 0$, and from the above relations we can derive

$$B_{0,\chi} = \frac{1}{f_\chi} \sum_{a=1}^{f_\chi} \chi(a)$$

for all χ. Therefore

$$B_{0,\chi} = \begin{cases} 0, & \text{if } \chi \neq 1 \\ 1, & \text{if } \chi = 1 . \end{cases}$$

The ordinary Bernoulli polynomials, $B_n(t)$, $n \in \mathbf{Z}$, $n \geq 0$, are defined by

$$\frac{xe^{tx}}{e^x - 1} = \sum_{n=0}^{\infty} B_n(t) \frac{x^n}{n!}, \quad |x| < 2\pi, \tag{5}$$

and the Bernoulli numbers, B_n, $n \in \mathbf{Z}$, $n \geq 0$,

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}, \quad |x| < 2\pi .$$

From this we obtain the values $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_4 = -1/30, \ldots$, with $B_n = 0$ for odd $n \geq 3$. For even $n \geq 2$, we have

$$B_n = -\frac{1}{n+1} \sum_{m=0}^{n-1} \binom{n+1}{m} B_m .$$

Note that we again have the relations $B_n(0) = B_n$ and

$$B_n(t) = \sum_{m=0}^{n} \binom{n}{m} B_{n-m} t^m,$$

as we did for the generalized Bernoulli polynomials.

Some of the more important properties of Bernoulli polynomials are that

$$B_n(t+1) - B_n(t) = nt^{n-1} \tag{6}$$

for all $n \geq 1$, and

$$B_n(1-t) = (-1)^n B_n(t)$$

for $n \geq 0$. Each of these results can be derived from the generating function (5) above.

Similar to (4) for the generalized Bernoulli polynomials, whenever $m, n \in \mathbf{Z}$, $m \geq 1$, $n \geq 1$,

$$\frac{1}{n}(B_n(m) - B_n) = \sum_{a=0}^{m-1} a^{n-1},$$

where we take 0^0 to be 1 in the case of $a = 0$ and $n = 1$. Note that this can be derived from (6) since

$$B_n(m) - B_n = \sum_{a=0}^{m-1} (B_n(a+1) - B_n(a)) .$$

The Bernoulli numbers are rational numbers, and, in fact, the von Staudt-Clausen theorem states that for even $n \geq 2$,

$$B_n + \sum_{\substack{p \text{ prime} \\ (p-1)|n}} \frac{1}{p} \in \mathbf{Z}.$$

Thus the denominator of each B_n must be square-free.

The ordinary Bernoulli numbers are related to the generalized Bernoulli numbers in that for $\chi = 1$ we have

$$\frac{xe^x}{e^x - 1} = \sum_{n=0}^{\infty} B_{n,1} \frac{x^n}{n!}, \quad |x| < 2\pi,$$

and since

$$\frac{xe^x}{e^x - 1} = x + \frac{x}{e^x - 1},$$

we see that $B_{n,1} = B_n$ for all $n \neq 1$, and $B_{1,1} = -B_1$. In fact, this can be written as $B_{n,1} = (-1)^n B_n$, and for the polynomials, $B_{n,1}(t) = (-1)^n B_n(-t)$.

2.3 Dirichlet L-functions

For χ a Dirichlet character with conductor f_χ, the Dirichlet L-function for χ is defined by

$$L(s;\chi) = \sum_{b=1}^{\infty} \frac{\chi(b)}{b^s},$$

for $s \in \mathbf{C}$ such that $\Re(s) > 1$. Note that $L(s;\chi)$ can be continued analytically to all of $\mathbf{C}$, except for a pole of order 1 at $s=1$ when $\chi = 1$.

Let $\tau(\chi)$ be a Gauss sum,

$$\tau(\chi) = \sum_{a=1}^{f_\chi} \chi(a) e^{2\pi i a/f_\chi},$$

where $i^2 = -1$, and let

$$\delta_\chi = \begin{cases} 0, & \text{if } \chi(-1) = 1 \\ 1, & \text{if } \chi(-1) = -1 . \end{cases}$$

Then $L(s;\chi)$ satisfies the functional equation

$$(7)\quad \left(\frac{f_\chi}{\pi}\right)^{s/2} \Gamma\left(\frac{s+\delta_\chi}{2}\right) L(s;\chi) = W_\chi \left(\frac{f_\chi}{\pi}\right)^{(1-s)/2} \Gamma\left(\frac{1-s+\delta_\chi}{2}\right) L(1-s;\overline{\chi}),$$

where $\Gamma(s)$ is the gamma function, and $W_\chi = \frac{\tau(\chi)}{i^{\delta_\chi}\cdot\sqrt{f_\chi}}$, having the property that $|W_\chi| = 1$. Since $\Gamma(s)$ has simple poles at the negative integers, $L(s;\chi)$ must be zero for $s = 1-n$, where $n \in \mathbf{Z}$, $n \geq 1$, such that $n \not\equiv \delta_\chi \pmod 2$, except when $\chi = 1$ and $n = 1$. $L(s;\chi)$ can also be described by means of the Euler product $L(s;\chi) = \prod_{p \text{ prime}} \left(1 - \chi(p)p^{-s}\right)^{-1}$, for $s \in \mathbf{C}$ such that $\Re(s) > 1$. Thus $L(s;\chi) \neq 0$ in this domain.

The generalized Bernoulli numbers, $B_{n,\chi}$, and the Dirichlet L-function, $L(s;\chi)$, share the following relationship, a proof of this being found in [13]:

Theorem 2.1. *Let χ be a Dirichlet character, and let $n \in \mathbf{Z}$, $n \geq 1$. Then $L(1-n;\chi) = -\frac{1}{n}B_{n,\chi}$.*

Thus we have a way to express certain values of a function defined in terms of an infinite sum as quantities that can be found by a finite process.

2.4 THE p-ADIC NUMBER FIELD

Let p be prime. We shall use $\mathbf{Z}_p$ to represent the p-adic integers, and $\mathbf{Q}_p$ the p-adic rationals. Let $|\cdot|_p$ denote the p-adic absolute value on $\mathbf{Q}_p$, normalized so that $|p|_p = p^{-1}$. Let $\overline{\mathbf{Q}}_p$ be the algebraic closure of $\mathbf{Q}_p$. The absolute value on $\mathbf{Q}_p$ extends uniquely to $\overline{\mathbf{Q}}_p$, however $\overline{\mathbf{Q}}_p$ is not complete with respect to the absolute value. Let $\mathbf{C}_p$ be the completion of $\overline{\mathbf{Q}}_p$ with respect to this absolute value. Then the absolute value extends to $\mathbf{C}_p$, and $\overline{\mathbf{Q}}_p$ is dense in $\mathbf{C}_p$. We also have $\mathbf{C}_p$ algebraically closed. Furthermore, on $\mathbf{C}_p$ the absolute value is non-Archimedean, and so

$$|a+b|_p \leq \max\{|a|_p, |b|_p\}$$

for any $a, b \in \mathbf{C}_p$. Note that the two fields $\mathbf{C}$ and $\mathbf{C}_p$ are algebraically isomorphic, and any one of the two can be embedded in the other. We denote two particular subrings of $\mathbf{C}_p$ in the following manner

$$\mathfrak{o} = \{a \in \mathbf{C}_p : |a|_p \leq 1\}, \qquad \mathfrak{p} = \{a \in \mathbf{C}_p : |a|_p < 1\}.$$

Then $\mathfrak{p}$ is a maximal ideal of $\mathfrak{o}$. If $\tau \in \mathbf{C}_p$ such that $|\tau|_p \leq |p|_p^s$, where $s \in \mathbf{Q}$, then $\tau \in p^s\mathfrak{o}$, and so we shall also write this as $\tau \equiv 0 \pmod{p^s\mathfrak{o}}$.

Any $n \in \mathbf{Z}$, $n > 0$, can be uniquely expressed in the form $n = \sum_{m=0}^{k} a_m p^m$, where $a_m \in \mathbf{Z}$, $0 \leq a_m \leq p-1$, for $m = 0, 1, \ldots, k$, and $a_k \neq 0$. For such n, we define

$$s_p(n) = \sum_{m=0}^{k} a_m,$$

the sum of the p-adic digits of n, and also define $s_p(0) = 0$. For any $n \in \mathbf{Z}$, let $v_p(n)$ be the highest power of p dividing n. This function is additive, and relates to the function $s_p(n)$ by means of the identity

$$v_p(n!) = \frac{n - s_p(n)}{p-1}, \tag{8}$$

which holds for all $n \geq 0$. Note that for $n \geq 1$ this implies that

$$v_p(n!) \leq \frac{n-1}{p-1}.$$

The definition of this function can be extended to all of $\mathbf{Q}$ by taking $v_p(1/n) = -v_p(n)$.

Throughout we let $q = 4$ if $p = 2$, and $q = p$ otherwise. Note that there exist $\phi(q)$ distinct solutions, modulo q, to the equation $x^{\phi(q)} - 1 = 0$, and each solution must be congruent to one of the values $a \in \mathbf{Z}$, where $1 \leq a \leq q$,

$(a,p)=1$. Thus, by Hensel's Lemma, given $a \in \mathbf{Z}$ with $(a,p)=1$, there exists a unique $\omega(a) \in \mathbf{Z}_p$, where $\omega(a)^{\phi(q)}=1$, such that

$$\omega(a) \equiv a \pmod{q\mathbf{Z}_p}.$$

Letting $\omega(a)=0$ for $a \in \mathbf{Z}$ such that $(a,p) \neq 1$, we see that ω is actually a Dirichlet character, called the Teichmüller character, having conductor $f_\omega = q$. Let us define

$$\langle a \rangle = \omega^{-1}(a)a.$$

Then $\langle a \rangle \equiv 1 \pmod{q\mathbf{Z}_p}$. For $p \geq 3$, $\lim_{n\to\infty} a^{p^n} = \omega(a)$, since $a^{p^n} \equiv a \pmod{p}$ and $a^{p^n(p-1)} \equiv 1 \pmod{p^{n+1}}$.

For our purposes we shall need to make a slight extension of the definition of the Teichmüller character ω. If $t \in \mathbf{C}_p$ such that $|t|_p \leq 1$, then for any $a \in \mathbf{Z}$, $a+qt \equiv a \pmod{q\mathfrak{o}}$. Thus we define

$$\omega(a+qt) = \omega(a)$$

for these values of t. We also define

$$\langle a+qt \rangle = \omega^{-1}(a)(a+qt)$$

for such t.

Fix an embedding of the algebraic closure of $\mathbf{Q}$, $\overline{\mathbf{Q}}$, into $\mathbf{C}_p$. We may then consider the values of a Dirichlet character χ as lying in $\mathbf{C}_p$. For $n \in \mathbf{Z}$ we define the product $\chi_n = \chi\omega^{-n}$ in the sense of the product of characters. This implies that $f_{\chi_n} \mid f_\chi q$. However, since we can write $\chi = \chi_n\omega^n$, we also have $f_\chi \mid f_{\chi_n} q$. Thus f_χ and f_{χ_n} differ by a factor that is a power of p. In fact, either $f_{\chi_n}/f_\chi \in \mathbf{Z}$ and divides q, or $f_\chi/f_{\chi_n} \in \mathbf{Z}$ and divides q.

Let $\mathbf{Q}_p(\chi)$ denote the field generated over $\mathbf{Q}_p$ by all values $\chi(a)$, $a \in \mathbf{Z}$. In this context we can state the following, found in [13] (pp. 14–15).

LEMMA 2.2. *In the field $\mathbf{Q}_p(\chi)$, for all $n \in \mathbf{Z}$, $n \geq 0$,*

$$B_{n,\chi} = \frac{1}{n+1} \lim_{h\to\infty} \frac{1}{p^h f_\chi} \left(B_{n+1,\chi}\left(p^h f_\chi\right) - B_{n+1,\chi}(0) \right).$$

From this we can obtain

LEMMA 2.3. *Let $\tau \in \mathbf{C}_p$. In the field $\mathbf{Q}_p(\chi, \tau)$, for all $n \in \mathbf{Z}$, $n \geq 0$,*

$$B_{n,\chi_n}(\tau) = \lim_{h\to\infty} \frac{1}{p^h f_\chi} \sum_{a=1}^{p^h f_\chi} \chi_n(a)(a+\tau)^n .$$

Proof. By applying Lemma 2.2 to (4), we obtain

$$B_{n,\chi} = \lim_{h\to\infty} \frac{1}{p^h f_\chi} \sum_{a=1}^{p^h f_\chi} \chi(a) a^n .$$

Therefore, by (2),

$$\begin{aligned} B_{n,\chi_n}(\tau) &= \sum_{m=0}^{n} \binom{n}{m} \tau^{n-m} \lim_{h\to\infty} \frac{1}{p^h f_{\chi_n}} \sum_{a=1}^{p^h f_{\chi_n}} \chi_n(a) a^m \\ &= \lim_{h\to\infty} \frac{1}{p^h f_{\chi_n}} \sum_{a=1}^{p^h f_{\chi_n}} \chi_n(a) \sum_{m=0}^{n} \binom{n}{m} \tau^{n-m} a^m . \end{aligned}$$

Since f_χ and f_{χ_n} differ by a factor that is a power of p, we must have

$$B_{n,\chi_n}(\tau) = \lim_{h\to\infty} \frac{1}{p^h f_\chi} \sum_{a=1}^{p^h f_\chi} \chi_n(a)(a+\tau)^n,$$

and the proof is complete. □

2.5 p-ADIC FUNCTIONS

Let K be an extension of $\mathbf{Q}_p$ contained in $\mathbf{C}_p$. An infinite series $\sum_{n=0}^{\infty} a_n$, $a_n \in K$, converges in K if and only if $|a_n|_p \to 0$ as $n \to \infty$. Let $K[[x]]$ be the algebra of formal power series in x. Then it follows that a power series

$$A(x) = \sum_{n=0}^{\infty} a_n x^n,$$

in $K[[x]]$, converges at $x = \xi$, $\xi \in \mathbf{C}_p$, if and only if $|a_n \xi^n|_p \to 0$ as $n \to \infty$. Therefore whenever a power series $A(x)$ converges at some $\xi_0 \in \mathbf{C}_p$, then it must converge at all $\xi \in \mathbf{C}_p$ such that $|\xi|_p \leq |\xi_0|_p$. The following result, for double series in K, can be found in [8].

PROPOSITION 2.4. *Let $b_{n,m} \in K$, and suppose that for each $\epsilon > 0$ there exists $N \in \mathbf{Z}$, depending on ϵ, such that if* $\max\{n, m\} \geq N$, *then* $|b_{n,m}|_p \leq \epsilon$. *Then both series*

$$\sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty} b_{n,m}\right) \quad \textit{and} \quad \sum_{m=0}^{\infty}\left(\sum_{n=0}^{\infty} b_{n,m}\right)$$

converge, and their sums are equal.

There are two power series that we wish to make note of in particular. First we define the p-adic exponential function, $\exp(x)$, in $\mathbf{Q}_p[[x]]$, by

$$\exp(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n. \tag{9}$$

From (8) we can conclude that this power series converges in $\{x \in \mathbf{C}_p : |x|_p < p^{-1/(p-1)}\}$. The p-adic logarithm function, $\log(x)$, in $\mathbf{Q}_p[[x]]$, is defined by

$$\log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n, \tag{10}$$

the power series converging in the domain $\{x \in \mathbf{C}_p : |x|_p < 1\}$. For $|x|_p < p^{-1/(p-1)}$, we have $\log(\exp(x)) = x$ and $\exp(\log(1+x)) = 1 + x$.

The following property is a uniqueness property for power series, found in [13].

LEMMA 2.5. *Let $A(x), B(x) \in K[[x]]$, such that each converges in a neighborhood of 0 in $\mathbf{C}_p$. If $A(\xi_n) = B(\xi_n)$ for a sequence $\{\xi_n\}_{n=0}^{\infty}$, $\xi_n \neq 0$, in $\mathbf{C}_p$, such that $\xi_n \to 0$, then $A(x) = B(x)$.*

Let U be an open subset of $\mathbf{C}_p$, contained in the domain of the p-adic function f. We say that f is differentiable at $x \in U$ if the limit

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists. If this limit exists for each $x \in U$, then we say that f is differentiable in U.

The relationship between the derivatives of a function and its power series expansion is given in the following result, found in [8].

PROPOSITION 2.6. *Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with coefficients in $\mathbf{C}_p$, and suppose that*

$$f(x) = \sum_{n=0}^{\infty} a_n (x-\alpha)^n$$

converges on some closed ball B in $\mathbf{C}_p$. Then

i) *For each $x \in B$, the k^{th} derivative $f^{(k)}(x)$ exists, and is given by*

$$f^{(k)}(x) = k! \sum_{n=k}^{\infty} \binom{n}{k} a_n (x-\alpha)^{n-k},$$

and we have

$$a_k = \frac{1}{k!} f^{(k)}(\alpha).$$

ii) *Let $\beta \in B$. Then there exists a series $\sum_{n=0}^{\infty} b_n x^n$ such that*

$$f(x) = \sum_{n=0}^{\infty} b_n (x-\beta)^n$$

for any $x \in B$. Both series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ have the same region of convergence.

Now let K be a finite extension of $\mathbf{Q}_p$. For $A(x) \in K[[x]]$, $A(x) = \sum_{n=0}^{\infty} a_n x^n$, where $a_n \in K$, define

$$\|A\| = \sup_n |a_n|_p.$$

Let $P_K = \{A(x) \in K[[x]] : \|A\| < \infty\}$. Then $\|\cdot\|$ defines a norm on P_K, and so $K[x] \subset P_K \subset K[[x]]$. Furthermore P_K is complete in this norm.

Let $\{b_n\}_{n=0}^{\infty}$ be a sequence of elements of K, and let the sequence $\{c_n\}_{n=0}^{\infty}$ be defined by

$$c_n = \sum_{m=0}^{n} \binom{n}{m} (-1)^{n-m} b_m \tag{11}$$

for each $n \in \mathbf{Z}$, $n \geq 0$. Then $c_n \in K$ for each $n \geq 0$. Note that (11) implies that these sequences must satisfy

$$\sum_{n=0}^{\infty} c_n \frac{t^n}{n!} = e^{-t} \sum_{n=0}^{\infty} b_n \frac{t^n}{n!}.$$

This implies that

$$\sum_{n=0}^{\infty} b_n \frac{t^n}{n!} = e^t \sum_{n=0}^{\infty} c_n \frac{t^n}{n!},$$

and so we have the relationship

$$b_n = \sum_{m=0}^{n} \binom{n}{m} c_m \tag{12}$$

for each $n \in \mathbf{Z}$, $n \geq 0$. We can reverse this process to derive (11) given (12). Thus (11) and (12) must be equivalent. The following relate to sequences that satisfy (11) and (12), and are found in [13].

THEOREM 2.7. *Let $\{b_n\}_{n=0}^{\infty}$ and $\{c_n\}_{n=0}^{\infty}$ be defined as in the above relation. Let $\rho \in \mathbf{R}$ such that $0 < \rho < |p|_p^{1/(p-1)}$. If $|c_n|_p \leq C\rho^n$ for all $n \geq 0$, where $C > 0$, then there exists a unique power series $A(x) \in P_K$ such that $A(x)$ converges at every $\xi \in \mathbf{C}_p$ with $|\xi|_p < |p|_p^{1/(p-1)}\rho^{-1}$, and $A(n) = b_n$ for every $n \geq 0$.*

COROLLARY 2.8. *Let $A(x)$ be the power series from the theorem. Then for each $\xi \in \mathbf{C}_p$ such that $|\xi|_p < |p|_p^{1/(p-1)}\rho^{-1}$, we have*

$$A(\xi) = \sum_{n=0}^{\infty} c_n \binom{\xi}{n}.$$

Theorem 2.7 can be applied to the sequence $\{b_n\}_{n=0}^{\infty}$ in $K = \mathbf{Q}_p(\chi)$, where

$$b_n = \left(1 - \chi_n(p)p^{n-1}\right) B_{n,\chi_n},$$

in order to obtain a power series $A_\chi(s)$ satisfying $A_\chi(n) = b_n$, and converging on the domain $\{s \in \mathbf{C}_p : |s|_p < |p|_p^{1/(p-1)}|q|_p^{-1}\}$. (Since $|p|_p^{1/(p-1)}|q|_p^{-1} > 1$ and $|n|_p \leq 1$ for each $n \in \mathbf{Z}$, all of $\mathbf{Z}$ is contained in this domain.) From this a p-adic function, $L_p(s;\chi)$, can be derived that interpolates the values

$$L_p(1-n;\chi) = -\frac{1}{n} b_n,$$

and which converges in $\{s \in \mathbf{C}_p : |s-1|_p < |p|_p^{1/(p-1)}|q|_p^{-1}\}$, except $s \neq 1$ if $\chi = 1$. Note that if χ is odd, then χ_n is even when n is odd, and χ_n is odd when n is even. Thus the quantity $(1-\chi_n(p)p^{n-1})B_{n,\chi_n} = 0$ for all $n \in \mathbf{Z}$, $n \geq 1$, as we saw from the properties of generalized Bernoulli numbers. Therefore $L_p(s;\chi)$ vanishes on a sequence such as $\{-p^m\}_{m=0}^{\infty}$, which has 0 as a limit point, implying that for such χ we must have $L_p(s;\chi) \equiv 0$.

3. The p-adic L-function $L_p(s,t;\chi)$

In the following, we apply Theorem 2.7 to the sequence $\{b_n(\tau)\}_{n=0}^{\infty}$, where $b_n(\tau) = B_{n,\chi_n}(q\tau) - \chi_n(p)p^{n-1}B_{n,\chi_n}\left(p^{-1}q\tau\right)$, for $\tau \in \overline{\mathbf{Q}}_p$, $|\tau|_p \le 1$, to show that there exists a power series $A_\chi(s,\tau) \in K_\tau[[s]]$, $K_\tau = \mathbf{Q}_p(\chi,\tau)$, which converges on $\{s \in \mathbf{C}_p : |s|_p < |p|_p^{1/(p-1)}|q|_p^{-1}\}$. From this we can prove the existence of a p-adic function, $L_p(s,\tau;\chi)$, that interpolates the values $L_p(1-n,\tau;\chi) = -\frac{1}{n}b_n(\tau)$ for $n \in \mathbf{Z}$, $n \ge 1$, and converges in $\{s \in \mathbf{C}_p : |s-1|_p < |p|_p^{1/(p-1)}|q|_p^{-1}\}$, except $s \ne 1$ if $\chi = 1$. After this we will show that there exists $L_p(s,\tau;\chi)$ for each $\tau \in \mathbf{C}_p$, $|\tau|_p \le 1$, satisfying

$$L_p(1-n,\tau;\chi) = -\frac{1}{n}b_n(\tau),$$

and converging in the domain above.

3.1 $L_p(s,\tau;\chi)$ for $\tau \in \overline{\mathbf{Q}}_p$, $|\tau|_p \le 1$

Let p be prime, and let χ be a Dirichlet character with conductor f_χ. Let $\tau \in \overline{\mathbf{Q}}_p$, $|\tau|_p \le 1$, and let $K_\tau = \mathbf{Q}_p(\chi,\tau)$, the field generated over $\mathbf{Q}_p$ by adjoining τ and the values $\chi(a)$, $a \in \mathbf{Z}$. Since τ and each of the $\chi(a)$ are in $\overline{\mathbf{Q}}_p$, we see that K_τ is a finite extension of $\mathbf{Q}_p$ in $\overline{\mathbf{Q}}_p$. For each $\tau \in \overline{\mathbf{Q}}_p$, $|\tau|_p \le 1$, we shall derive our L-function $L_p(s,\tau;\chi)$ in a manner similar to that given for the derivation of $L_p(s;\chi)$ found in Chapter 3 of [13].

For $\tau \in \overline{\mathbf{Q}}_p$, $|\tau|_p \le 1$, define the sequences $\{b_n(\tau)\}_{n=0}^{\infty}$ and $\{c_n(\tau)\}_{n=0}^{\infty}$ in K_τ according to

$$b_n(\tau) = B_{n,\chi_n}(q\tau) - \chi_n(p)p^{n-1}B_{n,\chi_n}\left(p^{-1}q\tau\right),$$

and

$$c_n(\tau) = \sum_{m=0}^{n}\binom{n}{m}(-1)^{n-m}b_m(\tau).$$

In order to derive our L-function $L_p(s,\tau;\chi)$, we will prove a particular bound on the magnitude of $c_n(\tau)$, but to do so, we shall need the following:

LEMMA 3.1. *Let $m, r \in \mathbf{Z}$, with $m \ge 0$ and $r \ge 1$. Then*

$$\sum_{a=0}^{p^r-1} a^m \equiv 0 \pmod{p^{r-1}},$$

where we take $0^0 = 1$ in the case of $a = 0$ and $m = 0$.

Proof. This is obvious for $m = 0$, so assume that $m \geq 1$. We shall prove this result for the remaining values of m by induction on r.

Since any sum of elements of $\mathbf{Z}$ must also be in $\mathbf{Z}$, the lemma is true for $r = 1$. Now assume that the lemma holds for some $r \in \mathbf{Z}$, $r \geq 1$. By rewriting the sum

$$\sum_{a=0}^{p^{r+1}-1} a^m = \sum_{v=0}^{p-1} \sum_{u=0}^{p^r-1} \left(u + p^r v\right)^m ,$$

and reducing this modulo p^r, we obtain

$$\begin{aligned}\sum_{a=0}^{p^{r+1}-1} a^m &\equiv \sum_{v=0}^{p-1} \sum_{u=0}^{p^r-1} u^m \pmod{p^r}\\ &\equiv p \sum_{u=0}^{p^r-1} u^m \pmod{p^r}.\end{aligned}$$

By our induction hypothesis we must then have

$$\sum_{a=0}^{p^{r+1}-1} a^m \equiv 0 \pmod{p^r},$$

and the lemma follows. $\square$

LEMMA 3.2. *Let $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$, and let $n \in \mathbf{Z}$, $n \geq 0$. For all $h \in \mathbf{Z}$, $h \geq 1$,*

$$\frac{1}{q^h f_\chi} \sum_{\substack{a=1\\(a,p)=1}}^{q^h f_\chi} \chi(a) \left(\langle a + q\tau \rangle - 1\right)^n \equiv 0 \pmod{f_\chi^{-1} p^{-1} q^{n-1} \mathfrak{o}}.$$

Proof. This is obvious for $n = 0$ since writing

$$\sum_{\substack{a=1\\(a,p)=1}}^{q^h f_\chi} \chi(a) = \sum_{a=1}^{q^h f_\chi} \chi(a) - \sum_{a=1}^{p^{-1} q^h f_\chi} \chi(pa)$$

allows us to derive

$$\sum_{\substack{a=1\\(a,p)=1}}^{q^h f_\chi} \chi(a) = \begin{cases} p^{-1} q^h (p-1), & \text{if } \chi = 1\\ 0, & \text{if } \chi \neq 1 . \end{cases}$$

So let us assume that $n \geq 1$.

Let $h = 1$. Then $\langle a+q\tau\rangle \equiv 1 \pmod{q\mathfrak{o}}$ for all $a \in \mathbf{Z}$ such that $(a,p) = 1$ implies that

$$\left(\langle a + q\tau\rangle - 1\right)^n \equiv 0 \pmod{q^n\mathfrak{o}},$$

and the lemma holds for this case.

Now assume that $h \geq 1$. We can rewrite our sum as follows:

$$\sum_{\substack{a=1\\(a,p)=1}}^{q^h f_\chi} \chi(a)(\langle a+q\tau\rangle - 1)^n = \sum_{v=0}^{q^{h-1}-1} \sum_{\substack{u=1\\(u+vqf_\chi,p)=1}}^{qf_\chi} \chi(u+vqf_\chi)(\langle u+vqf_\chi+q\tau\rangle - 1)^n .$$

Since $|\tau|_p \leq 1$, we can write

$$\begin{aligned}\langle u + vqf_\chi + q\tau\rangle &= \left(u + vqf_\chi + q\tau\right)\omega^{-1}\left(u + vqf_\chi + q\tau\right)\\ &= (u + q\tau)\,\omega^{-1}(u + q\tau) + vqf_\chi\omega^{-1}(u + q\tau)\\ &= \langle u + q\tau\rangle + vqf_\chi\omega^{-1}(u).\end{aligned}$$

Thus

$$\begin{aligned}&\sum_{\substack{a=1\\(a,p)=1}}^{q^h f_\chi} \chi(a)\left(\langle a + q\tau\rangle - 1\right)^n\\ &\qquad = \sum_{\substack{u=1\\(u,p)=1}}^{qf_\chi} \chi(u) \sum_{v=0}^{q^{h-1}-1} \left(\langle u + q\tau\rangle - 1 + vqf_\chi\omega^{-1}(u)\right)^n .\end{aligned}$$

By expanding, the inner sum on the right can be written

$$\begin{aligned}&\sum_{v=0}^{q^{h-1}-1} (\langle u + q\tau\rangle - 1 + vqf_\chi\omega^{-1}(u))^n\\ &\qquad = \sum_{k=0}^{n} \binom{n}{k} (\langle u + q\tau\rangle - 1)^{n-k} q^k f_\chi^k \omega^{-k}(u) \sum_{v=0}^{q^{h-1}-1} v^k .\end{aligned}$$

Since $(u,p) = 1$, we obtain the equivalence

$$q^k\left(\langle u + q\tau\rangle - 1\right)^{n-k} \equiv 0 \pmod{q^n\mathfrak{o}}$$

for each k, $0 \leq k \leq n$. Furthermore, by Lemma 3.1

$$\sum_{v=0}^{q^{h-1}-1} v^k \equiv 0 \pmod{p^{-1}q^{h-1}}$$

for each such k. Therefore

$$\sum_{v=0}^{q^{h-1}-1} \left(\langle u+q\tau\rangle - 1 + vqf_\chi\omega^{-1}(u)\right)^n \equiv 0 \pmod{p^{-1}q^{n+h-1}\mathfrak{o}}.$$

This implies that

$$\sum_{\substack{a=1\\(a,p)=1}}^{q^h f_\chi} \chi(a)(\langle a+q\tau\rangle - 1)^n \equiv 0 \pmod{p^{-1}q^{n+h-1}\mathfrak{o}},$$

yielding the result. □

We now derive our bound on the magnitude of $c_n(\tau)$.

PROPOSITION 3.3. *For all* $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$, *and for* $n \in \mathbf{Z}$, $n \geq 0$, *we have* $|c_n(\tau)|_p \leq |pqf_\chi|_p^{-1}|q|_p^n$.

Proof. This follows in a manner similar to that given for the proof of the bound $|c_n(0)|_p \leq \left|q^2 f_\chi\right|_p^{-1}|q|_p^n$ found in [13] (Lemma 4 of Chapter 3). However, in this case we use Lemma 2.3 and the properties of χ and ω to derive

$$b_n(\tau) = \lim_{h\to\infty} \frac{1}{q^h f_\chi} \sum_{\substack{a=1\\(a,p)=1}}^{q^h f_\chi} \chi(a)\langle a+q\tau\rangle^n$$

for each $n \geq 0$, and thus

$$c_n(\tau) = \lim_{h\to\infty} \frac{1}{q^h f_\chi} \sum_{\substack{a=1\\(a,p)=1}}^{q^h f_\chi} \chi(a)\left(\langle a+q\tau\rangle - 1\right)^n$$

for each such n. From Lemma 3.2 we obtain

$$c_n(\tau) \equiv 0 \pmod{f_\chi^{-1}p^{-1}q^{n-1}\mathfrak{o}},$$

and thus the result. □

For our immediate concern we only need this proposition to hold for all $\tau \in \overline{\mathbf{Q}}_p$ such that $|\tau|_p \leq 1$. However, later on we shall need it in the form in which we have it.

We are now ready to begin the construction of our L-function.

THEOREM 3.4. *For each* $\tau \in \overline{\mathbf{Q}}_p$, *with* $|\tau|_p \leq 1$, *there exists a power series* $A_\chi(s,\tau)$ *in* $K_\tau[[s]]$ *such that the power series converges on* $\{s \in \mathbf{C}_p : |s|_p < |p|_p^{1/(p-1)}|q|_p^{-1}\}$, *and for each* $n \in \mathbf{Z}$, $n \geq 0$, $A_\chi(n,\tau)$ *satisfies*

$$A_\chi(n,\tau) = B_{n,\chi_n}(q\tau) - \chi_n(p)p^{n-1}B_{n,\chi_n}\left(p^{-1}q\tau\right).$$

Proof. By Proposition 3.3, $|c_n(\tau)|_p \leq C|q|_p^n$ for all $n \geq 0$, where $C = |pqf_\chi|_p^{-1}$. Therefore we can apply Theorem 2.7 to the sequences $\{b_n(\tau)\}_{n=0}^\infty$ and $\{c_n(\tau)\}_{n=0}^\infty$ in $K_\tau = \mathbf{Q}_p(\chi,\tau)$, and for $\rho = |q|_p < |p|_p^{1/(p-1)}$, yielding this result. □

Let us denote $\mathfrak{D} = \{s \in \mathbf{C}_p : |s-1|_p < |p|_p^{1/(p-1)}|q|_p^{-1}\}$.

THEOREM 3.5. *For each* $\tau \in \overline{\mathbf{Q}}_p$, *with* $|\tau|_p \leq 1$, *there exists a unique* p*-adic, meromorphic function* $L_p(s,\tau;\chi)$ *that can be expressed in the form*

$$L_p(s,\tau;\chi) = \frac{a_{-1}(\tau)}{s-1} + \sum_{n=0}^{\infty} a_n(\tau)(s-1)^n,$$

where the power series converges in the domain $\mathfrak{D}$, *having coefficients* $a_n(\tau) \in \mathbf{Q}_p(\chi,\tau)$, *with*

$$a_{-1}(\tau) = \begin{cases} 1-\frac{1}{p}, & \text{if } \chi = 1 \\ 0, & \text{if } \chi \neq 1. \end{cases}$$

Furthermore, for each $n \in \mathbf{Z}$, $n \geq 1$,

$$L_p(1-n,\tau;\chi) = -\frac{1}{n}\left(B_{n,\chi_n}(q\tau) - \chi_n(p)p^{n-1}B_{n,\chi_n}\left(p^{-1}q\tau\right)\right).$$

Proof. Let

(13) $$L_p(s,\tau;\chi) = \frac{1}{s-1}A_\chi(1-s,\tau)$$

with the $A_\chi(s,\tau)$ as in Theorem 3.4. Then from the properties of $A_\chi(s,\tau)$, the power series must converge in the given domain, and for $n \in \mathbf{Z}$, $n \geq 1$,

$$L_p(1-n,\tau;\chi) = -\frac{1}{n}A_\chi(n,\tau) = -\frac{1}{n}\left(B_{n,\chi_n}(q\tau) - \chi_n(p)p^{n-1}B_{n,\chi_n}\left(p^{-1}q\tau\right)\right).$$

Note that

$$\begin{aligned} a_{-1}(\tau) &= A_\chi(0,\tau) = B_{0,\chi}(q\tau) - \chi(p)p^{-1}B_{0,\chi}\left(p^{-1}q\tau\right) \\ &= (1-\chi(p)p^{-1})B_{0,\chi}, \end{aligned}$$

and thus

$$a_{-1}(\tau) = \begin{cases} 1 - \frac{1}{p}, & \text{if } \chi = 1 \\ 0, & \text{if } \chi \neq 1 . \end{cases}$$

The uniqueness of $L_p(s, \tau; \chi)$ follows from Lemma 2.5. $\square$

At this point we have not completed our goal of showing that the p-adic function $L_p(s, \tau; \chi)$ exists for each $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$. In order to prove this, we will need to study the coefficients, $a_n(\tau)$, of the power series expansion of $L_p(s, \tau; \chi)$ for each $\tau \in \overline{\mathbf{Q}}_p$, $|\tau|_p \leq 1$. From the results of this we will show that the function $L_p(s, \tau; \chi)$ exists for each $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$, and for any sequence $\{\tau_i\}_{i=0}^{\infty}$ in $\overline{\mathbf{Q}}_p$, with $|\tau_i|_p \leq 1$, converging to τ, the values $L_p(1-n, \tau_i; \chi)$ converge to $L_p(1-n, \tau; \chi)$ for each $n \in \mathbf{Z}$, $n \geq 1$.

3.2 $L_p(s, \tau; \chi)$ FOR $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$

Our previous work has been for $\tau \in \overline{\mathbf{Q}}_p$, $|\tau|_p \leq 1$. To extend this result to all $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$, we need to find a way to express $a_n(\tau)$ so that it can be defined for these values of τ.

For $k \in \mathbf{Z}$, $k \geq 0$, the Stirling numbers of the first kind, $s(n,k)$, are defined by the generating function

$$\sum_{n=0}^{\infty} s(n,k)\frac{t^n}{n!} = \frac{1}{k!}\left(\log(1+t)\right)^k . \tag{14}$$

Since the power series expansion of $\log(1+t)$ lacks a constant term, we must have $s(n,k) = 0$ whenever $0 \leq n < k$. We also have $s(n,n) = 1$ for all $n \geq 0$. The $s(n,k)$ are integers, where $n, k \in \mathbf{Z}$, $n \geq 0$, $k \geq 0$, and they satisfy the relation

$$\binom{x}{n} = \frac{1}{n!}\sum_{k=0}^{n} s(n,k)x^k . \tag{15}$$

For additional information on Stirling numbers of the first kind we refer the reader to [6], pp. 214–217.

LEMMA 3.6. *Let $\tau \in \overline{\mathbf{Q}}_p$, $|\tau|_p \leq 1$. For $n \in \mathbf{Z}$, $n \geq -1$,*

$$a_n(\tau) = (-1)^{n+1} \sum_{m=n+1}^{\infty} \frac{1}{m!} s(m, n+1) c_m(\tau) .$$

Proof. From Corollary 2.8 we can write

$$A_\chi(s,\tau) = \sum_{m=0}^{\infty} \binom{s}{m} c_m(\tau),$$

where $s \in \mathbf{C}_p$ such that $|s|_p < |p|_p^{1/(p-1)}|q|_p^{-1}$. Now, expanding the quantity $\binom{s}{m}$ according to (15) yields

$$A_\chi(s,\tau) = \sum_{m=0}^{\infty}\sum_{n=0}^{m} \frac{1}{m!} s(m,n) c_m(\tau) s^n,$$

where $s(m,n) \in \mathbf{Z}$ is a Stirling number of the first kind. At this point we wish to switch the order of summation in this expression, but before doing so we must show that the terms in the summation converge to 0 at a sufficient rate.

Let $\epsilon > 0$ and let $\xi \in \mathbf{C}_p$ such that $|\xi|_p < |p|_p^{1/(p-1)}|q|_p^{-1}$. Then there exists $\delta \in \mathbf{R}$, $0 \le \delta < 1$, such that

$$|\xi|_p = \delta \cdot |p|_p^{1/(p-1)}|q|_p^{-1}.$$

Let $N, M \in \mathbf{Z}$, $N > 0$, $M > 0$, such that if $n \ge N$ then $|pqf_\chi|_p^{-1}\delta^n < \epsilon$, and if $m \ge M$ then $|pqf_\chi|_p^{-1}|p|_p^{-m/(p-1)}|q|_p^m < \epsilon$ (such an M exists since $0 \le |p|_p^{-1/(p-1)}|q|_p < 1$).

Let $m, n \in \mathbf{Z}$, $m \ge 0$, $n \ge 0$. If $n > m$, then $s(m,n) = 0$, and so

$$\left| \frac{1}{m!} s(m,n) c_m(\tau) \xi^n \right|_p = 0.$$

Thus we can assume that $m = \max\{m,n\}$. Consider

$$\left| \frac{1}{m!} s(m,n) c_m(\tau) \xi^n \right|_p \le |m!|_p^{-1} |c_m(\tau)|_p |\xi|_p^n.$$

Utilizing Proposition 3.3 and the fact that $v_p(m!) \le m/(p-1)$, we can write

$$|m!|_p^{-1}|c_m(\tau)|_p|\xi|_p^n \le |pqf_\chi|_p^{-1}|p|_p^{-(m-n)/(p-1)}|q|_p^{m-n}\delta^n.$$

Suppose that $m \ge M + N$. If $m - n < M$, then

$$M + N \le m < M + n,$$

so that $n > N$. Thus

$$|m!|_p^{-1}|c_m(\tau)|_p|\xi|_p^n \le |pqf_\chi|_p^{-1}\delta^n < \epsilon.$$

If $m - n \ge M$, then

$$|m!|_p^{-1}|c_m(\tau)|_p|\xi|_p^n \leq |pqf_\chi|_p^{-1}|p|_p^{-(m-n)/(p-1)}|q|_p^{m-n} < \epsilon .$$

Either case implies that

$$\left|\frac{1}{m!}s(m,n)c_m(\tau)\xi^n\right|_p < \epsilon .$$

Therefore, whenever $\max\{m,n\} \geq M+N$, this bound must hold, implying that

$$\sum_{m=0}^{\infty}\sum_{n=0}^{m}\frac{1}{m!}s(m,n)c_m(\tau)\xi^n = \sum_{n=0}^{\infty}\sum_{m=n}^{\infty}\frac{1}{m!}s(m,n)c_m(\tau)\xi^n,$$

by Proposition 2.4.

Writing

$$A_\chi(s,\tau) = \sum_{n=0}^{\infty}s^n\sum_{m=n}^{\infty}\frac{1}{m!}s(m,n)c_m(\tau),$$

we have from (13),

$$\begin{aligned}L_p(s,\tau;\chi) &= \frac{1}{s-1}\sum_{n=0}^{\infty}(1-s)^n\sum_{m=n}^{\infty}\frac{1}{m!}s(m,n)c_m(\tau)\\ &= \sum_{n=-1}^{\infty}(-1)^{n+1}(s-1)^n\sum_{m=n+1}^{\infty}\frac{1}{m!}s(m,n+1)c_m(\tau),\end{aligned}$$

which implies the lemma, since we must have convergence for the inner sum. □

Since we have only derived $L_p(s,\tau;\chi)$ for $\tau \in \overline{\mathbf{Q}}_p$, $|\tau|_p \leq 1$, we cannot say that $a_n(\tau)$ is defined for all $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$. For $n \in \mathbf{Z}$, $n \geq -1$, let us define

$$a_n(\tau) = (-1)^{n+1}\sum_{m=n+1}^{\infty}\frac{1}{m!}s(m,n+1)c_m(\tau), \tag{16}$$

for these values of τ. Note that in the proof of Lemma 3.6, the only influence generated by the value of τ is in the bound of the value of $|c_m(\tau)|_p$, which was determined in Proposition 3.3. However, this proposition holds for all $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$. Thus this sum converges and $a_n(\tau)$ is well-defined for all $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$.

THEOREM 3.7. *Let* $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$, *and let* $\{\tau_i\}_{i=1}^{\infty}$ *be a sequence in* $\overline{\mathbf{Q}}_p$, *with* $|\tau_i|_p \leq 1$, *such that* $\tau_i \to \tau$. *Then for* $n \in \mathbf{Z}$, $n \geq -1$,

$$\lim_{i\to\infty} a_n(\tau_i) = a_n(\tau).$$

Proof. By definition, for $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$, and for $n \in \mathbf{Z}$, $n \geq -1$, we have the expansion

$$a_n(\tau) = (-1)^{n+1} \sum_{m=n+1}^{\infty} \frac{1}{m!} s(m, n+1) c_m(\tau),$$

and as we have seen, regardless of the value of τ,

$$\left|\frac{1}{m!} s(m, n+1) c_m(\tau)\right|_p \leq |pqf_\chi|_p^{-1} |p|_p^{-m/(p-1)} |q|_p^m \to 0$$

as $m \to \infty$. Therefore given $\epsilon > 0$ there must exist some $m_0 \in \mathbf{Z}$, $m_0 \geq n+1$, such that

$$\left|\sum_{m=m_0+1}^{\infty} \frac{1}{m!} s(m, n+1) c_m(\tau)\right|_p < \epsilon.$$

Thus for any sequence $\{\tau_i\}_{i=1}^{\infty}$ in $\mathbf{Q}_p$, with $|\tau_i|_p \leq 1$, such that $\tau_i \to \tau$,

$$|a_n(\tau) - a_n(\tau_i)|_p \leq \max_{n+1 \leq m \leq m_0} \left\{ \epsilon, \left|\frac{1}{m!} s(m, n+1) \left(c_m(\tau) - c_m(\tau_i)\right)\right|_p \right\}.$$

Since $\tau_i \to \tau$ and $c_m(\tau)$ is a polynomial in τ, we see that

$$\left|\frac{1}{m!} s(m, n+1) \left(c_m(\tau) - c_m(\tau_i)\right)\right|_p < \epsilon$$

for all m with $n+1 \leq m \leq m_0$ when i is sufficiently large, which implies that

$$|a_n(\tau) - a_n(\tau_i)|_p < \epsilon$$

for such i. Therefore the theorem must hold. $\square$

The purpose of the following three lemmas is to build an upper bound for the value of $|a_n(\tau)|_p$. After doing so we can define $L_p(s, \tau; \chi)$ for all $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$.

LEMMA 3.8. *Let p be prime. If $i, n \in \mathbf{Z}$ with $1 \leq i \leq n$, then*
$$\left|\binom{n}{i}p^i\right|_p \leq |np|_p .$$

Proof. For $i \in \mathbf{Z}$ such that $1 \leq i \leq n$, (8) implies that $v_p(i!) \leq i-1$, or equivalently, $|i!|_p \geq |p|_p^{i-1}$. Therefore by combining this with
$$\left|\binom{n}{i}p^i\right|_p = \left|\frac{n(n-1)\cdots(n-i+1)}{i!}\right|_p |p|_p^i \leq |i!|_p^{-1}|n|_p|p|_p^i ,$$
the result will follow. □

LEMMA 3.9. *Let p be prime. Then for $m, n \in \mathbf{Z}$, $m > n \geq 0$,*
$$\left|\frac{n!}{m!}s(m,n)q^m\right|_p \leq |np|_p|q|_p^n .$$

Proof. From (14), the generating function for the $s(m,n)$, we obtain
$$\sum_{m=0}^{\infty} \frac{n!}{m!}s(m,n)q^m t^m = (\log(1+qt))^n .$$
Thus we wish to evaluate the power of p that divides the coefficient of t^m in the expansion of $(\log(1+qt))^n$. The power series expansion of the logarithm function (10) yields
$$(\log(1+qt))^n = \left(\sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i}q^i t^i\right)^n ,$$
and by factoring qt out of the sum,
$$(\log(1+qt))^n = q^n t^n\left(1 + pt\sum_{i=2}^{\infty}\frac{(-1)^{i-1}}{i}p^{-1}q^{i-1}t^{i-2}\right)^n .$$
For $i \geq 2$, we see that $p^{-1}q^{i-1}/i \in \mathbf{Z}_p$. Therefore
$$(\log(1+qt))^n = q^n t^n(1+ptf(t))^n,$$
where $f(t) \in \mathbf{Z}_p[[t]]$. Now, this can be written
$$(\log(1+qt))^n = q^n t^n + q^n t^n \sum_{i=1}^{n}\binom{n}{i}p^i t^i f(t)^i,$$
and from Lemma 3.8, the p-adic absolute value of the coefficients of the terms in the sum on the right must be bounded above by $|np|_p|q|_p^n$. Thus, for $m > n$, the coefficient of t^m must also be bounded above by this quantity, implying the result. □

LEMMA 3.10. *Let $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$. Then for $n \in \mathbf{Z}$, $n \geq 0$,*

$$f_\chi n! a_n(\tau) \equiv \frac{(-1)^{n+1}}{n+1} f_\chi c_{n+1}(\tau) \pmod{q^n \mathfrak{o}}.$$

Proof. From (16), we see that for $n \in \mathbf{Z}$, $n \geq 0$,

$$f_\chi n! a_n(\tau) = (-1)^{n+1} \sum_{m=n+1}^{\infty} \frac{n!}{m!} f_\chi s(m, n+1) c_m(\tau).$$

Proposition 3.3 implies that

$$f_\chi c_m(\tau) \equiv 0 \pmod{p^{-1} q^{m-1} \mathfrak{o}}.$$

By Lemma 3.9, when $m \geq n+2$,

$$\frac{n!}{m!} s(m, n+1) \equiv 0 \pmod{p q^{n-m+1} \mathfrak{o}}.$$

Thus

$$f_\chi n! a_n(\tau) \equiv \frac{(-1)^{n+1}}{n+1} f_\chi c_{n+1}(\tau) \pmod{q^n \mathfrak{o}}. \qquad \square$$

We are nearing our goal of defining $L_p(s, \tau; \chi)$ for all $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$. The final step before doing so is proving the following lemma on the convergence of a specific infinite sum.

LEMMA 3.11. *Let $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$. Then the sum*

$$\sum_{n=0}^{\infty} a_n(\tau)(s-1)^n$$

converges for all $s \in \mathfrak{D}$.

Proof. Let $\xi \in \mathfrak{D}$. Then $|\xi - 1|_p < |p|_p^{1/(p-1)} |q|_p^{-1}$. Thus there must be some $\delta \in \mathbf{R}$, $0 \leq \delta < 1$, such that

$$|\xi - 1|_p = \delta \cdot |p|_p^{1/(p-1)} |q|_p^{-1}.$$

Let $n \in \mathbf{Z}$, $n \geq 0$. From Lemma 3.10

$$f_\chi (n+1)! a_n(\tau) \equiv (-1)^{n+1} f_\chi c_{n+1}(\tau) \pmod{(n+1) q^n \mathfrak{o}},$$

and from Proposition 3.3,

$$|f_\chi c_{n+1}(\tau)|_p \leq |p|_p^{-1} |q|_p^n.$$

Therefore

$$|f_\chi(n+1)!a_n(\tau)|_p \leq |p|_p^{-1}|q|_p^n,$$

which implies that

$$|a_n(\tau)|_p \leq |f_\chi(n+1)!p|_p^{-1}|q|_p^n.$$

Thus

$$|a_n(\tau)(\xi-1)^n|_p \leq |f_\chi(n+1)!p|_p^{-1}|p|_p^{n/(p-1)}\delta^n.$$

Now,

$$v_p((n+1)!) \leq \frac{n}{p-1},$$

so that

$$|a_n(\tau)(\xi-1)^n|_p \leq |f_\chi p|_p^{-1}\delta^n.$$

Since $0 \leq \delta < 1$, we see that $|a_n(\tau)(\xi-1)^n|_p \to 0$ as $n \to \infty$. Thus the sum

$$\sum_{n=0}^{\infty} a_n(\tau)(\xi-1)^n$$

must converge. $\square$

Note that from this proof we have obtained the bound

$$|a_n(\tau)|_p \leq |f_\chi(n+1)!p|_p^{-1}|q|_p^n, \tag{17}$$

for each $n \in \mathbf{Z}$, $n \geq -1$, and for all $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$.

Now let us define

$$L_p(s,\tau;\chi) = \frac{a_{-1}(\tau)}{s-1} + \sum_{n=0}^{\infty} a_n(\tau)(s-1)^n$$

for $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$, and $s \in \mathfrak{D}$, $s \neq 1$ if $\chi = 1$. This definition is consistent with what we already have for $\tau \in \overline{\mathbf{Q}}_p$, $|\tau|_p \leq 1$. We will now show that, for all $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$, this function satisfies

$$L_p(1-n,\tau;\chi) = -\frac{1}{n}\left(B_{n,\chi_n}(q\tau) - \chi_n(p)p^{n-1}B_{n,\chi_n}\left(p^{-1}q\tau\right)\right),$$

for each $n \in \mathbf{Z}$, $n \geq 1$. To do this, we prove the following:

LEMMA 3.12. *Let $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$, and let $\{\tau_i\}_{i=1}^{\infty}$ be a sequence in $\overline{\mathbf{Q}}_p$, with $|\tau_i|_p \leq 1$, such that $\tau_i \to \tau$. Then for each $n \in \mathbf{Z}$, $n \geq 1$,*

$$\lim_{i\to\infty} L_p(1-n, \tau_i; \chi) = L_p(1-n, \tau; \chi).$$

Proof. We can write

$$L_p(s, \tau; \chi) = \frac{a_{-1}(\tau)}{s-1} + \sum_{m=0}^{\infty} a_m(\tau)(s-1)^m,$$

where the power series converges for each $s \in \mathfrak{D}$.

Let $\epsilon > 0$, and let $n \in \mathbf{Z}$, $n \geq 1$. Then we must have $1 - n \in \mathfrak{D}$, and thus the power series converges for $s = 1 - n$. Also, by (17)

$$\left|a_m(\tau)(-n)^m\right|_p \leq \left|f_\chi(m+1)!p\right|_p^{-1} |nq|_p^m \to 0$$

independently of τ as $m \to \infty$. Therefore, for $m_0 \in \mathbf{Z}$ sufficiently large,

$$\left|\sum_{m=m_0}^{\infty} a_m(\tau)(-n)^m\right|_p < \epsilon.$$

For $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$, let $\{\tau_i\}_{i=1}^{\infty}$ be in $\overline{\mathbf{Q}}_p$, with $|\tau_i|_p \leq 1$, such that $\tau_i \to \tau$. Consider

$$\left|L_p(1-n, \tau; \chi) - L_p(1-n, \tau_i; \chi)\right|_p \leq \max_{0 \leq m < m_0} \left\{\epsilon, \left|(a_m(\tau) - a_m(\tau_i))(-n)^m\right|_p\right\}.$$

Since $a_m(\tau_i) \to a_m(\tau)$ as $\tau_i \to \tau$, we have

$$\left|L_p(1-n, \tau; \chi) - L_p(1-n, \tau_i; \chi)\right|_p < \epsilon$$

for i sufficiently large. Thus the lemma must hold. $\square$

At this point we have finally proven

THEOREM 3.13. *For each $\tau \in \mathbf{C}_p$, with $|\tau|_p \leq 1$, there exists a unique p-adic, meromorphic function $L_p(s, \tau; \chi)$ that satisfies*

$$L_p(1-n, \tau; \chi) = -\frac{1}{n}\left(B_{n,\chi_n}(q\tau) - \chi_n(p)p^{n-1}B_{n,\chi_n}\left(p^{-1}q\tau\right)\right),$$

for each $n \in \mathbf{Z}$, $n \geq 1$. Furthermore, this function can be expressed in the form

$$L_p(s, \tau; \chi) = \frac{a_{-1}(\tau)}{s-1} + \sum_{n=0}^{\infty} a_n(\tau)(s-1)^n,$$

where the power series converges in the domain $\mathfrak{D}$, *and*

$$a_{-1}(\tau) = \begin{cases} 1 - \frac{1}{p}, & \text{if } \chi = 1 \\ 0, & \text{if } \chi \neq 1 . \end{cases} \qquad \square$$

Since $L_p(s, \tau; \chi)$ is defined for each $\tau \in \mathbf{C}_p$ such that $|\tau|_p \leq 1$, we now have a p-adic function of two variables, $L_p(s, t; \chi)$, where $s \in \mathfrak{D}$, $s \neq 1$ if $\chi = 1$, and $t \in \mathbf{C}_p$ with $|t|_p \leq 1$.

4. Properties of $L_p(s, t; \chi)$

Most of the properties that follow are direct consequences of similar properties that hold for the generalized Bernoulli polynomials. In all of the following we will take p prime and χ a Dirichlet character with conductor f_χ.

4.1 A symmetry property in t

The first property we obtain regarding $L_p(s, t; \chi)$ is a direct consequence of the generalized Bernoulli polynomials being either odd or even functions, except when $\chi = 1$. Recall that $L_p(s, t; \chi)$ interpolates the values

$$L_p(1 - n, t; \chi) = -\frac{1}{n} b_n(t) , \tag{18}$$

for $n \in \mathbf{Z}$, $n \geq 1$, and $t \in \mathbf{C}_p$, $|t|_p \leq 1$, where

$$b_n(t) = B_{n,\chi_n}(qt) - \chi_n(p) p^{n-1} B_{n,\chi_n}\left(p^{-1} qt\right) , \tag{19}$$

and we define

$$c_n(t) = \sum_{m=0}^{n} \binom{n}{m} (-1)^{n-m} b_m(t) . \tag{20}$$

LEMMA 4.1. *For all* $n \in \mathbf{Z}$, $n \geq 0$, *we have*

$$B_{n,1}(-t) = (-1)^n B_{n,1}(t) - (-1)^n n t^{n-1} .$$

Proof. This holds for $n = 0$ since $B_{0,1}(t) = 1$. Now assume that $n \geq 1$. Because $B_{n,1} = 0$ for odd $n \geq 3$, we can write (2) in the form

$$B_{n,1}(t) = \sum_{\substack{m=0 \\ n-m \text{ even}}}^{n} \binom{n}{m} B_{n-m,1} t^m + n B_{1,1} t^{n-1} .$$

Any m such that $n-m$ is even must have the same parity as n. Thus

$$\begin{aligned} B_{n,1}(-t) &= (-1)^n \sum_{\substack{m=0 \\ n-m \text{ even}}}^{n} \binom{n}{m} B_{n-m,1} t^m + (-1)^{n-1} n B_{1,1} t^{n-1} \\ &= (-1)^n B_{n,1}(t) - 2(-1)^n n B_{1,1} t^{n-1} . \end{aligned}$$

From the value $B_{1,1} = -B_1 = 1/2$, the lemma then follows. □

LEMMA 4.2. *For all* $n \in \mathbf{Z}$, $n \geq 0$,

$$b_n(-t) = \chi(-1) b_n(t).$$

Proof. This is obviously true for $n=0$ since

$$b_0(t) = \left(1 - \chi(p)p^{-1}\right) B_{0,\chi} ,$$

and $B_{0,\chi} = 0$ except when $\chi = 1$, in which case $B_{0,1} = 1$. So we can assume that $n \geq 1$.

First consider the case of $\chi_n = 1$. This implies that $\chi = \omega^n$. By Lemma 4.1,

$$\begin{aligned} b_n(-t) &= B_{n,1}(-qt) - p^{n-1} B_{n,1}\left(-p^{-1}qt\right) \\ &= (-1)^n B_{n,1}(qt) - (-1)^n n (qt)^{n-1} \\ &\quad - p^{n-1}\left((-1)^n B_{n,1}\left(p^{-1}qt\right) - (-1)^n n \left(p^{-1}qt\right)^{n-1}\right) \\ &= (-1)^n \left(B_{n,1}(qt) - p^{n-1} B_{n,1}\left(p^{-1}qt\right)\right) \\ &= (-1)^n b_n(t) . \end{aligned}$$

Since $\chi = \omega^n$ and $\omega(-1) = -1$, the lemma holds for $\chi_n = 1$.

Now suppose that $\chi_n \neq 1$. Then, from (3),

$$\begin{aligned} b_n(-t) &= B_{n,\chi_n}(-qt) - \chi_n(p) p^{n-1} B_{n,\chi_n}\left(-p^{-1}qt\right) \\ &= (-1)^n \chi_n(-1) \left(B_{n,\chi_n}(qt) - \chi_n(p) p^{n-1} B_{n,\chi_n}\left(p^{-1}qt\right)\right) \\ &= (-1)^n \chi_n(-1) b_n(t) . \end{aligned}$$

Note that $\chi_n = \chi\omega^{-n}$, which implies that $\chi_n(-1) = (-1)^n \chi(-1)$. Thus the lemma also holds for $\chi_n \neq 1$.

Since the lemma holds for both $\chi_n = 1$ and $\chi_n \neq 1$, the proof must be complete. □

Using this result, we can prove

THEOREM 4.3. *Let* $t \in \mathbf{C}_p$, $|t|_p \leq 1$, *and* $s \in \mathfrak{D}$, *except* $s \neq 1$ *if* $\chi = 1$. *Then*

$$L_p(s, -t; \chi) = \chi(-1) L_p(s, t; \chi).$$

Proof. From Lemma 4.2 we see that

$$b_n(-t) = \chi(-1) b_n(t).$$

Also, (20) implies that

$$c_n(-t) = \chi(-1) c_n(t).$$

From (16), whenever $n \geq -1$,

$$a_n(-t) = \chi(-1) a_n(t),$$

which implies that

$$L_p(s, -t; \chi) = \chi(-1) L_p(s, t; \chi). \qquad \square$$

If $\chi(-1) = -1$ and $t = 0$, then

$$L_p(s, 0; \chi) = -L_p(s, 0; \chi),$$

which implies that

$$L_p(s; \chi) = -L_p(s; \chi),$$

and thus $L_p(s; \chi) = 0$ for all $s \in \mathfrak{D}$, as we would expect.

4.2 $L_p(s, t; \chi)$ AS A POWER SERIES IN $t - \alpha$, $\alpha \in \mathbf{C}_p$, $|\alpha|_p \leq 1$

To develop $L_p(s, t; \chi)$ in terms of a power series in t will enable us to find a derivative of this function with respect to this second variable. All this we shall do, but before doing so we need to specify some notation.

LEMMA 4.4. *Let* $t \in \mathbf{C}_p$, $|t|_p \leq 1$. *Then for* $n \in \mathbf{Z}$, $n \geq 1$,

$$\lim_{s \to 1-n} \binom{-s}{n} L_p(s + n, t; \chi) = -\frac{1}{n} \left(1 - \chi(p) p^{-1}\right) B_{0,\chi}.$$

Proof. Recall that, from Theorem 3.13, we can write

$$L_p(s, t; \chi) = \frac{a_{-1}(t)}{s - 1} + \sum_{m=0}^{\infty} a_m(t)(s - 1)^m,$$

where $a_{-1}(t) = (1 - \chi(p) p^{-1}) B_{0,\chi}$. Thus

$$\lim_{s\to 1}(s-1)L_p(s,t;\chi) = \left(1-\chi(p)p^{-1}\right) B_{0,\chi}\,.$$

Now let $n \in \mathbf{Z}$, $n \geq 1$, and consider

$$\lim_{s\to 1-n}\binom{-s}{n}L_p(s+n,t;\chi) = \lim_{s\to 1}\binom{n-s}{n}L_p(s,t;\chi)\,.$$

If $n=1$, then we write this as

$$\lim_{s\to 1}(1-s)L_p(s,t;\chi) = -\left(1-\chi(p)p^{-1}\right) B_{0,\chi}\,.$$

If $n\geq 2$, then

$$\frac{1}{n!}\lim_{s\to 1}\prod_{i=0}^{n-2}(n-s-i) = \frac{1}{n},$$

which implies that

$$\begin{aligned}\lim_{s\to 1-n}\binom{-s}{n}L_p(s+n,t;\chi) &= \frac{1}{n!}\left(\lim_{s\to 1}\prod_{i=0}^{n-2}(n-s-i)\right)\left(\lim_{s\to 1}(1-s)L_p(s,t;\chi)\right)\\ &= -\frac{1}{n}\left(1-\chi(p)p^{-1}\right) B_{0,\chi}\,.\end{aligned}$$

Therefore the lemma holds for all $n\geq 1$. $\square$

Now, because $L_p(s,t;1)$ is undefined when $s=1$, the quantity

$$\binom{-s}{n}L_p(s+n,t;1)$$

is undefined when $s=1-n$, for $n\in\mathbf{Z}$, $n\geq 1$. However, Lemma 4.4 shows that this quantity exists as $s\to 1-n$. In the following we will encounter expressions that involve $\binom{-s}{n}L_p(s+n,t;\chi)$, and because of Lemma 4.4 we shall assume the understanding that

$$\left.\binom{-s}{n}L_p(s+n,t;\chi)\right|_{s=1-n} = -\frac{1}{n}\left(1-\chi(p)p^{-1}\right) B_{0,\chi}$$

for $n\in\mathbf{Z}$, $n\geq 1$.

THEOREM 4.5. *Let* $t\in\mathbf{C}_p$, $|t|_p\leq 1$, *and* $s\in\mathfrak{D}$, *except* $s\neq 1$ *if* $\chi=1$. *Then*

$$L_p(s,t;\chi) = \sum_{m=0}^{\infty}\binom{-s}{m}q^m t^m L_p(s+m;\chi_m)\,. \tag{21}$$

Proof. Let $t\in\mathbf{C}_p$, $|t|_p\leq 1$, and let $k\in\mathbf{Z}$, $k\geq 1$. Then

$$\sum_{m=0}^{\infty}\binom{k-1}{m}q^m t^m L_p(1-k+m;\chi_m) = -\frac{1}{k}q^k t^k(1-\chi_k(p)p^{-1})B_{0,\chi_k}$$
$$+\sum_{m=0}^{k-1}\binom{k-1}{m}q^m t^m L_p(1-(k-m);\chi_m).$$

By evaluating the L-function, we obtain

$$\binom{k-1}{m}L_p(1-(k-m);\chi_m) = -\frac{1}{k}\binom{k}{m}\left(1-\chi_k(p)p^{k-m-1}\right)B_{k-m,\chi_k},$$

and thus

$$\sum_{m=0}^{\infty}\binom{k-1}{m}q^m t^m L_p\big(1-(k-m);\chi_m\big)$$
$$= -\frac{1}{k}\sum_{m=0}^{k}\binom{k}{m}q^m t^m\left(1-\chi_k(p)p^{k-m-1}\right)B_{k-m,\chi_k},$$

which implies that the sum converges for $s=1-k$. Breaking this into two sums

$$\sum_{m=0}^{\infty}\binom{k-1}{m}q^m t^m L_p(1-(k-m);\chi_m)$$
$$= -\frac{1}{k}\sum_{m=0}^{k}\binom{k}{m}B_{k-m,\chi_k}q^m t^m + \frac{1}{k}\chi_k(p)p^{k-1}\sum_{m=0}^{k}\binom{k}{m}B_{k-m,\chi_k}p^{-m}q^m t^m$$
$$= -\frac{1}{k}\left(B_{k,\chi_k}(qt) - \chi_k(p)p^{k-1}B_{k,\chi_k}\left(p^{-1}qt\right)\right)$$
$$= L_p(1-k,t;\chi).$$

Thus (21) holds for a sequence $\{1-k\}_{k=1}^{\infty}$ that has 0 as a limit point. Lemma 2.5 then implies that Theorem 4.5 holds for all s in any neighborhood about 0 common to the domains of the functions on either side of (21).

Now we will show that the domains, in s, of each of the functions on either side of (21) contain $\mathfrak{D}$, except $s \neq 1$ when $\chi = 1$.

This is obvious for the function $L_p(s,t;\chi)$. Consider the function

$$\sum_{m=0}^{\infty}\binom{-s}{m}q^m t^m L_p(s+m;\chi_m) = \sum_{m=0}^{\infty}\sum_{n=-1}^{\infty}\binom{-s}{m}q^m t^m a_{n,\chi_m}(s+m-1)^n.$$

We have seen that this sum converges for $s=1-k$, where $k\in\mathbf{Z}$, $k\geq 1$. Now we need to show that it converges for $s=\xi$, where $\xi\in\mathfrak{D}$, $\xi\neq 1$ if $\chi=1$, and $\xi\neq 1-k$ for $k\in\mathbf{Z}$, $k\geq 1$. So let ξ satisfy these restrictions, and let $\epsilon > 0$. Note that $|\xi - 1|_p < r$, where $r = |p|_p^{1/(p-1)}|q|_p^{-1}$. Let $r_0 \in \mathbf{R}$, $0 \le r_0 < r$, such that $|\xi - 1|_p = r_0$. Then for any $m \in \mathbf{Z}$, $m \ge 0$,

$$\begin{aligned} |\xi + m - 1|_p &\le \max\left\{|m|_p, |\xi - 1|_p\right\} \\ &\le \max\{1, r_0\}, \end{aligned}$$

implying that $\xi + m \in \mathfrak{D}$, $\xi + m \ne 1$. Let $\delta \in \mathbf{R}$ such that $r^\delta = \max\{1, r_0\}$. Then $0 \le \delta < 1$, and

$$|\xi + m - 1|_p \le r^\delta. \tag{22}$$

Let $N_1 \in \mathbf{Z}$ such that

$$|p^{-1}q|_p |p|_p^{-(1-\delta)(N_1-1)/(p-1)} |q|_p^{(1-\delta)(N_1-1)} < \epsilon.$$

Then for any $m \in \mathbf{Z}$, $m \ge 1$, such that $m \ge N_1$, we must also have

$$|p^{-1}q|_p |p|_p^{-(1-\delta)(m-1)/(p-1)} |q|_p^{(1-\delta)(m-1)} < \epsilon.$$

For $m \in \mathbf{Z}$, $m \ge 1$, consider

$$\left|\binom{-\xi}{m} q^m t^m a_{-1,\chi_m} (\xi + m - 1)^{-1}\right|_p \le |p|_p^{-1}|q|_p^m \left|\binom{-\xi}{m}(\xi + m - 1)^{-1}\right|_p.$$

Note that, by (22),

$$\begin{aligned} \left|\binom{-\xi}{m}(\xi + m - 1)^{-1}\right|_p &= |\xi + m - 1|_p^{-1} \prod_{i=1}^{m} \frac{|-\xi - (i-1)|_p}{|i|_p} \\ &\le |m!|_p^{-1} r^{\delta(m-1)}. \end{aligned}$$

Therefore

$$\left|\binom{-\xi}{m} q^m t^m a_{-1,\chi_m} (\xi + m - 1)^{-1}\right|_p \le |p|_p^{-1}|q|_p^m |m!|_p^{-1} r^{\delta(m-1)},$$

and from the bound

$$|m!|_p \ge |p|_p^{(m-1)/(p-1)},$$

we obtain

$$\left|\binom{-\xi}{m} q^m t^m a_{-1,\chi_m} (\xi + m - 1)^{-1}\right|_p \le |p^{-1}q|_p |p|_p^{-(1-\delta)(m-1)/(p-1)} |q|_p^{(1-\delta)(m-1)}.$$

Thus if $m \ge N_1$, then

$$\left|\binom{-\xi}{m} q^m t^m a_{-1,\chi_m} (\xi + m - 1)^{-1}\right|_p < \epsilon.$$

Now let $N_2 \in \mathbf{Z}$ such that
$$|f_\chi p|_p^{-1} |p|_p^{-(1-\delta)N_2/(p-1)} |q|_p^{(1-\delta)N_2} < \epsilon .$$
Then we must also have
$$|f_\chi p|_p^{-1} |p|_p^{-(1-\delta)(m+n)/(p-1)} |q|_p^{(1-\delta)(m+n)} < \epsilon$$
for any $m,n \in \mathbf{Z}$ such that $m \geq 0$, $n \geq 0$, and $\max\{m,n\} \geq N_2$. Let us consider
$$\left| \binom{-\xi}{m} q^m t^m a_{n,\chi_m} (\xi + m - 1)^n \right|_p \leq \left| \binom{-\xi}{m} \right|_p |q|_p^m |a_{n,\chi_m}|_p |\xi + m - 1|_p^n ,$$
where $m,n \in \mathbf{Z}$, $m \geq 0$, $n \geq 0$. For all $m \geq 0$,
$$\left| \binom{-\xi}{m} \right|_p \leq |m!|_p^{-1} r^{\delta m},$$
and by utilizing this along with (17) and (22), our expression becomes
$$\left| \binom{-\xi}{m} q^m t^m a_{n,\chi_m} (\xi + m - 1)^n \right|_p \leq |m!(n+1)!|_p^{-1} |f_\chi p|_p^{-1} r^{\delta(m+n)} |q|_p^{m+n} .$$
Since
$$|m!(n+1)!|_p \geq |p|_p^{(m+n)/(p-1)},$$
we obtain
$$\left| \binom{-\xi}{m} q^m t^m a_{n,\chi_m} (\xi + m - 1)^n \right|_p \leq |f_\chi p|_p^{-1} |p|_p^{-(1-\delta)(m+n)/(p-1)} |q|_p^{(1-\delta)(m+n)} .$$
Thus if $\max\{m,n\} \geq N_2$, then
$$\left| \binom{-\xi}{m} q^m t^m a_{n,\chi_m} (\xi + m - 1)^n \right|_p < \epsilon .$$

Let $N = \max\{N_1, N_2\}$, and let $m,n \in \mathbf{Z}$, $m \geq 0$, $n \geq -1$. Then for $\max\{m,n\} \geq N$, it must be true that
$$\left| \binom{-\xi}{m} q^m t^m a_{n,\chi_m} (\xi + m - 1)^n \right|_p < \epsilon .$$
Thus, by Proposition 2.4, the sum
$$\sum_{m=0}^{\infty} \sum_{n=-1}^{\infty} \binom{-\xi}{m} q^m t^m a_{n,\chi_m} (\xi + m - 1)^n$$
must converge. This implies that the function on the right of (21) must converge for all $s \in \mathfrak{D}$, except $s \neq 1$ if $\chi = 1$, and the theorem must then hold. $\square$

Since we can now express $L_p(s,t;\chi)$ in terms of a power series in t, we can take a derivative of this function with respect to t.

LEMMA 4.6. *Let $t \in \mathbf{C}_p$, $|t|_p \leq 1$, and $s \in \mathfrak{D}$, except $s \neq 1$ if $\chi = 1$. Then*

$$\frac{\partial^n}{\partial t^n} L_p(s, t; \chi) = n! q^n \binom{-s}{n} L_p(s+n, t; \chi_n),$$

for $n \in \mathbf{Z}$, $n \geq 0$.

Proof. If $n = 0$, then the lemma is obviously true. So consider $n = 1$. Applying Proposition 2.6 to (21),

$$\frac{\partial}{\partial t} L_p(s, t; \chi) = \sum_{m=1}^{\infty} \binom{-s}{m} q^m m t^{m-1} L_p(s+m; \chi_m).$$

Now,

$$m\binom{-s}{m} = -s\binom{-s-1}{m-1},$$

so that

$$\begin{aligned} \frac{\partial}{\partial t} L_p(s, t; \chi) &= \sum_{m=1}^{\infty} (-s)\binom{-s-1}{m-1} q^m t^{m-1} L_p(s+m; \chi_m) \\ &= -qs \sum_{m=0}^{\infty} \binom{-s-1}{m} q^m t^m L_p\left(s+1+m; \chi_{1+m}\right) \\ &= -qs L_p(s+1, t; \chi_1). \end{aligned}$$

Now suppose that

$$\frac{\partial^n}{\partial t^n} L_p(s, t; \chi) = n! q^n \binom{-s}{n} L_p(s+n, t; \chi_n)$$

for some $n \in \mathbf{Z}$, $n \geq 1$. Then

$$\begin{aligned} \frac{\partial^{n+1}}{\partial t^{n+1}} L_p(s, t; \chi) &= \frac{\partial}{\partial t}\left(\frac{\partial^n}{\partial t^n} L_p(s, t; \chi)\right) \\ &= n! q^n \binom{-s}{n} \frac{\partial}{\partial t} L_p(s+n, t; \chi_n). \end{aligned}$$

From the case for $n = 1$, we see that

$$\begin{aligned} n! q^n \binom{-s}{n} \frac{\partial}{\partial t} L_p(s+n, t; \chi_n) &= n! q^n \binom{-s}{n} (-s-n) q L_p\left(s+n+1, t; \chi_{n+1}\right) \\ &= (n+1)! q^{n+1} \binom{-s}{n+1} L_p\left(s+n+1, t; \chi_{n+1}\right). \end{aligned}$$

Therefore

$$\frac{\partial^{n+1}}{\partial t^{n+1}} L_p(s, t; \chi) = (n+1)! q^{n+1} \binom{-s}{n+1} L_p\left(s+n+1, t; \chi_{n+1}\right),$$

and the lemma must hold by induction. □

With this result, we can derive a more general power series expansion of $L_p(s,t;\chi)$.

THEOREM 4.7. *Let* $t \in \mathbf{C}_p$, $|t|_p \le 1$, *and* $s \in \mathfrak{D}$, *except* $s \neq 1$ *if* $\chi = 1$. *Then for* $\alpha \in \mathbf{C}_p$, $|\alpha|_p \le 1$,

$$L_p(s,t;\chi) = \sum_{m=0}^{\infty} \binom{-s}{m} q^m (t-\alpha)^m L_p\left(s+m,\alpha;\chi_m\right).$$

REMARK. Note that Theorem 4.5 is the case of $\alpha = 0$ here.

Proof. It follows from the Taylor series expansion of $L_p(s,t;\chi)$ in the variable t about α (see Proposition 2.6) that we can write $L_p(s,t;\chi)$ in the form

$$L_p(s,t;\chi) = \sum_{m=0}^{\infty} \beta_m (t-\alpha)^m,$$

where

$$\beta_m = \frac{1}{m!}\frac{\partial^m}{\partial t^m} L_p(s,t;\chi)\bigg|_{t=\alpha}.$$

From Lemma 4.6

$$\frac{1}{m!}\frac{\partial^m}{\partial t^m} L_p(s,t;\chi) = \binom{-s}{m} q^m L_p(s+m,t;\chi_m),$$

and so

$$\beta_m = \binom{-s}{m} q^m L_p(s+m,\alpha;\chi_m),$$

completing the proof. □

4.3 RELATING $L_p(s,t;\chi)$ TO SOME FINITE SUMS

From (4) it becomes obvious that the generalized Bernoulli polynomials have a considerable significance in regard to sums of consecutive nonnegative integers, each raised to the same power, itself a nonnegative integer. The following illustrates how this can be extended with the use of $L_p(s,t;\chi)$.

For the character χ, let $F_0 = \operatorname{lcm}(f_\chi, q)$. Then $f_{\chi_n} \mid F_0$ for each $n \in \mathbf{Z}$. Also, let F be a positive multiple of $pq^{-1}F_0$.

THEOREM 4.8. *Let* $t \in \mathbf{C}_p$, $|t|_p \leq 1$, *and* $s \in \mathfrak{D}$, *except* $s \neq 1$ *if* $\chi = 1$. *Then*

$$L_p(s, t+F; \chi) - L_p(s, t; \chi) = -\sum_{\substack{a=1 \\ (a,p)=1}}^{qF} \chi_1(a) \langle a + qt \rangle^{-s}. \tag{23}$$

Proof. Let $t \in \mathbf{C}_p$, $|t|_p \leq 1$, and let $n \in \mathbf{Z}$, $n \geq 1$. Then from (18),

$$L_p(1-n, t+F; \chi) - L_p(1-n, t; \chi) = -\frac{1}{n}\,(b_n(t+F) - b_n(t)).$$

Now, (19) implies

$$\begin{aligned} b_n(t+F) - b_n(t) &= \left(B_{n,\chi_n}(q(t+F)) - \chi_n(p)p^{n-1}B_{n,\chi_n}\left(p^{-1}q(t+F)\right)\right) \\ &\quad - \left(B_{n,\chi_n}(qt) - \chi_n(p)p^{n-1}B_{n,\chi_n}\left(p^{-1}qt\right)\right) \\ &= \left(B_{n,\chi_n}(q(t+F)) - B_{n,\chi_n}(qt)\right) \\ &\quad - \chi_n(p)p^{n-1}\left(B_{n,\chi_n}\left(p^{-1}q(t+F)\right) - B_{n,\chi_n}\left(p^{-1}qt\right)\right). \end{aligned}$$

Thus, by (4), we can write

$$\begin{aligned} b_n(t+F)&-b_n(t) \\ &= n\sum_{a=1}^{qF} \chi_n(a)(a+qt)^{n-1} - n\chi_n(p)p^{n-1}\sum_{a=1}^{p^{-1}qF} \chi_n(a)(a+p^{-1}qt)^{n-1} \\ &= n\sum_{a=1}^{qF} \chi_n(a)(a+qt)^{n-1} - n\sum_{\substack{a=1 \\ p\mid a}}^{qF} \chi_n(a)(a+qt)^{n-1}. \end{aligned}$$

Therefore,

$$L_p(1-n, t+F; \chi) - L_p(1-n, t; \chi) = -\sum_{\substack{a=1 \\ (a,p)=1}}^{qF} \chi_n(a)(a+qt)^{n-1}.$$

Now, $\chi_n = \chi_1\omega^{-(n-1)}$, so that

$$\begin{aligned} \chi_n(a)(a+qt)^{n-1} &= \chi_1(a)\omega^{-(n-1)}(a)(a+qt)^{n-1} \\ &= \chi_1(a)\langle a+qt\rangle^{n-1}. \end{aligned}$$

Thus

$$L_p(1-n, t+F; \chi) - L_p(1-n, t; \chi) = -\sum_{\substack{a=1 \\ (a,p)=1}}^{qF} \chi_1(a)\langle a+qt\rangle^{n-1},$$

and (23) holds for all $s = 1 - n$, where $n \in \mathbf{Z}$, $n \geq 1$. Therefore, since the negative integers have 0 as a limit point, Lemma 2.5 implies that Theorem 4.8 holds for all s in any neighborhood about 0 common to the domains of the functions on either side of (23).

It is obvious that the domains, in the variable s, of the functions on the left of (23) contain $\mathfrak{D}$, except $s \neq 1$ when $\chi = 1$. Consider now the function

$$- \sum_{\substack{a=1 \\ (a,p)=1}}^{qF} \chi_1(a) \langle a + qt \rangle^{-s} = - \sum_{\substack{a=1 \\ (a,p)=1}}^{qF} \chi_1(a) \langle a + qt \rangle^{-1} \langle a + qt \rangle^{1-s}.$$

Since it consists of a finite sum of functions of the form $\langle a + qt \rangle^{1-s}$, where $a \in \mathbf{Z}$, $(a, p) = 1$, we need only show that each such function is analytic on $\mathfrak{D}$, and the proof will be complete.

The quantity $\langle a + qt \rangle^{1-s}$ can be written as

$$\langle a + qt \rangle^{1-s} = \exp\left((1 - s) \log \langle a + qt \rangle \right),$$

and by (9), the Taylor series expansion of the exponential function,

$$\langle a + qt \rangle^{1-s} = \sum_{m=0}^{\infty} \frac{1}{m!} (1 - s)^m \left(\log \langle a + qt \rangle \right)^m.$$

Since $\langle a + qt \rangle \equiv 1 \pmod{q\mathfrak{o}}$ for $a \in \mathbf{Z}$, $(a, p) = 1$, and $t \in \mathbf{C}_p$, $|t|_p \leq 1$, we must also have $\log \langle a + qt \rangle \equiv 0 \pmod{q\mathfrak{o}}$ for such a and t. Thus

$$\left| \frac{1}{m!} (1 - s)^m \left(\log \langle a + qt \rangle \right)^m \right|_p \leq \left| \frac{1}{m!} q^m (s - 1)^m \right|_p$$

for all m. By (8) we can write

$$\begin{aligned} \left| \frac{1}{m!} q^m (s - 1)^m \right|_p &\leq \left| p^{-m/(p-1)} q^m (s - 1)^m \right|_p \\ &= \left| p^{-1/(p-1)} q (s - 1) \right|_p^m. \end{aligned}$$

Thus if

$$\left| p^{-1/(p-1)} q (s - 1) \right|_p < 1,$$

then

$$\left| \frac{1}{m!} (1 - s)^m \left(\log \langle a + qt \rangle \right)^m \right|_p \to 0$$

as $m \to \infty$. So whenever $|s - 1|_p < |p|_p^{1/(p-1)} |q|_p^{-1}$, meaning that $s \in \mathfrak{D}$, we have convergence for the power series. Therefore, the functions on either side of (23) have domains that contain $\mathfrak{D}$, except possibly for $s = 1$ when $\chi = 1$, and the theorem must hold. $\square$

COROLLARY 4.9. *Let $s \in \mathfrak{D}$, except $s \neq 1$ if $\chi = 1$. Then*

$$L_p(s, F; \chi) = L_p(s; \chi) - \sum_{\substack{a=1 \\ (a,p)=1}}^{qF} \chi_1(a)\langle a\rangle^{-s}.$$

Proof. This follows from Theorem 4.8 since $L_p(s, 0; \chi) = L_p(s; \chi)$ for any character χ. □

We shall now consider how Corollary 4.9 can be utilized to derive a collection of congruences related to the generalized Bernoulli polynomials. Let Δ_c denote the forward difference operator, $\Delta_c x_n = x_{n+c} - x_n$. Repeated application of this operator can be expressed in the form

$$\Delta_c^k x_n = \sum_{m=0}^{k} \binom{k}{m} (-1)^{k-m} x_{n+mc}.$$

Recall that $F_0 = \operatorname{lcm}(f_\chi, q)$. For $n \in \mathbf{Z}$, $n \geq 1$, denote

$$\beta_{n,\chi}(t) = -\frac{1}{n}\left(B_{n,\chi_n}(qt) - \chi_n(p)p^{n-1}B_{n,\chi_n}\left(p^{-1}qt\right)\right).$$

This is the polynomial structure that we utilized with respect to generalizing the p-adic L-functions. We will incorporate this structure in an extension of the Kummer congruences, but the results that we derive will be without restriction on either χ or p.

THEOREM 4.10. *Let n, c, and k be positive integers, and let $\tau \in \mathbf{Z}_p$ such that $|\tau|_p \leq |pq^{-1}F_0|_p$. Then the quantity $q^{-k}\Delta_c^k\beta_{n,\chi}(\tau) - q^{-k}\Delta_c^k\beta_{n,\chi}(0) \in \mathbf{Z}_p[\chi]$, and, modulo $q\mathbf{Z}_p[\chi]$, is independent of n.*

Proof. Since Δ_c is a linear operator, Corollary 4.9 implies that

$$\Delta_c^k L_p(1-n, F; \chi) = \Delta_c^k L_p(1-n; \chi) - \sum_{\substack{a=1 \\ (a,p)=1}}^{qF} \chi_1(a)\Delta_c^k\langle a\rangle^{n-1},$$

where F is a positive multiple of $pq^{-1}F_0$. Thus

$$\Delta_c^k\beta_{n,\chi}(F) - \Delta_c^k\beta_{n,\chi}(0) = -\sum_{\substack{a=1 \\ (a,p)=1}}^{qF} \chi_1(a)\langle a\rangle^{-1}\Delta_c^k\langle a\rangle^{n}.$$

Note that

$$\Delta_c^k \langle a \rangle^n = \sum_{m=0}^{k} \binom{k}{m} (-1)^{k-m} \langle a \rangle^{n+mc} = \langle a \rangle^n \left(\langle a \rangle^c - 1 \right)^k . \tag{24}$$

Now, $\langle a \rangle \equiv 1 \pmod{q\mathbf{Z}_p}$, which implies that $\langle a \rangle^c \equiv 1 \pmod{q\mathbf{Z}_p}$, and thus

$$\Delta_c^k \langle a \rangle^n \equiv 0 \pmod{q^k \mathbf{Z}_p} .$$

Therefore

$$\Delta_c^k \beta_{n,\chi}(F) - \Delta_c^k \beta_{n,\chi}(0) \equiv 0 \pmod{q^k \mathbf{Z}_p[\chi]} ,$$

and so $q^{-k}\Delta_c^k \beta_{n,\chi}(F) - q^{-k}\Delta_c^k \beta_{n,\chi}(0) \in \mathbf{Z}_p[\chi]$. Also, since $\langle a \rangle^n \equiv 1 \pmod{q\mathbf{Z}_p}$,

$$q^{-k}\Delta_c^k \beta_{n,\chi}(F) - q^{-k}\Delta_c^k \beta_{n,\chi}(0) = - \sum_{\substack{a=1 \\ (a,p)=1}}^{qF} \chi_1(a) \langle a \rangle^{n-1} \left(\frac{\langle a \rangle^c - 1}{q} \right)^k \tag{25}$$

implies that the value of $q^{-k}\Delta_c^k \beta_{n,\chi}(F) - q^{-k}\Delta_c^k \beta_{n,\chi}(0)$ modulo $q\mathbf{Z}_p[\chi]$ is independent of n.

Let $\tau \in pq^{-1}F_0\mathbf{Z}_p$. Since the set of positive integers in $pq^{-1}F_0\mathbf{Z}$ is dense in $pq^{-1}F_0\mathbf{Z}_p$, there exists a sequence $\{\tau_i\}_{i=1}^{\infty}$ in $pq^{-1}F_0\mathbf{Z}$, with $\tau_i > 0$ for each i, such that $\tau_i \to \tau$. Now, $\beta_{n,\chi}(t)$ is a polynomial, which implies that $\beta_{n,\chi}(\tau_i) \to \beta_{n,\chi}(\tau)$. Therefore

$$\lim_{i\to\infty} \left(\Delta_c^k \beta_{n,\chi}(\tau_i) - \Delta_c^k \beta_{n,\chi}(0) \right) = \Delta_c^k \beta_{n,\chi}(\tau) - \Delta_c^k \beta_{n,\chi}(0) .$$

The left side of this equality is 0 modulo $q^k\mathbf{Z}_p[\chi]$, which implies that

$$\Delta_c^k \beta_{n,\chi}(\tau) - \Delta_c^k \beta_{n,\chi}(0) \equiv 0 \pmod{q^k \mathbf{Z}_p[\chi]} ,$$

and so $q^{-k}\Delta_c^k \beta_{n,\chi}(\tau) - q^{-k}\Delta_c^k \beta_{n,\chi}(0) \in \mathbf{Z}_p[\chi]$. Furthermore, for n' a positive integer,

$$\begin{aligned}
&\lim_{i\to\infty} \left(\left(q^{-k}\Delta_c^k \beta_{n,\chi}(\tau_i) - q^{-k}\Delta_c^k \beta_{n,\chi}(0) \right) - \left(q^{-k}\Delta_c^k \beta_{n',\chi}(\tau_i) - q^{-k}\Delta_c^k \beta_{n',\chi}(0) \right) \right) \\
&\quad = \left(\left(q^{-k}\Delta_c^k \beta_{n,\chi}(\tau) - q^{-k}\Delta_c^k \beta_{n,\chi}(0) \right) - \left(q^{-k}\Delta_c^k \beta_{n',\chi}(\tau) - q^{-k}\Delta_c^k \beta_{n',\chi}(0) \right) \right) .
\end{aligned}$$

Since $\tau_i \in pq^{-1}F_0\mathbf{Z}$ for each i, the quantity on the left must also be 0 modulo $q\mathbf{Z}_p[\chi]$. Therefore the value of $q^{-k}\Delta_c^k \beta_{n,\chi}(\tau) - q^{-k}\Delta_c^k \beta_{n,\chi}(0)$ modulo $q\mathbf{Z}_p[\chi]$ is independent of n. □

THEOREM 4.11. *Let n, c, k, and k' be positive integers with $k \equiv k' \pmod{p-1}$, and let $\tau \in \mathbf{Z}_p$ such that $|\tau|_p \le |pq^{-1}F_0|_p$. Then*

$$q^{-k}\Delta_c^k\beta_{n,\chi}(\tau) - q^{-k}\Delta_c^k\beta_{n,\chi}(0) \equiv q^{-k'}\Delta_c^{k'}\beta_{n,\chi}(\tau) - q^{-k'}\Delta_c^{k'}\beta_{n,\chi}(0) \pmod{p\mathbf{Z}_p[\chi]}.$$

Proof. Let k and k' be positive integers such that $k \equiv k' \pmod{p-1}$. Without loss of generality, we can assume that $k \ge k'$. From (25),

$$\begin{aligned}
&\left(q^{-k}\Delta_c^k\beta_{n,\chi}(F) - q^{-k}\Delta_c^k\beta_{n,\chi}(0)\right) - \left(q^{-k'}\Delta_c^{k'}\beta_{n,\chi}(F) - q^{-k'}\Delta_c^{k'}\beta_{n,\chi}(0)\right) \\
&\quad = -\sum_{\substack{a=1\\(a,p)=1}}^{qF} \chi_1(a)\langle a\rangle^{n-1}\left(\frac{\langle a\rangle^c - 1}{q}\right)^k + \sum_{\substack{a=1\\(a,p)=1}}^{qF} \chi_1(a)\langle a\rangle^{n-1}\left(\frac{\langle a\rangle^c - 1}{q}\right)^{k'} \\
&\quad = -\sum_{\substack{a=1\\(a,p)=1}}^{qF} \chi_1(a)\langle a\rangle^{n-1}\left(\frac{\langle a\rangle^c - 1}{q}\right)^{k'}\left(\left(\frac{\langle a\rangle^c - 1}{q}\right)^{k-k'} - 1\right),
\end{aligned}$$

where F is a positive multiple of $pq^{-1}F_0$. If a is such that

$$\langle a\rangle^c - 1 \not\equiv 0 \pmod{pq\mathbf{Z}_p},$$

then

$$\left(\frac{\langle a\rangle^c - 1}{q}\right)^{k-k'} - 1 \equiv 0 \pmod{p\mathbf{Z}_p},$$

since $k - k' \equiv 0 \pmod{p-1}$. Thus

$$q^{-k}\Delta_c^k\beta_{n,\chi}(F) - q^{-k}\Delta_c^k\beta_{n,\chi}(0) \equiv q^{-k'}\Delta_c^{k'}\beta_{n,\chi}(F) - q^{-k'}\Delta_c^{k'}\beta_{n,\chi}(0) \pmod{p\mathbf{Z}_p[\chi]}.$$

Now let $\tau \in pq^{-1}F_0\mathbf{Z}_p$. Then there exists a sequence $\{\tau_i\}_{i=1}^{\infty}$ in $pq^{-1}F_0\mathbf{Z}$, with $\tau_i > 0$ for each i, such that $\tau_i \to \tau$. Consider

$$\begin{aligned}
&\lim_{i\to\infty}\left((q^{-k}\Delta_c^k\beta_{n,\chi}(\tau_i) - q^{-k}\Delta_c^k\beta_{n,\chi}(0)) - (q^{-k'}\Delta_c^{k'}\beta_{n,\chi}(\tau_i) - q^{-k'}\Delta_c^{k'}\beta_{n,\chi}(0))\right) \\
&\quad = \left(q^{-k}\Delta_c^k\beta_{n,\chi}(\tau) - q^{-k}\Delta_c^k\beta_{n,\chi}(0)\right) - \left(q^{-k'}\Delta_c^{k'}\beta_{n,\chi}(\tau) - q^{-k'}\Delta_c^{k'}\beta_{n,\chi}(0)\right).
\end{aligned}$$

Since the left side of this equality must be 0 modulo $p\mathbf{Z}_p[\chi]$, the theorem must hold. $\square$

THEOREM 4.12. *Let n, c, and k be positive integers, and let $\tau \in \mathbf{Z}_p$ such that $|\tau|_p \leq |pq^{-1}F_0|_p$. Then the quantity*

$$\binom{q^{-1}\Delta_c}{k}\beta_{n,\chi}(\tau) - \binom{q^{-1}\Delta_c}{k}\beta_{n,\chi}(0) \in \mathbf{Z}_p[\chi],$$

and, modulo $q\mathbf{Z}_p[\chi]$, is independent of n.

Proof. We are once again working with a linear operator, so Corollary 4.9 implies that

$$\binom{q^{-1}\Delta_c}{k}L_p(1-n,F;\chi) = \binom{q^{-1}\Delta_c}{k}L_p(1-n;\chi) - \sum_{\substack{a=1\\(a,p)=1}}^{qF} \chi_1(a)\binom{q^{-1}\Delta_c}{k}\langle a\rangle^{n-1},$$

where F is a positive multiple of $pq^{-1}F_0$. Then

$$\binom{q^{-1}\Delta_c}{k}\beta_{n,\chi}(F) - \binom{q^{-1}\Delta_c}{k}\beta_{n,\chi}(0) = -\sum_{\substack{a=1\\(a,p)=1}}^{qF} \chi_1(a)\langle a\rangle^{-1}\binom{q^{-1}\Delta_c}{k}\langle a\rangle^{n}.$$

Utilizing (15), we can write

$$\begin{aligned}\binom{q^{-1}\Delta_c}{k}\langle a\rangle^{n} &= \frac{1}{k!}\sum_{m=0}^{k} s(k,m)q^{-m}\Delta_c^m\langle a\rangle^{n}\\ &= \frac{1}{k!}\sum_{m=0}^{k} s(k,m)q^{-m}\langle a\rangle^{n}\left(\langle a\rangle^{c}-1\right)^{m},\end{aligned}$$

which follows from (24). This can then be rewritten as

$$\binom{q^{-1}\Delta_c}{k}\langle a\rangle^{n} = \langle a\rangle^{n}\binom{q^{-1}(\langle a\rangle^{c}-1)}{k}.$$

Since $q^{-1}(\langle a\rangle^{c}-1) \in \mathbf{Z}_p$ for each $a \in \mathbf{Z}$ with $(a,p)=1$, we see that

$$\langle a\rangle^{n}\binom{q^{-1}(\langle a\rangle^{c}-1)}{k} \in \mathbf{Z}_p.$$

This then implies that

$$\binom{q^{-1}\Delta_c}{k}\beta_{n,\chi}(F) - \binom{q^{-1}\Delta_c}{k}\beta_{n,\chi}(0) \in \mathbf{Z}_p[\chi].$$

Furthermore, since $\langle a\rangle^{n} \equiv 1 \pmod{q\mathbf{Z}_p}$, the value of this quantity modulo $q\mathbf{Z}_p[\chi]$ is independent of n.

Now let $\tau \in pq^{-1}F_0\mathbf{Z}_p$, and let $\{\tau_i\}_{i=1}^{\infty}$ be a sequence in $pq^{-1}F_0\mathbf{Z}$, with $\tau_i > 0$ for each i, such that $\tau_i \to \tau$. We are working with polynomials, so that

$$\lim_{i\to\infty}\left(\binom{q^{-1}\Delta_c}{k}\beta_{n,\chi}(\tau_i) - \binom{q^{-1}\Delta_c}{k}\beta_{n,\chi}(0)\right) = \binom{q^{-1}\Delta_c}{k}\beta_{n,\chi}(\tau) - \binom{q^{-1}\Delta_c}{k}\beta_{n,\chi}(0),$$

which must be in $\mathbf{Z}_p[\chi]$ since the limit of any sequence in $\mathbf{Z}_p[\chi]$ must also be in $\mathbf{Z}_p[\chi]$. Now let n' be a positive integer, and consider

$$\lim_{i\to\infty}\left(\left(\tbinom{q^{-1}\Delta_c}{k}\beta_{n,\chi}(\tau_i) - \tbinom{q^{-1}\Delta_c}{k}\beta_{n,\chi}(0)\right) - \left(\tbinom{q^{-1}\Delta_c}{k}\beta_{n',\chi}(\tau_i) - \tbinom{q^{-1}\Delta_c}{k}\beta_{n',\chi}(0)\right)\right)$$
$$= \left(\left(\tbinom{q^{-1}\Delta_c}{k}\beta_{n,\chi}(\tau) - \tbinom{q^{-1}\Delta_c}{k}\beta_{n,\chi}(0)\right) - \left(\tbinom{q^{-1}\Delta_c}{k}\beta_{n',\chi}(\tau) - \tbinom{q^{-1}\Delta_c}{k}\beta_{n',\chi}(0)\right)\right).$$

The quantity on the left must be 0 modulo $q\mathbf{Z}_p[\chi]$, which implies that the value of

$$\binom{q^{-1}\Delta_c}{k}\beta_{n,\chi}(\tau) - \binom{q^{-1}\Delta_c}{k}\beta_{n,\chi}(0)$$

modulo $q\mathbf{Z}_p[\chi]$ is independent of n. □

4.4 Generalized Bernoulli power series

In [9] we find a definition of ordinary Bernoulli numbers of negative index, B_{-n}, where $n \in \mathbf{Z}$, $n \geq 1$, in the field $\mathbf{Q}_p$, given by

$$B_{-n} = \lim_{k\to\infty} B_{\phi(p^k)-n}, \tag{26}$$

where the limit is taken in a p-adic sense. Note that $\phi(p^k) \to 0$ in $\mathbf{Z}_p$ as $k \to \infty$. Since $|B_m|_p$ is bounded for all $m \in \mathbf{Z}$, $m \geq 0$, we must have

$$\begin{aligned} B_{-n} &= \lim_{k\to\infty}\left(1 - p^{\phi(p^k)-n-1}\right)B_{\phi(p^k)-n} \\ &= \lim_{k\to\infty} -\left(\phi\left(p^k\right)-n\right)L_p\left(1-\left(\phi\left(p^k\right)-n\right);\omega^{-n}\right) \\ &= nL_p\left(n+1;\omega^{-n}\right). \end{aligned}$$

implying that the limit exists and can be described in familiar terms.

Recall that $B_m = 0$ for any odd $m \in \mathbf{Z}$, $m \geq 3$. Thus (26) implies that $B_{-n} = 0$ for any odd $n \in \mathbf{Z}$, $n \geq 1$. Furthermore, we have the following:

THEOREM 4.13. *Let $n \in \mathbf{Z}$ be even, $n \geq 2$. Then*

$$B_{-n} + \sum_{\substack{r \text{ prime} \\ (r-1)|n}} \frac{1}{r} \in \mathbf{Z}_p ,$$

where each prime r is taken to be a rational prime.

REMARK. Since $1/r \in \mathbf{Z}_p$ for any rational prime $r \neq p$, this implies that $B_{-n} + 1/p \in \mathbf{Z}_p$ whenever $(p-1) \mid n$, and $B_{-n} \in \mathbf{Z}_p$ otherwise.

Proof. By the von Staudt-Clausen theorem, we know that

$$B_m + \sum_{\substack{r \text{ prime} \\ (r-1)|m}} \frac{1}{r} \in \mathbf{Z}$$

for any even $m \in \mathbf{Z}$, $m \geq 2$.

Let $n \in \mathbf{Z}$ be even, $n \geq 2$. For any integer $k \geq 2$, $\phi(p^k)$ is even and $(p-1) \mid \phi(p^k)$. Thus $\phi(p^k) - n$ is even, and $(p-1) \mid n$ if and only if $(p-1) \mid (\phi(p^k) - n)$. Therefore, if k is sufficiently large,

$$B_{\phi(p^k)-n} + \sum_{\substack{r \text{ prime} \\ (r-1)|n}} \frac{1}{r} \in \mathbf{Z}_p ,$$

and the result follows from (26). $\square$

In a similar manner we define generalized Bernoulli numbers of negative index, $B_{-n,\chi}$, where $n \in \mathbf{Z}$, $n \geq 1$, in the field $\mathbf{C}_p$ according to

$$B_{-n,\chi} = \lim_{k \to \infty} B_{\phi(p^k)-n,\chi} , \tag{27}$$

where the limit is once again taken in a p-adic sense. For each $m \in \mathbf{Z}$, $m \geq 0$, the quantity $|B_{m,\chi}|_p$ is bounded. Thus, since $\chi_{\phi(p^k)} = \chi$ for all characters χ and for all $k \in \mathbf{Z}$, $k \geq 1$, we can write

$$\begin{aligned} B_{-n,\chi} &= \lim_{k \to \infty} \left(1 - \chi_{\phi(p^k)}(p) p^{\phi(p^k)-n-1}\right) B_{\phi(p^k)-n,\chi_{\phi(p^k)}} \\ &= \lim_{k \to \infty} -\left(\phi\left(p^k\right) - n\right) L_p\left(1 - \left(\phi\left(p^k\right) - n\right); \chi_n\right) \\ &= n L_p\left(n+1; \chi_n\right), \end{aligned}$$

so that the limit exists. Since $B_{\phi(p^k)-n,1} = B_{\phi(p^k)-n}$ for $n, k \in \mathbf{Z}$, with $n \geq 1$ and k sufficiently large, we obtain $B_{-n,1} = B_{-n}$ for all such n.

If $k \geq 2$, then $\phi(p^k)$ is even. Thus n and $\phi(p^k) - n$ are of the same parity. Recall that

$$\delta_\chi = \begin{cases} 1, & \text{if } \chi \text{ is odd} \\ 0, & \text{if } \chi \text{ is even}. \end{cases}$$

Then $B_{\phi(p^k)-n,\chi} = 0$ whenever $n \not\equiv \delta_\chi \pmod 2$, provided $\phi(p^k) - n > 1$. Because of this, the relation (27) implies that $B_{-n,\chi} = 0$ whenever $n \not\equiv \delta_\chi \pmod 2$ for all $n \in \mathbf{Z}$, $n \geq 1$. Furthermore, we can obtain

THEOREM 4.14. *Let χ be such that $\chi \neq 1$, and let $n \in \mathbf{Z}$, $n \geq 1$. Then $f_\chi B_{-n,\chi} \in \mathbf{Z}_p[\chi]$.*

Proof. Recall that when $\chi \neq 1$, $f_\chi B_{m,\chi} \in \mathbf{Z}[\chi]$ for all $m \in \mathbf{Z}$, $m \geq 0$. Thus

$$f_\chi B_{-n,\chi} = \lim_{k \to \infty} f_\chi B_{\phi(p^k)-n,\chi}$$

must be in the p-adic completion of $\mathbf{Z}[\chi]$ for any $n \in \mathbf{Z}$, $n \geq 1$. Since the p-adic completion of $\mathbf{Z}[\chi]$ is $\mathbf{Z}_p[\chi]$, the theorem must hold. $\square$

We now define what we shall refer to as generalized Bernoulli power series of negative index in $\mathbf{Z}_p[\chi]$. For $n \in \mathbf{Z}$, $n \geq 1$, and for $t \in \mathbf{C}_p$, $|t|_p \leq |q|_p$, let

$$B_{-n,\chi}(t) = \lim_{k \to \infty} B_{\phi(p^k)-n,\chi}(t).$$

Then

$$\begin{aligned} B_{-n,\chi}(qt) &= \lim_{k \to \infty} \left(B_{\phi(p^k)-n,\chi_{\phi(p^k)}}(qt) - \chi_{\phi(p^k)}(p) p^{\phi(p^k)-n-1} B_{\phi(p^k)-n,\chi_{\phi(p^k)}}(p^{-1}qt)\right) \\ &= \lim_{k \to \infty} -(\phi(p^k) - n) L_p(1 - (\phi(p^k) - n), t; \chi_n) \\ &= n L_p(n+1, t; \chi_n). \end{aligned}$$

Since $L_p(n+1, t; \chi_n)$ exists for each $n \in \mathbf{Z}$, $n \geq 1$, and $t \in \mathbf{C}_p$, $|t|_p \leq 1$, we see that $B_{-n,\chi}(qt)$ must also exist for such t. Thus $B_{-n,\chi}(t)$ exists for $t \in \mathbf{C}_p$, $|t|_p \leq |q|_p$. Now, by Theorem 4.5, we can expand this quantity as a power series, obtaining

$$\begin{aligned} B_{-n,\chi}(qt) &= n \sum_{m=0}^{\infty} \binom{-(n+1)}{m} q^m t^m L_p(n+m+1; \chi_{n+m}) \\ &= n \sum_{m=0}^{\infty} \binom{-(n+1)}{m} q^m t^m \frac{1}{n+m} B_{-(n+m),\chi} \\ &= \sum_{m=0}^{\infty} \binom{-n}{m} B_{-(n+m),\chi} q^m t^m. \end{aligned}$$

Since $|B_{-(n+m),\chi}|_p \leq \max\{|p|_p^{-1}, |f_\chi|_p^{-1}\}$ and

$$\binom{-n}{m} = (-1)^m \binom{n+m-1}{m},$$

this sum converges for $|qt|_p < 1$. Thus we have the relation

$$B_{-n,\chi}(t) = \sum_{m=0}^{\infty} \binom{-n}{m} B_{-n-m,\chi} t^m, \tag{28}$$

converging for all $t \in \mathbf{C}_p$, $|t|_p < 1$. Note that this is in the same form as (2) for the generalized Bernoulli polynomials having positive index, which we can rewrite as

$$B_{n,\chi}(t) = \sum_{m=0}^{\infty} \binom{n}{m} B_{n-m,\chi} t^m,$$

since $\binom{n}{m} = 0$ for $m, n \in \mathbf{Z}$, $m > n \geq 0$. By setting $t = 0$ in (28), we see that $B_{-n,\chi}(0) = B_{-n,\chi}$ for all $n \in \mathbf{Z}$, $n \geq 1$.

THEOREM 4.15. *Let $n \in \mathbf{Z}$, $n \geq 1$. Then for any $m \in \mathbf{Z}$, $m \geq 1$, such that $q \mid mf_\chi$,*

$$B_{-n,\chi}\left(mf_\chi\right) - B_{-n,\chi}(0) = -n \sum_{\substack{a=1 \\ (a,p)=1}}^{mf_\chi} \chi(a) a^{-n-1}.$$

Proof. By definition, since $|mf_\chi|_p \leq |q|_p$,

$$\begin{aligned} B_{-n,\chi}\left(mf_\chi\right) - B_{-n,\chi}(0) &= \lim_{k\to\infty} \left(B_{\phi(p^k)-n,\chi}\left(mf_\chi\right) - B_{\phi(p^k)-n,\chi}(0)\right) \\ &= \lim_{k\to\infty} \left(\phi\left(p^k\right) - n\right) \sum_{a=1}^{mf_\chi} \chi(a) a^{\phi(p^k)-n-1}, \end{aligned}$$

following from (4). Now, $v_p(\phi(p^k)) = k-1$, and $a^{\phi(p^k)} \equiv 1 \pmod{p^k}$ for $(a,p) = 1$. These imply that

$$\lim_{k\to\infty} \left(\phi\left(p^k\right) - n\right) \sum_{a=1}^{mf_\chi} \chi(a) a^{\phi(p^k)-n-1} = -n \sum_{\substack{a=1 \\ (a,p)=1}}^{mf_\chi} \chi(a) a^{-n-1},$$

completing the proof. □

THEOREM 4.16. *Let $n \in \mathbf{Z}$, $n \geq 1$. Then for all χ and for all $t \in \mathbf{C}_p$, $|t|_p < 1$,*

$$B_{-n,\chi}(-t) = (-1)^n \chi(-1) B_{-n,\chi}(t)\,.$$

Proof. Since

$$B_{-n,\chi}(t) = \sum_{m=0}^{\infty} \binom{-n}{m} B_{-n-m,\chi} t^m,$$

and $B_{-n-m,\chi} = 0$ whenever $n+m \not\equiv \delta_\chi \pmod 2$ for each $m \in \mathbf{Z}$, $m \geq 1$, we see that $B_{-n,\chi}(t)$ is either an odd or an even function according to whether $n+\delta_\chi$ is odd or even, respectively. Thus

$$\begin{aligned} B_{-n,\chi}(-t) &= (-1)^{n+\delta_\chi} B_{-n,\chi}(t) \\ &= (-1)^n \chi(-1) B_{-n,\chi}(t)\,, \end{aligned}$$

and the proof is complete. □

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Glenn J. Fox

Department of Mathematics
and Computer Science
Emory University
Atlanta, GA 30322
U. S. A.
e-mail: fox@mathcs.emory.edu