6. Discrete groups : Chern character

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Let G be a connected semi-simple Lie group with finite center. The lemma of this section elucidates the role of $H\backslash G$ in the Atiyah-Schmid geometric construction of the discrete series [4]. Atiyah and Schmid obtain the discrete series representations by using the Dirac operator on $H\backslash G$. As noted in the introduction $K_0[C^*G]$ contains a free abelian group with one generator for each (irreducible) discrete series representation. By the lemma, however, all of $K^*(\cdot, G)$ is obtained from $H\backslash G$. If (as conjectured in §2 above) $K^*(\cdot, G) \cong K_*(C^*G)$, then not only the discrete series, but *all* of $K_*(C^*G)$ can be obtained from $H\backslash G$.

At this juncture one might ask, "*Why not simply define* $K^i(X, G) = K^i_H(X)$ *?*" We believe that there are compelling reasons for not doing this. First, this misses the dimension-shift by $\epsilon = \dim(H\backslash G)$. Second, this overlooks the issue of whether or not the action of H on $(\mathfrak{h}\backslash\mathfrak{g})^*$ is Spin^c. Third, this greatly obscures the relation of K-theory to index theory. Finally, in the case of discrete groups and foliations there is no maximal compact subgroup so that if this were done there would be no unified theory for Lie groups, discrete groups, and foliations.

6. Discrete groups: Chern character

In this section G is a discrete group which is either finite or countable infinite. For a G-manifold X, $K^*(X, G)$ was defined in §2 above. As in §3 there is the natural map

$$K_*^{\tau}([EG \times X]/G) \to K^*(X, G),$$

where $\tau = [EG \times T^*X]/G$.

PROPOSITION 1. *Let G be a discrete group and X a G-manifold. Then*

$$K_*^{\tau}([EG \times X]/G) \otimes_{\mathbf{Z}} \mathbf{Q} \to K^*(X, G) \otimes_{\mathbf{Z}} \mathbf{Q}$$

is injective.

REMARK 2. When X is a point, Proposition 1 asserts that

$$K_*(BG) \otimes_{\mathbf{Z}} \mathbf{Q} \to K^*(\cdot, G) \otimes_{\mathbf{Z}} \mathbf{Q}$$

is injective.

The proposition is proved by defining the *Chern character*. This is a natural map

$$K^*(X,G) \to H_*^\tau([EG \times X]/G; \mathbf{C}).$$

Here $H_*^\tau([EG\times X]/G;\mathbf{C})$ denotes the homology (with coefficients the complex numbers $\mathbf{C}$) of the pair $(B\tau, S\tau)$ where $B\tau, S\tau$ are the unit ball and unit sphere bundles of τ with respect to any continuous Euclidean structure chosen for τ

$$H_*^\tau([EG \times X]/G; \mathbf{C}) = H_*((B\tau, S\tau), \mathbf{C}).$$

The Chern character can be defined by the following five-step procedure, which is similar to a procedure used by M.F. Atiyah [2].

Step 1. Let (Z,ξ,f) be a K-cocycle for (X,G). Form the commutative diagram

$$\begin{array}{ccc} Z & \xrightarrow{h} & X \times Z \\ & f\searrow \quad \swarrow \pi_1 & \\ & X & \end{array}$$

where $h(z) = (fz, z)$ and $\pi_1(x,z) = x$. Consider $h_!(\xi) \in K_G^*(T^*(X \times Z) \oplus \pi_1^*T^*X)$. Now $T^*(X\times Z) = \pi_1^*T^*X \oplus \pi_2^*T^*Z$ where $\pi_2(x,z) = z$. Since $\pi_1^*T^*X \oplus \pi_1^*T^*X$ has a G-invariant Spin^c-structure, the Thom isomorphism theorem gives an isomorphism

$$K_G^*(T^*(X \times Z) \oplus \pi_1^*T^*X) \cong K_G^*(\pi_2^*T^*Z).$$

Via this isomorphism $h_!(\xi)$ determines $\xi' \in K_G^*(\pi_2^*T^*Z)$. Using G-invariant connections and the Chern-Weil curvature theory of characteristic classes, let ω be the differential form on $X \times T^*Z = \pi_2 T^*Z$ which represents the Atiyah-Singer answer for the index of a family of elliptic operators [7]. Thus ω is a G-invariant closed differential form with G-compact support which represents $\mathrm{ch}(\xi') \cup \pi_2^* \, \mathrm{Td}(\mathbf{C} \otimes_{\mathbf{R}} T^*Z)$. Here ch is the usual Chern character and Td is the Todd polynomial.

Step 2. The action of G on $X \times T^*Z = \pi_2^*T^*Z$ is proper. This implies that the quotient space $\pi_2^*T^*Z/G$ is a rational homology manifold. The differential form ω of Step 1 is closed, G-invariant and has G-compact support. Hence ω descends to determine a cohomology class $\underline{\omega}$, with compact support, on $\pi_2^*T^*Z/G$

$$\underline{\omega} \in H_c^*(\pi_2^*T^*Z/G; \mathbf{C}).$$

Step 3. On $X \times Z$ choose a G-invariant Euclidean structure for $\pi_1^* T^* X$ and let $B\pi_1^* T^* X$, $S\pi_1^* T^* X$ be the unit ball and unit sphere bundles. The rational homology manifold $T^*(X \times Z)/G$ is oriented. This gives a Poincaré duality isomorphism

$$H_c^*(\pi_2^* T^* Z/G; \mathbf{C}) \cong H_*((B\pi_1^* T^* X/G, S\pi_1^* T^* X/G); \mathbf{C}).$$

Using this isomorphism, $\underline{\omega} \in H_c^*(\pi_2^* T^* Z/G; \mathbf{C})$ determines

$$\mathrm{Dual}(\underline{\omega}) \in H_*((B\pi_1^* T^* X/G, S\pi_1^* T^* X/G); \mathbf{C}).$$

Step 4. On $[EG \times X \times Z]/G$ let $\tilde{\tau}$ be the vector bundle $[EG \times \pi_1^* T^* X]/G$. Consider the evident map

$$\tilde{\tau} = [EG \times \pi_1^* T^* X]/G \to \pi_1^* T^* X/G.$$

A typical fibre of this map is of the form $B\Gamma$ where Γ is an isotropy group for the action of G on $\pi_1^* T^* X$. Since this action is proper, Γ is a finite group and $H_i(B\Gamma; \mathbf{Q}) = 0$ for $i > 0$. Hence the map

$$\tilde{\tau} \to \pi_1^* T^* X/G$$

is an isomorphism in rational homology. This gives an isomorphism:

$$H_*((B\tilde{\tau}, S\tilde{\tau}); \mathbf{C}) \cong H_*((B\pi_1^* T^* X/G, S\pi_1^* T^* X/G); \mathbf{C}).$$

By this isomorphism Dual $(\underline{\omega}) \in H_*((B\pi_1^* T^* X/G, S\pi_1^* T^* X/G); \mathbf{C})$ determines $\underline{\omega} \in H_*((B\tilde{\tau}, S\tilde{\tau}); \mathbf{C})$.

Step 5. The projection $\tilde{\tau} = [EG \times T^* X \times Z]/G \to [EG \times T^* X]/G = \tau$ induces a map of homology

$$H_*((B\tilde{\tau}, S\tilde{\tau}); \mathbf{C}) \to H_*((B\tau, S\tau); \mathbf{C}).$$

The image of $\tilde{\omega}$ under this map is, by definition, the Chern character of the original K-cocycle (Z, ξ, f).

Proof of Proposition 1. The following diagram is commutative

$$\begin{array}{ccc} K_*^{\tau}([EG \times X]/G) & \longrightarrow & K^*(X, G) \\ & \searrow \quad \swarrow & \\ & H_*^{\tau}([EG \times X]/G; \mathbf{C}) & \end{array}$$

where the left vertical arrow is the usual K-homology Chern character [9]. Since the usual K-homology Chern character is rationally injective, this forces the horizontal arrow to be rationally injective. $\square$

REMARK 3. For G discrete the reduced C^*-algebra of G, denoted C^*G, comes equipped with a trace. An element in C^*G is a formal sum $\sum_{g\in G} \lambda_g g$ where $\lambda_g \in \mathbf{C}$. The trace of such an element is λ_1 where 1 is the identity element of G. This trace then induces a map

$$\mathrm{tr}\colon K_0\, C^*G \to \mathbf{R}.$$

Let Z be a proper G-manifold and let D be a G-invariant elliptic operator on Z. If ξ is the symbol of D then (Z, ξ) is a K-cocycle for $(\cdot, G)$ and the Chern character defined above assigns to (Z, ξ)

$$\mathrm{ch}(Z, \xi) \in H_*(BG; \mathbf{C}).$$

Let $\epsilon\colon BG \to \cdot$ be the map of BG to a point. Identify $H_*(\cdot, \mathbf{C}) = \mathbf{C}$ and consider

$$\epsilon_*\, \mathrm{ch}(Z, \xi) \in \mathbf{C}.$$

The K-theory index of the elliptic operator D is an element of $K_0\, C^*G$

$$\mathrm{Index}(D) \in K_0\, C^*G.$$

We then have the following formula for $\mathrm{tr}[\mathrm{Index}(D)]$:

$$\mathrm{tr}[\mathrm{Index}(D)] = \epsilon_*\, \mathrm{ch}(Z, \xi).$$

For the special case when the action of G on Z is free this formula was obtained by M.F. Atiyah [3].

7. COROLLARIES OF THE ISOMORPHISM CONJECTURE

The conjecture stated in §2 above asserts that

$$\mu\colon K^*(X, G) \to K_*[C_0(X) \rtimes G]$$

is an isomorphism. Suppose that G is a discrete group and X is a point. The conjecture then asserts that $\mu\colon K^*(\cdot, G) \to K_* C^*G$ is an isomorphism where C^*G is the reduced C^*-algebra of G. Throughout this section G will be a discrete group and we shall consider some corollaries of the conjecture that $\mu\colon K^0(\cdot, G) \to K_0\, C^*G$ is an isomorphism. "*Proof*" will mean "*Proof modulo the conjecture*".