

# 1. Introduction

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## COUNTING PATHS IN GRAPHS

by Laurent BARTHOLDI

ABSTRACT. We give a simple combinatorial proof of a formula that extends a result by Grigorchuk [Gri78a, Gri78b] relating cogrowth and spectral radius of random walks. Our main result is an explicit equation determining the number of ‘bumps’ on paths in a graph: in a  $d$ -regular (not necessarily transitive) non-oriented graph let the series  $G(t)$  count all paths between two fixed points weighted by their length  $t^{\text{length}}$ , and  $F(u, t)$  count the same paths, weighted as  $u^{\text{number of bumps}} t^{\text{length}}$ . Then one has

$$\frac{F(1-u, t)}{1-u^2 t^2} = \frac{G\left(\frac{t}{1+u(d-u)t^2}\right)}{1+u(d-u)t^2}.$$

We then derive the circuit series of ‘free products’ and ‘direct products’ of graphs. We also obtain a generalized form of the Ihara-Selberg zeta function [Bas92, FZ98].

### 1. INTRODUCTION

Let  $\Gamma = \mathbf{F}_S/N$  be a group generated by a finite set  $S$ , where  $\mathbf{F}_S$  denotes the free group on  $S$ . Let  $f_n$  be the number of elements of the normal subgroup  $N$  of  $\mathbf{F}_S$  whose minimal representation as words in  $S \cup S^{-1}$  has length  $n$ ; let  $g_n$  be the number of (not necessarily reduced) words of length  $n$  in  $S \cup S^{-1}$  that evaluate to 1 in  $\Gamma$ ; and let  $d = |S \cup S^{-1}| = 2|S|$ . The numbers

$$\alpha = \limsup_{n \rightarrow \infty} \sqrt[n]{f_n}, \quad \nu = \frac{1}{d} \limsup_{n \rightarrow \infty} \sqrt[n]{g_n}$$

are called the *cogrowth* and *spectral radius* of  $(\Gamma, S)$ . The Grigorchuk Formula [Gri78b] states that

$$(1.1) \quad \nu = \begin{cases} \frac{\sqrt{d-1}}{d} \left( \frac{\alpha}{\sqrt{d-1}} + \frac{\sqrt{d-1}}{\alpha} \right) & \text{if } \alpha > \sqrt{d-1}, \\ \frac{2\sqrt{d-1}}{d} & \text{else.} \end{cases}$$

We generalize this result to a somewhat more general setting: we replace the group  $\Gamma$  by a regular graph  $\mathcal{X}$ , i.e. a graph with the same number of edges at each vertex. Fix a vertex  $\star$  of  $\mathcal{X}$ ; let  $g_n$  be the number of circuits (closed sequences of edges) of length  $n$  at  $\star$  and let  $f_n$  be the number of circuits of length  $n$  at  $\star$  with no backtracking (no edge followed twice consecutively). Then the same equation holds between the growth rates of  $f_n$  and  $g_n$ .

To a group  $\Gamma$  with fixed generating set one associates its Cayley graph  $\mathcal{X}$  (see Subsection 3.1).  $\mathcal{X}$  is a  $d$ -regular graph with distinguished vertex  $\star = 1$ ; paths starting at  $\star$  in  $\mathcal{X}$  are in one-to-one correspondence with words in  $S \cup S^{-1}$ , and paths starting at  $\star$  with no backtracking are in one-to-one correspondence with elements of  $\mathbf{F}_S$ . A circuit at  $\star$  in  $\mathcal{X}$  is then precisely a word evaluating to 1 in  $\Gamma$ , and a circuit without backtracking represents precisely one element of  $N$ . In this sense results on graphs generalize results on groups. The converse would not be true: there are even graphs with a vertex-transitive automorphism group that are not the Cayley graph of a group [Pas93].

Even more generally, we will show that, rather than counting circuits and proper circuits (those without backtracking) at a fixed vertex, we can count paths and proper paths between two fixed vertices and obtain the same formula relating their growth rates.

These relations between growth rates are consequences of a stronger result, expressed in terms of generating functions. Define the formal power series

$$F(t) = \sum_{n=0}^{\infty} f_n t^n, \quad G(t) = \sum_{n=0}^{\infty} g_n t^n.$$

Then assuming  $\mathcal{X}$  is  $d$ -regular we have

$$(1.2) \quad \frac{F(t)}{1-t^2} = \frac{G\left(\frac{t}{1+(d-1)t^2}\right)}{1+(d-1)t^2}.$$

This equation relates  $F$  and  $G$ , and so relates *a fortiori* their radii of convergence, which are  $1/\alpha$  and  $1/(d\nu)$ . We re-obtain thus the Grigorchuk Formula.

Finally, rather than counting paths and proper paths between two fixed vertices, we can count, for each  $m \geq 0$ , the number of paths with  $m$  backtrackings, i.e. with  $m$  occurrences of an edge followed twice in a row. Letting  $f_{m,n}$  be the number of paths of length  $n$  with  $m$  backtrackings, consider the two-variable formal power series

$$F(u, t) = \sum_{m,n=0}^{\infty} f_{m,n} u^m t^n.$$

Note that  $F(0, t) = F(t)$  and  $F(1, t) = G(t)$ . The following equation now holds:

$$\frac{F(1 - u, t)}{1 - u^2 t^2} = \frac{G\left(\frac{t}{1 + u(d - u)t^2}\right)}{1 + u(d - u)t^2}.$$

Setting  $u = 1$  in this equation reduces it to (1.2).

A generalization of the Grigorchuk Formula in a completely different direction can be attempted: consider again a finitely generated group  $\Gamma$ , and an exact sequence

$$1 \longrightarrow \Xi \longrightarrow \Pi \longrightarrow \Gamma \longrightarrow 1,$$

where this time  $\Pi$  is not necessarily free. Assume  $\Pi$  is generated as a monoid by a finite set  $S$ . Let again  $g_n$  be the number of words of length  $n$  in  $\Pi$  evaluating to 1 in  $\Gamma$ , and let  $f_n$  be the number of elements of  $\Xi$  whose minimal-length representation as a word in  $S$  has length  $n$ . Is there again a relation between the  $f_n$  and the  $g_n$ ? In Section 8 we derive such a relation when  $\Pi$  is the modular group  $\mathbf{PSL}_2(\mathbf{Z})$ .

Again there is a combinatorial counterpart; rather than considering graphs one considers a locally finite cellular complex  $\mathcal{K}$  such that all vertices have isomorphic neighbourhoods. As before,  $g_n$  counts the number of paths of length  $n$  in the 1-skeleton of  $\mathcal{K}$  between two fixed vertices; and  $f_n$  counts elements of the fundamental groupoid, i.e. homotopy classes of paths, between two fixed vertices whose minimal-length representation as a path in the 1-skeleton of  $\mathcal{K}$  has length  $n$ . We obtain a relation between these numbers when  $\mathcal{K}$  consists solely of triangles and arcs, with no two triangles nor two arcs meeting; these are precisely the complexes associated with quotients of the modular group.

The original motivation for our research was the study of cogrowth in group theory [Gri78a]; however, as it turned out, the more general problem in graph theory has applications to other domains of mathematics, like the Ihara-Selberg zeta function and its evaluation by Hyman Bass [Bas92].

## 2. MAIN RESULT

Let  $\mathcal{X}$  be a graph, that may have multiple edges and loops. We make the following typographical convention for the power series that will appear: a series in the formal variable  $t$  is written  $G(t)$ , or  $G$  for short, and  $G(x)$  refers to the series  $G$  with  $x$  substituted for  $t$ . Functions are written on the right, with  $(x)f$  or  $x^f$  denoting  $f$  evaluated at  $x$ .