# 5. Parallel transport in a cubical manifold and the proof of theorem 3

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **45 (1999)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **20.09.2024**

the sequence of curves $r \circ \gamma_m$ converges locally uniformly. By Proposition 3.4, the corresponding subsequence of the sequence of unit speed geodesics $\gamma_m$ converges locally uniformly. By definition, this means that the corresponding subsequence of $(x_m)$ converges to a point $\xi \in X(\infty)$.

Let $\phi \in \Gamma$ and choose $c = c_\phi$ as in Lemma 4.8. Let $t_0 > 0$ be given. By Lemma 4.8 we have $r \circ \gamma_m(t_0) \in F$ for all $m \geq m_0$. By Proposition 3.4 and Lemma 4.8, we have $d(\phi(\gamma_m(t_0)), \gamma_m(t_0)) \leq \sqrt{n} c_\phi$ for all such $m$. Now $c_\phi$ is independent of $t_0$, hence $\phi(\xi) = \xi$. □

We now complete the proof of Theorem 2 of the introduction. By Proposition 4.1, $\Delta \cong \ker h$ consists precisely of the elliptic elements of $\Gamma$. If indices of type 1 do not occur, then Proposition 4.3 applies: If $k = 0$, then $\Gamma \cong \Delta$ fixes a point of $X$ and possibility (1) holds. If $k > 0$, then possibility (2) holds. If indices of type 1 occur, then possibility (3) holds by Proposition 4.9 and Corollary 4.5. Note that $\mathrm{Stab}(x) \neq \Delta$ for any $x \in X$ in this case since $\Delta$ would have a fixed point otherwise.

## 5. Parallel transport in a cubical manifold and the proof of Theorem 3

Let $X$ be a cubical manifold of dimension $n$. Given two chambers $P$ and $Q$ in $X$ with a common face of dimension $n-1$, we define $t_{PQ}\colon P \to Q$ to be the *translation* which moves each point $p$ of $P$ along the unit geodesic segment starting at $p$ and orthogonal to the common $(n-1)$-face of $P$ to the end point in $Q$. The map $t_{PQ}$ is an isomorphism and isometry of $P$ with $Q$. Given a gallery $\pi = (P_1, \dots, P_n)$ in $X$, the *parallel transport* along $\pi$ is the isomorphism $t_\pi\colon P_1 \to P_n$ given by

$$t_\pi := t_{P_{n-1}P_n} \circ \cdots \circ t_{P_2P_3} \circ t_{P_1P_2} .$$

LEMMA 5.1. *Let $X$ be a simply connected cubical manifold and assume that the number of chambers adjacent to each face of codimension 2 in $X$ is divisible by 4. Then for any two chambers $P$ and $Q$ in $X$, the parallel transport $t_\pi$ along a gallery $\pi$ connecting $P$ and $Q$ is independent of $\pi$.*

*Proof.* It is enough to show that the parallel transport along any closed gallery is the identity. Let $\pi$ be such a gallery with initial and final chamber $P$.

Represent $\pi$ by a closed curve $c$ which starts and ends in some interior point $p$ of $P$, such that $c$ misses the $(n-2)$-skeleton of $X$ and crosses $(n-1)$-faces transversally and according to the pattern provided by $\pi$. Since $X$ is simply connected, the curve $c$ can be contracted in $X$ to the point $p$. Since $X$ is a manifold, the links of the vertices in $X$ are $(n-2)$-connected. Hence the contraction of $c$ can be chosen to be generic in the sense that it misses the $(n-3)$-skeleton of $X$ and crosses the $(n-2)$-skeleton transversally. Following the curve $c$ along this contraction, we get a sequence of modifications of the gallery $\pi$. These modifications occur when $c$ crosses an $(n-2)$-face of $X$. The condition that the number of chambers adjacent to such faces is divisible by 4 implies that the parallel transport $t_\pi$ does not change under these modifications. Since the parallel transport along the trivial gallery is the identity, $t_\pi = \mathrm{id}_P$. □

From now on we assume that $X$ is a simply connected cubical manifold such that the number of chambers adjacent to each face of codimension 2 in $X$ is divisible by 4. For chambers $P$ and $Q$ in $X$ define $t_{PQ} = t_\pi$, where $\pi$ is any gallery connecting $P$ with $Q$. The above lemma shows that $t_{PQ}$ is well defined.

We fix a chamber $P_0$ of $X$ and define a homomorphism $\phi\colon \Gamma \to \mathrm{Aut}\, P_0$ by

$$\phi(g) := t_{g(P_0)P_0} \circ g|_{P_0} .$$

The kernel $\Gamma' := \ker\phi$ is a finite index subgroup of $\Gamma$ and consists precisely of those automorphisms of $\Gamma$ whose restriction to any chamber commutes with the corresponding parallel transport.

## COORIENTATIONS

A *coorientation of a wall* in a chamber is a choice of one of the two half-chambers determined by the wall. Once and for all, we choose coorientations of the walls in the above chamber $P_0$. Now by Lemma 5.1, parallel transport gives rise to a consistent choice of coorientations for all walls in $X$.

By Corollary 1.3, $X$ is foldable. We fix a folding and denote by $\Lambda_i$ the set of hyperspaces of $X$ with label $i$. Note that $\Lambda_i$ is invariant under parallel transport. Along a hyperspace with label $i$, the half-chambers distinguished by the coorientation are all contained in the same halfspace with respect to the hyperspace. The above group $\Gamma'$ preserves the families $\Lambda_i$ together with the coorientations.

The index of intersection of an oriented curve $c$ at a transversal crossing of a hyperspace $H \in \Lambda_i$ is defined to be equal to $+1$ or $-1$ respectively, according to whether the orientation of $c$ coincides with the coorientation of $H$ or not. Fix a point $p_0$ in the interior of $P_0$ which does not belong to any wall and any of the chosen coorientations. For $p \in X$ define $f_i(p)$ to be the sum of the indices of intersection of an oriented curve $c$ connecting $p_0$ and $p$ with the hyperspaces from $\Lambda_i$. Here we assume that $c$ is generic, i.e. $c$ does not meet the $(n-2)$-skeleton and crosses hyperspaces transversally. The integer $f_i(p)$ does not depend on $c$ since $X$ is simply connected and any two such curves can be deformed into each other by a homotopy which misses the $(n-3)$-skeleton of $X$ and crosses the $(n-2)$-skeleton of $X$ transversally.

For $g \in \Gamma$ set $h_i(g) = f_i(g(p_0))$. Since the chosen system of coorientations is invariant under the action of $\Gamma'$, the maps $h_i \colon \Gamma' \to \mathbf{Z}$ are homomorphisms. We finish the proof of Theorem 3 by showing that the image of $h = (h_1, \dots, h_n)$ is of finite index in $\mathbf{Z}^n$.

We need to show that the image of $h$ contains $n$ linearly independent vectors. To that end, we show that the image contains non-zero vectors which span arbitrarily small angles with the unit vectors $e_i$ in $\mathbf{R}^n$, $1 \leq i \leq n$. Let $\sigma$ be a unit speed geodesic ray with $\sigma(0) = p_0$ which is perpendicular in $P_0$ to the wall with label $i$. By the choice of $p_0$, the ray $\sigma$ does not meet the $(n-2)$-skeleton of $X$ and is perpendicular to all $(n-1)$-faces and walls with label $i$ which it intersects. We have

$$f_j(\sigma(m)) = \delta_{ij} \cdot m, \quad m \geq 1.$$

By the cocompactness of the action of $\Gamma'$, there is an integer $k \geq 1$ such that for any $m \geq 1$ there is a $g_m \in \Gamma'$ with $d(\sigma(m), g_m(p_0)) \leq k$. By the definition of $h_j$ this implies $|h_j(g_m) - f_j(\sigma(m))| \leq k$. Theorem 3 follows.