5. Lower bounds

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Now we will consider generalized Følner sequences for functions f such that

$$Pf \geq \|P\| f \, .$$

This will enable us to obtain some lower bounds on the norm of random walk operators on graphs.

As in Section 3, let X be a connected, locally finite graph and let P be the simple random walk operator on X.

In this section we will prove the following lower bound on the norm $\|P\|$:

THEOREM 8. *Let X be a graph such that at each vertex there are at most k edges. Then*

$$\|P\| \geq \frac{2\sqrt{k-1}}{k} \, .$$

The norm of the random walk operator $\|P\|$ is equal to $\frac{2\sqrt{k-1}}{k}$ for the random walk on the tree which has k edges at each vertex. In [9] Kesten proved this lower bound in the case of Cayley graphs.

Proof of Theorem 8. Let us consider a graph X such that at each vertex there are at most k edges. We can suppose that $k \geq 3$ because for $k = 2$ we obtain subgraphs of $\mathbf{Z}$ or finite graphs, and necessarily $\|P\| = 1$. As it is enough to prove the desired bound for any connected component of X, we can suppose that X is connected.

In order to show that $\|P\|$ is large enough, we will construct a sequence of functions $f_n \in l^2(X, N)$ such that

$$\limsup_{n \to +\infty} \frac{\|Pf_n\|_{l^2(X,N)}}{\|f_n\|_{l^2(X,N)}} \geq \frac{2\sqrt{k-1}}{k} \, .$$

Let us endow the set of vertices of X with a metric. The distance between two vertices is the smallest number of edges needed to connect them. Let us choose a vertex e in X and for a vertex v let us denote by $|v|$ its distance from e.

Let f be the unique (up to translations and multiplications) radial eigenfunction of P on the homogeneous tree of degree k, corresponding to the eigenvalue $\frac{2\sqrt{k-1}}{k}$, which is the norm of P on this tree, i.e.

$$f(v) = g(|v|) = \left(\frac{k-2}{k}|v| + 1\right)\left(\frac{1}{\sqrt{k-1}}\right)^{|v|}. \tag{15}$$

Using (15) we can define f on X. We then prove

LEMMA 6. *For any vertex $v \in X$,*

$$Pf(v) \geq \frac{2\sqrt{k-1}}{k} f(v).$$

Proof. If $v = e$ the result is clearly true. Let us consider then a vertex $v \in X$ such that $n = |v| \geq 1$. Let the number of neighbors of v which are at a distance $n-1$ or n from e be equal respectively to p and q. So the number of neighbors of v which are at a distance $n+1$ is equal to $N(v) - p - q$. Hence

$$Pf(v) = \frac{1}{N(v)} \left(pg(n-1) + qg(n) + (N(v) - p - q)g(n+1)\right).$$

As $p \geq 1$ and g is a decreasing function,

$$Pf(v) \geq \frac{1}{N(v)} \left(g(n-1) + (N(v) - 1)g(n+1)\right).$$

As $N(v) \leq k$ and $g(n-1) \geq g(n+1)$,

$$Pf(v) \geq \frac{1}{k} \left(g(n-1) + (k-1)g(n+1)\right) = \frac{2\sqrt{k-1}}{k} g(n). \qquad \square$$

Let us denote by S_n and B_n the vertices which are respectively at a distance n and less than or equal to n.

LEMMA 7.

$$\frac{\sum_{v \in S_{n+1}} f^2(v) N(v)}{\sum_{v \in B_n} f^2(v) N(v)} \to_{n \to \infty} 0.$$

Proof. As $1 \leq N(v) \leq k$ it is enough to show that

$$\frac{\sum_{v \in S_{n+1}} f^2(v)}{\sum_{v \in B_n} f^2(v)} \to_{n \to \infty} 0.$$

Let us denote

$$a_n = \sum_{v \in S_n} f^2(v) = |S_n| g^2(n).$$

As $|S_{n+1}| \leq (k-1)|S_n|$ one has

$$(16)\ a_{n+1} = |S_{n+1}|g^2(n+1) \leq (k-1)|S_n|g^2(n+1) = \left(1 + \frac{k-2}{(k-2)n+k}\right)^2 a_n \, .$$

We have to show that

$$(17) \qquad \frac{\sum_{v \in S_{n+1}} f^2(v)}{\sum_{v \in B_n} f^2(v)} = \frac{a_{n+1}}{a_1 + \cdots + a_n} \to_{n \to \infty} 0 \, .$$

It is a standard exercise to show that (16) implies (17). □

Let f_n be the sequence of functions which are restrictions of f to the vertices that are at a distance not greater than n :

$$f_n = f|_{B_n} \, .$$

By Lemma 6 and Lemma 7 it follows that

$$\limsup_{n \to +\infty} \frac{\|Pf_n\|_{l^2(X,N)}}{\|f_n\|_{l^2(X,N)}} \geq \frac{2\sqrt{k-1}}{k} \, ,$$

which proves Theorem 8. □

Some examples of upper bounds on the norm of the simple random walk operator on graphs and their comparison with the lower bound from Theorem 8 can be found in [22].

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