

## 4.4 Wreath products

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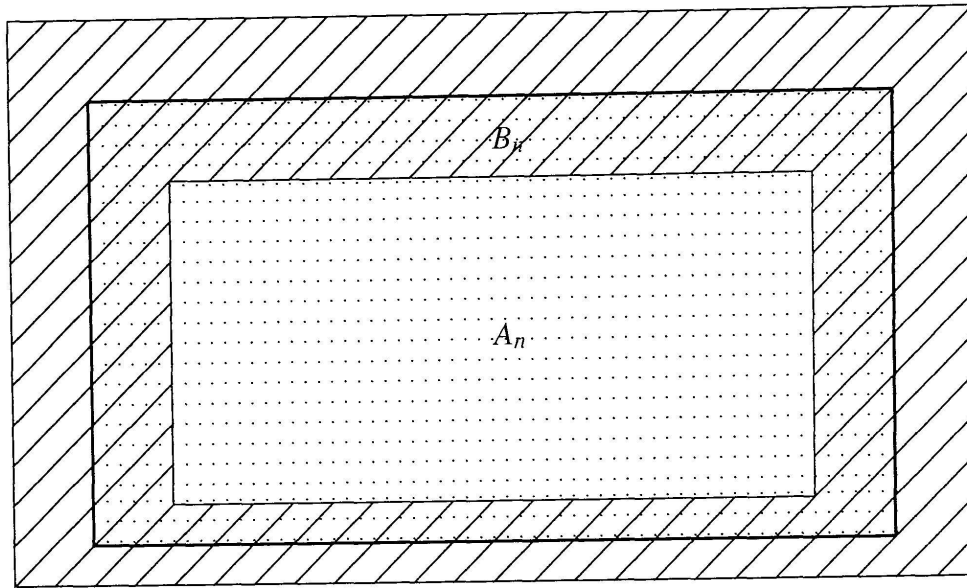


FIGURE 5  
Sets  $A_n$  and  $B_n$

$$\begin{aligned}
 B_n = & \{z \in H; -R \leq \operatorname{Re}(z) \leq R, e^R \geq \operatorname{Im}(z) \geq e^{-n-R}\} \\
 & \cup \{z \in H; -R+n \leq \operatorname{Re}(z) \leq n+R, e^R \geq \operatorname{Im}(z) \geq e^{-n-R}\} \\
 & \cup \{z \in H; -R \leq \operatorname{Re}(z) \leq n+R, e^R \geq \operatorname{Im}(z) \geq e^{-R}\} \\
 & \cup \{z \in H; -R \leq \operatorname{Re}(z) \leq n+R, e^{-n+R} \geq \operatorname{Im}(z) \geq e^{-n-R}\}.
 \end{aligned}$$

One can see that

$$|B_n|_{f^2} \approx n, \quad |A_n|_{f^2} \approx n^2.$$

This shows that  $\{A_n\}_{n=1}^\infty$  is a generalized Følner sequence. Thus

$$\|P\|_{L^2(H, d_{Hz}) \rightarrow L^2(H, d_{Hz})} = \int_{|z-i|=R} \sqrt{\operatorname{Im}(z)} \, dm_R(z).$$

#### 4.4 WREATH PRODUCTS

Let  $G$  and  $F$  be finitely generated groups. We define the wreath product  $G \wr F$  of these groups as follows. Elements of  $G \wr F$  are couples  $(g, \gamma_1)$  where  $g: F \rightarrow G$  is a function such that  $g(\gamma)$  is different from the identity element  $id_G$  of  $G$  only for finitely many elements  $\gamma$  in  $F$ , and where  $\gamma_1$  is an element of  $F$ . The multiplication in  $G \wr F$  is defined as follows:

$$(g_1, \gamma_1)(g_2, \gamma_2) = (g_3, \gamma_1\gamma_2)$$

where

$$g_3(\gamma) = g_1(\gamma)g_2(\gamma\gamma_1) \quad \text{for } \gamma \in F.$$

If  $S_G$  and  $S_F$  are generators of  $G$  and  $F$  respectively then

$$\{(g, \gamma) ; (g(F) = id_G, \gamma \in S_F) \text{ or } (g(F \setminus id_F) = id_G, g(id_F) \in S_G, \gamma = id_F)\}$$

is a generating subset for  $G \wr F$ .

Let  $\mu$  and  $\nu$  be symmetric, finitely supported probability measures on  $F$  and  $G$  respectively.

As there is a natural embedding of  $F$  and  $G$  into  $G \wr F$ , one can view the measures  $\mu$  and  $\nu$  as measures on  $G \wr F$ . More precisely:

$$\nu(g, \gamma) = \begin{cases} \nu(g(id_F)) & \text{if } \gamma = id_F \text{ and } g(F \setminus id_F) = id_G \\ 0 & \text{otherwise,} \end{cases}$$

$$\mu(g, \gamma) = \begin{cases} \mu(\gamma) & \text{if } g(F) = id_G \\ 0 & \text{otherwise.} \end{cases}$$

Then  $\mu * \nu * \mu$  is a symmetric measure on  $G \wr F$ . Explicitly we have:

$$\mu * \nu * \mu(g, \gamma) = \begin{cases} \mu(\gamma(\gamma_0)^{-1})\mu(\gamma_0)\nu(g(\gamma_0)) & \text{if } g(F \setminus \gamma_0) = id_G \\ 0 & \text{otherwise.} \end{cases}$$

We want to prove:

**THEOREM 7.** *Let  $F$  and  $G$  be finitely generated groups. If  $F$  is amenable then the spectral radius of  $\nu$  on  $G$  is the same as the spectral radius of  $\mu * \nu * \mu$  on  $G \wr F$ .*

*Proof.* We will prove Theorem 7 by constructing on  $G \wr F$  a positive function  $\tilde{f}$  which is an eigenfunction for the convolution by  $\mu * \nu * \mu$  with eigenvalue  $\|\nu\|_{l^2(G) \rightarrow l^2(G)}$  and for which there exists a generalized Følner sequence.

Let  $f$  be a positive eigenfunction for the operator which is a convolution on  $l^2(G)$  by  $\nu$ , corresponding to the eigenvalue  $\|\nu\|$ , i.e.

$$(12) \quad f * \nu = \|\nu\|f.$$

We can normalize  $f$  so that

$$(13) \quad f(id_G) = 1.$$

By Theorem 3 (and the remark after its proof) there exists a sequence of finite subsets  $A_n \subset G$ , such that

$$\frac{\sum_{\gamma \in \partial A_n} f^2(\gamma)}{\sum_{\gamma \in A_n} f^2(\gamma)} \rightarrow_{n \rightarrow \infty} 0.$$

As the group  $F$  is amenable there exists a sequence of finite subsets  $B_n \subset F$ , such that

$$\frac{\#\partial B_n}{\#B_n} \rightarrow_{n \rightarrow \infty} 0.$$

For technical reasons let us choose the sequences  $B_n$  and  $A_n$  in such a way that

$$(14) \quad \frac{\#\partial B_n}{\#B_n} < \frac{1}{n} \quad \text{and} \quad \frac{\sum_{\gamma \in \partial A_n} f^2(\gamma)}{\sum_{\gamma \in A_n} f^2(\gamma)} < \frac{1}{n(\#B_n)}.$$

Now, on  $G \wr F$  we define  $\tilde{f}$  as follows

$$\tilde{f}(g, \gamma_1) = \prod_{\gamma \in F} f(g(\gamma)).$$

The function  $\tilde{f}$  is well defined because by (13),  $f(g(\gamma))$  is different from 1 only for finitely many  $\gamma \in F$ . This function is of course positive and does not depend on  $\gamma_1$ . From (12) one has

$$\tilde{f} * \mu * \nu * \mu = \tilde{f} * \nu * \mu = \|\nu\| \tilde{f} * \mu = \|\nu\| \tilde{f}.$$

To complete the proof of Theorem 7 it is enough to construct a generalized Følner sequence  $C_n \subset G \wr F$  for  $\tilde{f}$ . We define  $C_n$  as follows:

$$C_n = \{(g, \gamma_1); \gamma_1 \in B_n, g(B_n) \subset A_n, g^{-1}(G \setminus id_G) \subset B_n\}.$$

LEMMA 5. *The sequence  $C_n \subset G \wr F$  is a generalized Følner sequence for  $\tilde{f}$ .*

*Proof.* Let us define sets  $D_n$  and  $\partial D_n$  as follows:

$$D_n = \{g: F \rightarrow G; g(B_n) \subset A_n, g^{-1}(G \setminus id_G) \subset B_n\},$$

$$\partial D_n = \{g: F \rightarrow G; \text{there exists } \gamma_0 \in B_n \text{ such that } g(\gamma_0) \in \partial A_n,$$

$$g(B_n \setminus \gamma_0) \subset A_n, g^{-1}(G \setminus id_G) \subset B_n\}.$$

Thus

$$C_n = D_n \times B_n,$$

$$\partial C_n = (\partial D_n \times B_n) \cup (D_n \times \partial B_n).$$

We have then

$$\begin{aligned} \sum_{(g,\gamma_1) \in C_n} (\tilde{f}(g, \gamma_1))^2 &= \sum_{(g,\gamma_1) \in C_n} \left( \prod_{\gamma \in F} f(g(\gamma)) \right)^2 \\ &= \sum_{(g,\gamma_1) \in D_n \times B_n} \left( \prod_{\gamma \in F} f(g(\gamma)) \right)^2 = \#B_n \sum_{g \in D_n} \left( \prod_{\gamma \in F} f(g(\gamma)) \right)^2. \end{aligned}$$

On the other hand

$$\begin{aligned} \sum_{(g,\gamma_1) \in \partial C_n} (\tilde{f}(g, \gamma_1))^2 &= \sum_{(g,\gamma_1) \in \partial C_n} \left( \prod_{\gamma \in F} f(g(\gamma)) \right)^2 \\ &= \sum_{(g,\gamma_1) \subset (\partial D_n \times B_n) \cup (D_n \times \partial B_n)} \left( \prod_{\gamma \in F} f(g(\gamma)) \right)^2 \\ &= \#\partial B_n \sum_{g \in D_n} \left( \prod_{\gamma \in F} f(g(\gamma)) \right)^2 + \#B_n \sum_{g \in \partial D_n} \left( \prod_{\gamma \in F} f(g(\gamma)) \right)^2 \\ &= \frac{\#\partial B_n}{\#B_n} \sum_{(g,\gamma_1) \in C_n} (\tilde{f}(g, \gamma_1))^2 \\ &\quad + \frac{\sum_{g \in \partial D_n} \left( \prod_{\gamma \in F} f(g(\gamma)) \right)^2}{\sum_{g \in D_n} \left( \prod_{\gamma \in F} f(g(\gamma)) \right)^2} \sum_{(g,\gamma_1) \in C_n} (\tilde{f}(g, \gamma_1))^2. \end{aligned}$$

But

$$\sum_{g \in \partial D_n} \left( \prod_{\gamma \in F} f(g(\gamma)) \right)^2 = \#B_n \frac{\sum_{\alpha \in \partial A_n} f^2(\alpha)}{\sum_{\alpha \in A_n} f^2(\alpha)} \sum_{g \in D_n} \left( \prod_{\gamma \in F} f(g(\gamma)) \right)^2.$$

Thus by (14)

$$\begin{aligned} \sum_{(g,\gamma_1) \in \partial C_n} (\tilde{f}(g, \gamma_1))^2 &= \left( \frac{\#\partial B_n}{\#B_n} + \#B_n \frac{\sum_{\alpha \in \partial A_n} f^2(\alpha)}{\sum_{\alpha \in A_n} f^2(\alpha)} \right) \sum_{(g,\gamma_1) \in C_n} (\tilde{f}(g, \gamma_1))^2 \\ &\leq \frac{2}{n} \sum_{(g,\gamma_1) \in C_n} (\tilde{f}(g, \gamma_1))^2, \end{aligned}$$

which shows that  $C_n$  is a generalized Følner sequence for  $\tilde{f}$ .  $\square$

This ends the proof of Theorem 7.