7. The Plancherel Formula

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **45 (1999)**

Heft 3-4: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **24.09.2024**

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7. The Plancherel Formula

Let L_l denote the differential operator

$$L_l = \frac{d^2}{dt^2} + [(4n-1)\coth t + 3\tanh t]\frac{d}{dt} + \frac{4l(l+1)}{\cosh^2 t}\,. \tag{7.56}$$

Proposition 4.3, Part 3, and Formula (4.39) prove that the restriction to $A \equiv \mathbf{R}$ of the τ_l-spherical function $\zeta_{l,s}$ is the unique solution to the differential equation

$$L_l u = (s^2 - \rho^2)u \tag{7.57}$$

satisfying $u(0) = 1$ and $u'(0) = 0$. If s is not an integer, two linearly independent solutions of (7.57) are also the functions $\Phi_{l,\pm s}$ defined by

$$\Phi_{l,\pm s}(t) = (2\cosh t)^{\pm s-\rho} F\Big(\frac{\rho \mp s}{2} + l, \frac{\rho \mp s}{2} - l - 1; 1 \mp s; 1 - \tanh^2 t\Big). \tag{7.58}$$

$\zeta_{l,s}$ is an even function of $s \in \mathbf{C}$. Therefore if $s \in \mathbf{C}\setminus\mathbf{Z}$ there exists a constant $c_l(s)$ so that

$$\zeta_{l,s} = c_l(s)\Phi_{l,s} + c_l(-s)\Phi_{l,-s}\,. \tag{7.59}$$

Formula 2.9(33) in [E$^+$] gives $c_l(s)$ explicitly as the meromorphic function

$$c_l(s) = 2^{\rho-s}\frac{\Gamma(2n)\Gamma(s)}{\Gamma\left(\frac{\rho+s}{2}+l\right)\Gamma\left(\frac{\rho+s}{2}-l-1\right)}\,. \tag{7.60}$$

The function c_l determines the Plancherel measure for the τ_l- spherical transform. We immediately give the information that will make this measure explicit.

On the domain $S = \{s \in \mathbf{C} : \Re s \geq 0\}$, the function $c_l(s)$ has a simple pole at $s = 0$ and, if $2l \geq 2n-1$, $c_l(s)$ also has simple zeros at the points of the set

$$D_l = \{s_j = 2(l-j+1)-\rho = 2(l-j-n)+1 : \ j = 0,1,\dots \text{ and } s_j > 0\}\,. \tag{7.61}$$

The singularity of c_l^{-1} at $-s_j$ can be removed by setting

$$\begin{aligned} c_l(-s_j) :&= 2^{\rho+s_j}\frac{\Gamma(2n)}{\Gamma\left(\frac{\rho-s_j}{2}+l\right)}\,\frac{\operatorname*{Res}_{s=s_j}\Gamma(-s)}{\operatorname*{Res}_{s=s_j}\Gamma\left(\frac{\rho-s}{2}-l-1\right)} \\ &= 2^{\rho+s_j}(-1)^j\frac{\Gamma(2n)\Gamma(2(l-n+1)-j)}{\Gamma(2n+j)\Gamma(2(l-n-j+1))}\,. \end{aligned} \tag{7.62}$$

Since

$$\operatorname*{Res}_{s=s_j}\left[\frac{1}{c_l(s)}\right] = 2^{-\rho-s_j}\,(-1)^j\,\frac{\Gamma(2l+1-j)}{\Gamma(2n)\Gamma(2(l-n-j)+1)\Gamma(j+1)}\,,$$

we obtain from (7.62)

(7.63)

$$\operatorname*{Res}_{s=s_j}\left[\frac{1}{c_l(s)c_l(-s)}\right] = \frac{2^{-2\rho}}{\Gamma(2n)^2}\,\frac{\Gamma(j+2n)}{\Gamma(j+1)}\,\frac{\Gamma(2l+1-j)}{\Gamma(2(l-n-j)+1)}[2(l-n-j)+1]\,.$$

Setting $\alpha = 2n-1$, we can rewrite these residues in terms of the shifted factorials

$$(a)_\alpha := a(a+1)\cdots(a+\alpha-1)$$

as

$$\operatorname*{Res}_{s=s_j}\left[\frac{1}{c_l(s)c_l(-s)}\right] = 2^{-2\rho}\,[2(l-n-j)+1]\,\frac{(j+1)_\alpha(2(l-n+1)-j)_\alpha}{(\alpha!)^2}\,. \tag{7.64}$$

Moreover, if $s \in \mathbf{R}$, then $c_l(-is) = \overline{c_l(is)}$ and

$$|c_l(is)|^{-2} = \frac{2^{-2\rho}}{\pi\Gamma(2n)^2}\,s\sinh(\pi s)\left|\Gamma\Big(\frac{\rho+is}{2}+l\Big)\right|^2\left|\Gamma\Big(\frac{\rho+is}{2}-l-1\Big)\right|^2. \tag{7.65}$$

We have let the parameter l range in the set $\mathbf{N}/2$. Nevertheless, the explicit formulas for the functions $\zeta_{l,s}$, $\Phi_{l,s}$ and c_l allow us to consider l as a complex variable. Indeed, Formulas (4.39), (7.58) and (7.60) show, respectively, that for every fixed $t \in \mathbf{R}$, $\zeta_{l,s}$ is holomorphic in $(l,s) \in \mathbf{C}^2$, for fixed $t > 0$, $\Phi_{l,s}$ is holomorphic in $(l,s) \in \mathbf{C}^2$, and that $c_l(s)$ is an entire function of $l \in \mathbf{C}$ and a meromorphic function of $s \in \mathbf{C}$.

Because of Remark 2.3 and Formula (1.6), for every $f \in \mathcal{D}(G;\chi_l)$ we have

$$\widehat{f}_l(s) = C_l \int_0^\infty f(a_t)\zeta_{l,s}(t)\Delta(t)\,dt \tag{7.66}$$

where C_l is given by (6.55) and $f(t) := f(a_t)$. Formula (7.66) can be employed to define $\widehat{f}_l$ for every $l \in \mathbf{C}$ and every $f \in \mathcal{D}_+(\mathbf{R})$. Morera's Theorem then implies that $\widehat{f}_l(s)$ is a holomorphic function of $(l,s) \in \mathbf{C}^2$.

Lemmas 2.1, 2.2 and 2.3 in [K1] prove:

1. For each $r > 0$ there is a constant $K_r > 0$ such that

$$|c_l(s)|^{-1} \le K_r(1+|s|)^{2n-\frac{1}{2}} \tag{7.67}$$

if $\Re s \ge 0$ and $c_l(s') \ne 0$ for $|s'-s| \le r$.

2. For each $\delta > 0$ there is a constant $K_\delta > 0$ such that

$$|\Phi_{l,-s}(t)| \leq K_\delta e^{-(\Re s+\rho)t} \tag{7.68}$$

if $\Re s \geq 0$ and $t \geq \delta > 0$.

3. There exists a constant $K > 0$ such that

$$|\zeta_{l,s}(t)| \leq K(1+t)e^{(|\Re s|-\rho)t}$$

for all $t \geq 0$ and all $s \in \mathbf{C}$.

L_l is a symmetric operator on the space $L^2(\Delta(t)\,dt)$ of functions on $(0,\infty)$ which are L^2-integrable with respect to the measure $\Delta(t)\,dt$. Greeen's Formula and Equation (7.57) satisfied by $\zeta_{l,s}$ give for every $l, s \in \mathbf{C}$ and $f \in \mathcal{D}_+(\mathbf{R})$

$$(L_l^n f)_l^\wedge(s) = (s^2-\rho^2)^n \widehat{f}_l(s)$$

where $L_l^n = \underbrace{L_l \circ \cdots \circ L_l}_{n \text{ times}}$. Note that, for $k \geq 2n$, $L_l^k f(t)\,\Delta(t)$ is continuous and compactly supported on $[0,\infty)$. If $\operatorname{supp} f \subset [-R,R]$, then (7.68) gives

$$|(s^2-\rho^2)^n \widehat{f}_l(s)| \leq \int_0^\infty |L_l^n f(t)| \zeta_{l,s}(t)\Delta(t)\,dt \leq Ce^{|\Re s|R} \tag{7.69}$$

for some constant $C > 0$. The above estimates allow us to conclude: for every $r > 0$ and $\delta > 0$ there exists a constant $K_{r,\delta} > 0$ so that

$$\left|\widehat{f}_l(\mu+i\nu)\frac{\Phi_{l,-\mu+i\nu}(t)}{c_l(\mu+i\nu)}\right| \leq K_{r,\delta}(1+|\mu+i\nu|)^{2n-\frac{1}{2}-2n}e^{(R-t)\mu-\rho t} \tag{7.70}$$

if $\mu \geq 0$, $c_l(s') \neq 0$ for $|s'-(\mu+i\nu)| \leq r$, and $t \geq \delta > 0$.

Let D_l be as in (7.61), and set

$$\mu_0 := \begin{cases} \max D_l = 2(l-n)+1 & \text{if } 2l \geq 2n-1 \\ 0 & \text{otherwise.} \end{cases}$$

Fix $\mu_1 > \mu_0$ and define for $t > 0$

$$(\widehat{f}_l)_l^\vee(t) := \frac{1}{2\pi C_l} \int_{-\infty}^{\infty} \widehat{f}_l(\mu_1+i\nu)\Phi_{l,-\mu_1-i\nu}(t)\,\frac{d\nu}{c_l(\mu_1+i\nu)}$$

where C_l is given by (6.55). Observe that the integrand is a meromorphic function of $\mu+i\nu \in \mathbf{C}$ with singularities given by those of c_l^{-1}. Because of (7.70), Cauchy's Theorem gives

$$(\widehat{f}_l)_l^\vee(t) = \frac{1}{2\pi C_l} \int_{-\infty}^{+\infty} \widehat{f}_l(\mu+i\nu)\Phi_{l,-\mu-i\nu}(t)\,\frac{d\nu}{c_l(\mu+i\nu)} \tag{7.71}$$

for every $\mu > \mu_0$. Letting $\mu \to +\infty$, we find that $(\widehat{f}_l)_l^\vee(t) = 0$ for $t > R$.

7.1. THEOREM (Inversion formula, first form). *For every* $l \in \mathbf{C}$, $f \in \mathcal{D}_+(\mathbf{R})$ *and* $t > 0$

$$f(t) = (\widehat{f}_l)_l^\vee(t)\,. \tag{7.72}$$

Proof. Since $(\widehat{f}_l)_l^\vee$ is an entire function of l, it is enough to establish (7.72) when $0 \le 2l < 2n-1$. In this case, (7.71) holds also for $\mu = 0$. We then have to prove that for all $t > 0$

$$f(t) = \frac{1}{2\pi C_l} \int_{-\infty}^{\infty} \widehat{f}_l(i\nu)\zeta_{l,i\nu}(t)\, \frac{d\nu}{|c_l(i\nu)|^2}\,. \tag{7.73}$$

The method of Gangolli-Helgason-Rosenberg applies to this purpose without essential modifications. We therefore only sketch the proof, and refer to [Ros] and to [GV], § 6.6, for the details.

Endow $\mathcal{D}(\mathbf{R})$ with the usual inductive limit topology, and consider $\mathcal{D}_+(\mathbf{R})$ with the induced topology. Then the assignment

$$T(f) = \frac{1}{2\pi C_l} \int_{-\infty}^{\infty} \widehat{f}_l(i\nu)\, \frac{d\nu}{|c(i\nu)|_l^2}\,, \qquad f \in \mathcal{D}_+(\mathbf{R})\,,$$

defines a distribution T on $\mathcal{D}_+(\mathbf{R})$. As in the case of K-bi-invariant functions, it is possible to show that T is indeed a measure, and hence there is a constant C so that

$$Tf = Cf(e) \qquad \text{for all } f \in \mathcal{D}_+(\mathbf{R})\,. \tag{7.74}$$

If $f \in \mathcal{D}_+(\mathbf{R})$, so does its generalized translate

$$T_\tau f(t) = \int_0^\infty f(u)K_l(t,\tau,u)\Delta(u)\,du\,,$$

where the kernel $K_l(t,\tau,u)$ is as in (4.42). $T_\tau f$ satisfies $T_\tau f(t) = T_t f(\tau)$, $T_0 f = f$ and, because of (4.41),

$$(T_\tau f)_l^\wedge(s) = \zeta_{l,s}(\tau)\widehat{f}_l(s), \qquad s, l \in \mathbf{C},\ \tau \in \mathbf{R}$$

Since $T_t f(0) = f(t)$, Formula (7.72) follows from (7.73) provided $C = 1$, which can be proven true in the same lines of [Ros], p. 147. □

7.2. THEOREM (Inversion formula, second form). *Let D_l be the set defined by Formula (7.61) if $2l \geq 2n-1$ and $D_l = \varnothing$ otherwise. Let C_l be the constant in (6.55). For every $f \in \mathcal{D}(G;\chi_l)$ and $g \in G$,*

$$(7.75)\quad f(g) = \frac{1}{C_l}\sum_{s_j\in D_l}\left[\operatorname*{Res}_{s=s_j}\frac{1}{c_l(s)c_l(-s)}\right]\widehat{f}_l(s_j)\zeta_{l,s_j}(g) + \frac{1}{2\pi C_l}\int_0^\infty \widehat{f}_l(is)\zeta_{l,is}(g)\,\frac{ds}{|c_l(is)|^2}\,.$$

Proof. Fix $\mu > \mu_0$, and let γ_R be the rectangular contour of vertices $\pm iR$ and $\mu \pm iR$. Integrating the function

$$(7.76)\qquad s \longmapsto \frac{1}{2\pi C_l}\frac{\widehat{f}_l(s)\Phi_{l,-s}(t)}{c_l(s)}$$

along γ_R and letting $R\to\infty$, we obtain from (7.70), (7.61) and Theorem 7.1

$$f(a_t) = \frac{1}{C_l}\sum_{s_j\in D_l}\left[\operatorname*{Res}_{s=s_j}\frac{\Phi_{l,-s}(t)}{c_l(s)}\right]\widehat{f}_l(s_j) + \frac{1}{2\pi C_l}\int_{-\infty}^{+\infty}\widehat{f}_l(is)\Phi_{l,-s}(t)\frac{ds}{c_l(is)}$$

for all $t>0$. Equation (7.59) therefore proves the claim for $g = a_t$, $t > 0$, and hence for all t by continuity and evenness of the functions on both sides of (7.75). Formula (7.75) thus holds for arbitrary $g\in G$ because of (2.14) and (3.24). □

Since $\overline{\zeta_{l,s}} = \zeta_{l,s}$, the usual trick of replacing f with $f^* * f$ in (7.75) evaluated at e gives the Plancherel formula.

7.3. THEOREM (Plancherel Theorem). *Let D_l and C_l be as in Theorem 7.1. Define a measure σ_l on $i\mathbf{R}_+\cup D_l$ by*

$$(7.77)\quad \int_{i\mathbf{R}_+\cup D_l} g(s)\,d\sigma_l(s) = \frac{1}{C_l}\sum_{s_j\in D_l}\left[\operatorname*{Res}_{s=s_j}\frac{1}{c_l(s)c_l(-s)}\right]g(s_j) + \frac{1}{2\pi C_l}\int_0^\infty g(is)\,\frac{ds}{|c_l(is)|^2}\,.$$

Let $L^2(G;\chi_l)$ denote the closure of $\mathcal{D}(G;\chi_l)$ in $L^2(G)$, and let $L^2(d\sigma_l)$ be the space of L^2-integrable functions on $i\mathbf{R}_+\cup D_l$ with respect to the measure $d\sigma_l$. Then the map $f\mapsto\widehat{f}_l$ extends to an isometric isomorphism of $L^2(G;\chi_l)$ onto $L^2(d\sigma_l)$:

$$(7.78)\qquad \int_G |f(g)|^2\,dg = \int_{i\mathbf{R}_+\cup D_l}|\widehat{f}_l(s)|^2\,d\sigma_l(s)\,.$$

The techniques employed to prove the inversion formula (that is, Koornwinder's analytic continuation argument and the change of contour of integration) are the same used in [Shi] for the case of Hermitian symmetric pairs. Our choice of this method of proof is motivated by the propaedeutic nature of this paper. In fact, the computations involved in the proofs presented above are very much in the spirit of those required for the decomposition of the canonical representations in [DP].

We just mention a few alternative methods. First of all, because of Formula (7.74) and Part 3 of Proposition 4.3, the spectral theorem for the τ_l-spherical transform can be deduced from the spectral theorem for the differential operator L_l (see (7.56)) on a suitable domain in $L^2(\Delta(t)\,dt)$ on which it is self-adjoint. The latter theorem can be classically determined as an application of the Weyl-Titchmarsh Theorem. A second method is obtained observing the relation, ensured by Formula (4.39), between the τ_l-spherical transform and the Jacobi transform. Theorems 2.3 and 2.4 of [K2] are then quickly translated to our situation. Finally, observe that Koornwinder's method with the Abel transform can also be applied directly here because of Formula (6.54).

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