

6. The τ -ABEL transform

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5.5. COROLLARY. Let $E(\tau_l)$ denote the projection of $\mathcal{H}_{l,s}$ onto the K -isotypic subspace of type τ_l . Then

$$(5.43) \quad \zeta_{l,s}(g) = \frac{1}{d_l} \operatorname{tr}[E(\tau_l)T_{l,s}(g)].$$

$(T_{l,s}, \mathcal{H}_{l,s})$ is infinitesimally equivalent to a unitary representation if and only if the corresponding irreducible subquotient of $(\mathcal{H}'_{l,s})_K$ is unitarizable. The following theorem is thus a consequence of Theorems 5.1 and 5.3 and of Proposition 5.4.

5.6. THEOREM. $\zeta_{l,s} = \zeta_{l,-s}$ is positive definite if and only if one of the following cases occurs:

1. $s = i\nu$, $\nu \in \mathbf{R}$.
2. If $2l \geq 2n - 1$: $\pm s = s_j := 2(l - n - j) + 1$ for integers $j \geq 0$ so that $s_j > 0$. (discrete series)
3. If $2l < 2n - 1$: $s \in (2l - \rho + 2, -2l + \rho - 2)$. (complementary series)

The situation for s real and nonnegative is represented in Figure 6.1.

6. THE τ_l -ABEL TRANSFORM

Proposition 3.2 proves that the τ_l -Abel transform is a $*$ -homomorphism of $\mathcal{D}(G; \chi_l)$ into the convolution algebra $\mathcal{D}_+(\mathbf{R})$ consisting of the even C^∞ functions on $\mathbf{R}$ with compact support. The main theorem of this section states that the τ_l -Abel transform is also a bijection of $\mathcal{D}(G; \chi_l)$ onto $\mathcal{D}_+(\mathbf{R})$, and gives a formula for its inverse.

Identify A with $\mathbf{R}$ under the map $t \mapsto a_t$. Restriction to A then identifies $\mathcal{D}(G; \chi_l)$ with $\mathcal{D}_+(\mathbf{R})$. Let $\mathcal{D}([1, \infty))$ denote the set of the compactly supported C^∞ functions on $[1, \infty)$ (right differentiability at 1 is considered). Define a map H by

$$(Hf)(\cosh t) := f(a_t) \equiv f(t)$$

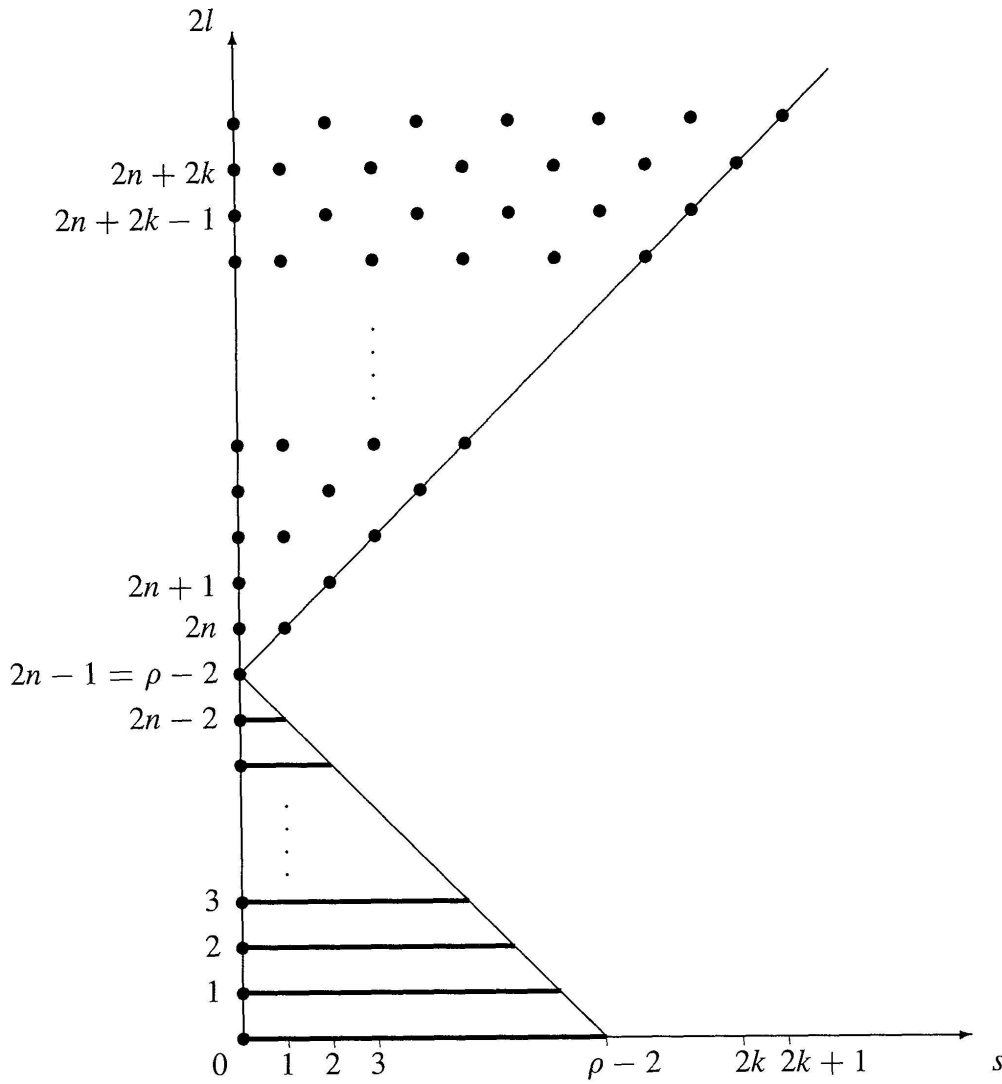


FIGURE 6.1

Positive definite $\zeta_{l,s}$ for real $s \geq 0$

for $f \in \mathcal{D}(G; \chi_l)$. Lemma 2 and its corollary in [Rou] imply

6.1. LEMMA. H is a bijection of $\mathcal{D}(G; \chi_l)$ onto $\mathcal{D}([1, \infty))$.

For every $\mu \in \mathbf{C}$ with $\Re \mu > 0$, the Weyl fractional integral transform of $\varphi \in \mathcal{D}([1, \infty))$ is defined by

$$(6.44) \quad \mathcal{W}_\mu \varphi(x) = \frac{1}{\Gamma(\mu)} \int_x^\infty \varphi(u)(u-x)^{\mu-1} du, \quad x \in [1, \infty).$$

Analytic continuation of $\mathcal{W}_\mu$ to $\Re \mu \leq 0$ is obtained via repeated integration by parts of (6.44): for every integer $m \geq 0$

$$\mathcal{W}_\mu \varphi(u) = \frac{(-1)^m}{\Gamma(\mu+m)} \int_u^\infty \frac{d^m \varphi}{dx^m}(x) (x-u)^{\mu+m-1} dx.$$

For every integer $m \geq 0$, the Gegenbauer transform (of dimension 4) of $\varphi \in \mathcal{D}([1, \infty))$ is defined by

$$(6.45) \quad \mathcal{G}_m \varphi(u) = \frac{4\pi}{m+1} \int_u^\infty \varphi(x) C_m^1\left(\frac{u}{x}\right) (x^2 - u^2)^{\frac{1}{2}} x \, dx, \quad u \in [1, \infty),$$

where

$$(6.46) \quad C_m^1(y) = (m+1)F\left(-m, m+2; \frac{3}{2}; \frac{1-y}{2}\right)$$

is the Gegenbauer polynomial of indices $(1, m)$ (cf. e.g. [E⁺], 3.15 (3)).

6.2. LEMMA ([K1], Theorem 3.2; [Dea], Formulas (28) and (29)).

1. For every $\mu \in \mathbf{C}$, $\mathcal{W}_\mu$ is a bijection of $\mathcal{D}([1, \infty))$ onto itself. The inverse mapping of $\mathcal{W}_\mu$ is $\mathcal{W}_{-\mu}$.

2. For every integer $m \geq 0$, $\mathcal{G}_m$ is a bijection of $\mathcal{D}([1, \infty))$ onto itself. The inverse mapping of $\mathcal{G}_m$ is given by

$$(6.47) \quad \mathcal{G}_m^{-1} \psi(x) = -\frac{1}{2\pi^2(m+1)} \frac{1}{x^2} \int_x^\infty \frac{d^3 \psi}{du^3}(u) C_m^1\left(\frac{u}{x}\right) (u^2 - x^2)^{\frac{1}{2}} du$$

for all $\psi \in \mathcal{D}([1, \infty))$ and all $x \in [1, \infty)$.

6.3. THEOREM. The τ_l -Abel transform is a bijection of $\mathcal{D}(G; \chi_l)$ onto $\mathcal{D}_+(\mathbf{R})$. It can be written as the composition

$$\mathcal{A}_l = \frac{(2\pi)^{2(n-1)}}{d_l^2} H^{-1} \circ \mathcal{W}_{2n-2} \circ \mathcal{G}_{2l} \circ H,$$

and its inverse is given by

$$\mathcal{A}_l^{-1} = \frac{d_l^2}{(2\pi)^{2(n-1)}} H^{-1} \circ \mathcal{G}_{2l}^{-1} \circ \mathcal{W}_{2-2n} \circ H.$$

Moreover, the support of the restriction to $A \equiv \mathbf{R}$ of $f \in \mathcal{D}(G; \chi_l)$ is contained in $[-R, R]$ if and only if the support of $\mathcal{A}_l f$ is contained in $[-R, R]$.

Proof. Identify the set of pure quaternions $w = i b + j c + k d \in \mathbf{H}$ with $\mathbf{R}^3$, and $\mathbf{H}^{n-1}$ with $\mathbf{R}^{4(n-1)}$. If $z \in \mathbf{H}^{n-1}$, then $-[z, z] \equiv |z|^2$ is the square of the Euclidean norm of z in $\mathbf{R}^{4(n-1)}$. For $a_t \in A$ and $n = n(w, z) \in N$ we have

$$(a_t n)_{00} = \cosh t + e^t (w + \tfrac{1}{2} |z|^2).$$

Let $f \in \mathcal{D}(G; \chi_l)$. Applying Lemma 3.3 and Formulas (1.5), we obtain

$$\begin{aligned}
\mathcal{A}f(t) &= \frac{1}{d_l^2} e^{\rho t} \int_N f(a_t n) \, dn \\
&= \frac{1}{d_l^3} e^{\rho t} \int_N \chi_l \left(\frac{(a_t n)_{00}}{|(a_t n)_{00}|} \right) Hf(|(a_t n)_{00}|) \, dn \\
&= \frac{1}{d_l^3} e^{\rho t} \int_{\mathbf{R}^{4(n-1)}} \int_{\mathbf{R}^3} \chi_l \left(\frac{\cosh t + e^t(w + \frac{1}{2}|z|^2)}{|\cosh t + e^t(w + \frac{1}{2}|z|^2)|} \right) \\
&\quad \times Hf(|\cosh t + e^t(w + \frac{1}{2}|z|^2)|) \, dz \, dw \\
&= \frac{1}{d_l^3} e^{\rho t} \int_{\mathbf{R}^{4(n-1)}} \int_{\mathbf{R}^3} C_{2l}^1 \left(\frac{\cosh t + \frac{1}{2}e^t|z|^2}{[(\cosh t + \frac{1}{2}e^t|z|^2)^2 + e^{2t}|w|^2]^{\frac{1}{2}}} \right) \\
&\quad \times Hf([(\cosh t + \frac{1}{2}e^t|z|^2)^2 + e^{2t}|w|^2]^{\frac{1}{2}}) \, dz \, dw \\
&\quad \text{(by Formula (4.40))}
\end{aligned}$$

$$\begin{aligned}
&= \frac{4^{n-1}}{d_l^3} \int_{\mathbf{R}^{4(n-1)}} \int_{\mathbf{R}^3} C_{2l}^1 \left(\frac{\cosh t + |X|^2}{[(\cosh t + |X|^2)^2 + |Y|^2]^{\frac{1}{2}}} \right) \\
&\quad \times Hf([(\cosh t + |X|^2)^2 + |Y|^2]^{\frac{1}{2}}) \, dX \, dY \\
&\quad \text{(by substituting } X = \frac{1}{\sqrt{2}}e^{\frac{t}{2}}z, Y = e^t w)
\end{aligned}$$

$$\begin{aligned}
&= \frac{2^\rho}{d_l^3} \frac{\pi^{2n-1}}{\Gamma(2n-2)} \int_0^\infty \int_0^\infty C_{2l}^1 \left(\frac{\cosh t + r^2}{[(\cosh t + r^2)^2 + s^2]^{\frac{1}{2}}} \right) \\
&\quad \times Hf([(\cosh t + r^2)^2 + s^2]^{\frac{1}{2}}) r^{4n-5} s^2 \, ds \, dr
\end{aligned}$$

(by passing to spherical coordinates in $\mathbf{R}^{4(n-1)}$ and in $\mathbf{R}^3$)

$$= \frac{2^{2n}}{d_l^3} \frac{\pi^{2n-1}}{\Gamma(2n-2)} \int_{\cosh t}^{\infty} \left[\int_0^{\infty} C_{2l}^1 \left(\frac{u}{[u^2 + s^2]^{\frac{1}{2}}} \right) Hf([u^2 + s^2]^{\frac{1}{2}}) s^2 ds \right] \\ \times (u - \cosh t)^{2n-3} du$$

(by setting $u = \cosh t + r^2$)

(6.48)

$$= \frac{2^{2n} \pi^{2n-1}}{d_l^3 \Gamma(2n-2)} \int_{\cosh t}^{\infty} \left[\int_u^{\infty} C_{2l}^1 \left(\frac{u}{x} \right) Hf(x) (x^2 - u^2)^{\frac{1}{2}} x dx \right] (u - \cosh t)^{2n-3} du$$

(by setting $x = [u^2 + s^2]^{\frac{1}{2}}$)

$$= \frac{(2\pi)^{2(n-1)}}{d_l^2 \Gamma(2n-1)} \int_{\cosh t}^{\infty} (\mathcal{G}_{2l} Hf)(u) (u - \cosh t)^{2n-3} du$$

$$= \frac{(2\pi)^{2(n-1)}}{d_l^2} \mathcal{W}_{2n-2} \mathcal{G}_{2l} Hf(\cosh t)$$

$$= \frac{(2\pi)^{2(n-1)}}{d_l^2} (H^{-1} \mathcal{W}_{2n-2} \mathcal{G}_{2l} Hf)(t),$$

i.e.

$$\mathcal{A}_l = \frac{(2\pi)^{2(n-1)}}{d_l^2} (H^{-1} \circ \mathcal{W}_{2n-2} \circ \mathcal{G}_{2l} \circ H).$$

The inversion formula immediately follows from Lemma 6.2.

The restriction to $A \equiv \mathbf{R}$ of $f \in \mathcal{D}(G; \chi_l)$ has its support $\text{supp } f$ contained in $[-R, R]$ if and only if $\text{supp } Hf \subset [1, \cosh R]$. Moreover, if $\text{supp } \varphi \subset [1, \cosh R]$, then $\text{supp } \mathcal{W}_\mu \varphi$, $\text{supp } \mathcal{G}_m \varphi$ and $\text{supp } \mathcal{G}_m^{-1} \varphi$ are also contained in $[1, \cosh R]$. The last statement then follows from the formulas for $\mathcal{A}_l$ and $\mathcal{A}_l^{-1}$. $\square$

The τ_l -spherical transform of $f \in \mathcal{D}(G; \chi_l)$ is the function $\widehat{f}_l$ on $\mathbf{C}$ defined by

$$\widehat{f}_l(s) = \int_G f(g) \zeta_{l,s}(g) dg, \quad s \in \mathbf{C}.$$

Let $S_l: f \mapsto \widehat{f}_l$ denote the τ_l -spherical transform, and let $\mathcal{F}$ denote the Fourier-Laplace transform on $\mathbf{R}$. Formulas (3.20) and (3.22) yield

$$(6.49) \quad S_l = \mathcal{F} \circ \mathcal{A}_l.$$

Let $\mathcal{H}_+^R(\mathbf{R})$ denote the set of even functions h on $\mathbf{C}$ which are entire rapidly decreasing functions of exponential type R : for every integer $N \geq 0$ there is a constant $C_N > 0$ so that

$$|h(s)| \leq C_N (1 + |s|)^{-N} e^{R|\Re s|} \quad \text{for all } s \in \mathbf{C}.$$

Set $\mathcal{H}_+(\mathbf{R}) := \bigcup_{R>0} \mathcal{H}_+^R(\mathbf{R})$. Theorem 6.3 and the Paley-Wiener Theorem for the Fourier-Laplace transform of even functions on $\mathbf{R}$ prove the following theorem.

6.4. THEOREM (Paley-Wiener Theorem). *The τ_l -spherical transform is a bijection of $\mathcal{D}(G; \chi_l)$ onto $\mathcal{H}_+(\mathbf{R})$. Moreover, the restriction of $f \in \mathcal{D}(G; \chi_l)$ to $A \equiv \mathbf{R}$ has support in $[-R, R]$ if and only if $\widehat{f}_l \in \mathcal{H}_+^R(\mathbf{R})$.*

We conclude this section by observing that the τ_l -Abel transform is related, as one should expect, to the Abel transform of [K2], §5.

Reversing the order of integration and substituting $x = \cosh \tau$ and $u = \cosh w$, we obtain from (6.48)

$$(6.50) \quad \mathcal{A}_l f(t) = \int_t^\infty A_l(t, \tau) f(\tau) d\tau$$

where

$$\begin{aligned} A_l(t, \tau) := & \frac{(2\pi)^{2n-1}}{d_l^3 \Gamma(2n-2)} \sinh(2\tau) \int_t^\tau C_{2l}^1 \left(\frac{\cosh w}{\cosh \tau} \right) (\cosh^2 \tau - \cosh^2 w)^{\frac{1}{2}} \\ & \times (\cosh w - \cosh t)^{2n-3} \sinh w dw. \end{aligned}$$

Substituting also $y = \frac{\cosh \tau - \cosh w}{\cosh \tau - \cosh t}$ and setting

$$\gamma(t, \tau) = \frac{\cosh \tau - \cosh t}{2 \cosh \tau} \quad \text{and} \quad K_l = \frac{(2\pi)^{2n-1}}{d_l^3 \Gamma(2n-2)},$$

we get from Formula (6.46)

$$\begin{aligned}
 A_l(t, \tau) &= \sqrt{2} K_l \sinh(2\tau) (\cosh \tau - \cosh t)^{2n-\frac{3}{2}} (\cosh \tau)^{\frac{1}{2}} \\
 &\quad \times \int_0^1 C_{2l}^1 (1 - 2\gamma(t, \tau)y) y^{\frac{1}{2}} (1-y)^{2n-3} (1 - \gamma(t, \tau)y)^{\frac{1}{2}} dy \\
 &= \sqrt{2} (2l+1) K_l \sinh(2\tau) (\cosh \tau - \cosh t)^{2n-\frac{3}{2}} (\cosh \tau)^{\frac{1}{2}} \\
 &\quad \times \int_0^1 F\left(\frac{3}{2} + 2l, -2l - \frac{1}{2}; \frac{3}{2}; \gamma(t, \tau)y\right) y^{\frac{1}{2}} (1-y)^{2n-3} (1 - \gamma(t, \tau)y)^{\frac{1}{2}} dy.
 \end{aligned}$$

If we now apply the relation ([E⁺], 2.9(2))

$$(6.51) \quad F(a, b; c; z) = (1-z)^{c-a-b} F(c-a, c-b; c; z)$$

and Bateman's Formula ([E⁺], 2.4(2))

$$\begin{aligned}
 (6.52) \quad F(a, b; c; z) &= \frac{\Gamma(c)}{\Gamma(s)\Gamma(c-s)} \int_0^1 x^{s-1} (1-x)^{c-s-1} F(a, b; s; xz) dx \\
 &\quad \text{for } \Re c > \Re s > 0, z \neq 1, |\arg(1-z)| < \pi,
 \end{aligned}$$

we finally obtain

$$\begin{aligned}
 (6.53) \quad A_l(t, \tau) &= \frac{(2\pi)^{2n-\frac{1}{2}}}{2\Gamma(2n-\frac{1}{2})} \frac{1}{d_l^2} \sinh(2\tau) (\cosh \tau - \cosh t)^{2n-\frac{3}{2}} (\cosh \tau)^{\frac{1}{2}} \\
 &\quad \times F\left(\frac{3}{2} + 2l, -2l - \frac{1}{2}, 2n - \frac{1}{2}, \gamma(t, \tau)\right).
 \end{aligned}$$

The comparison of Formula (6.53) with the kernel $A_{2n-1, 2l+1}(t, \tau)$ in [K2], Formula (5.60), gives

$$\begin{aligned}
 (6.54) \quad A_l(t, \tau) &= \frac{1}{2} \frac{1}{d_l^2} \left(\frac{\pi}{4}\right)^{2n} \frac{1}{\Gamma(2n)} 2^{-4l} (\cosh \tau)^{-2l} A_{2n-1, 2l+1}(t, \tau) \\
 &= 2^{-4l} C_l (\cosh \tau)^{-2l} A_{2n-1, 2l+1}(t, \tau)
 \end{aligned}$$

where we have set

$$(6.55) \quad C_l := \frac{1}{2} \frac{1}{(2l+1)^2} \left(\frac{\pi}{4}\right)^{2n} \frac{1}{\Gamma(2n)}.$$