4. From birational groups to algebraic groups

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Note that Φ is a regular morphism defined on U and Θ is a regular morphism defined on V. Since

$$\Phi\,\Theta(D_1, D_2) = (D_1, D_2) \quad \text{and} \quad \Theta\,\Phi(D_1, D_2) = (D_1, D_2)$$

whenever the left-hand sides are defined, the maps Φ and Θ induce regular morphisms $\Phi\colon U \cap \Phi^{-1}(V) \to V \cap \Theta^{-1}(U)$ and $\Theta\colon V \cap \Theta^{-1}(U) \to U \cap \Phi^{-1}(V)$. To show that Φ and Θ are birational inverses to each other, it is enough to check that $U \cap \Phi^{-1}(V)$ and $V \cap \Theta^{-1}(U)$ are non-empty.

Note that $(D_1, D_2) \in U \cap \Phi^{-1}(V)$ if and only if $(D_1, D_2) \in U$ and

$$l_{\mathfrak{m}}(m(D_1, D_2) - D_1 + \pi P_0) = 1, \quad l(m(D_1, D_2) - D_1 + \pi P_0 - \mathfrak{m}) = 0\,.$$

Since $m(D_1, D_2) \sim_{\mathfrak{m}} D_1 + D_2 - \pi P_0$, the above equations are equivalent to

$$l_{\mathfrak{m}}(D_2) = 1, \quad l(D_2 - \mathfrak{m}) = 0\,.$$

Applying Lemma 3.3 to the divisor $D_0 = 0$, we conclude that the set

$$V_0 = \{D \in (X - S)^{(\pi)} \mid l_{\mathfrak{m}}(D) = 0, \quad l(D - \mathfrak{m}) = 0\}$$

is open and non-empty. Since $(X - S)^{(\pi)} \times (X - S)^{(\pi)}$ is irreducible, the set $U \cap ((X-S)^{(\pi)} \times V_0)$ is also open and non-empty. This set is exactly $U \cap \Phi^{-1}(V)$. So $U \cap \Phi^{-1}(V)$ is non-empty.

Similarly $V \cap \Theta^{-1}(U)$ is also non-empty. This completes the proof of the proposition.

4. From birational groups to algebraic groups

Let k be an algebraically closed field, let V be a connected nonsingular variety over k, and let $m\colon V \times V \to V$, $(a, b) \mapsto ab$ be a rational map satisfying $(ab)c = a(bc)$. Assume the rational maps $\Phi(a, b) = (a, ab)$ and $\Psi(a, b) = (b, ab)$ are birational. Then there exist open subsets X_Φ, Y_Φ, X_Ψ and Y_Ψ in $V \times V$ such that Φ induces an isomorphism $X_\Phi \cong Y_\Phi$ and Ψ induces an isomorphism $X_\Psi \cong Y_\Psi$. Put $Z = X_\Phi \cap Y_\Phi \cap X_\Psi \cap Y_\Psi$.

It is convenient to write the formulae for Φ^{-1} and Ψ^{-1} as $\Phi^{-1}(a, b) = (a, a^{-1}b)$ and $\Psi^{-1}(a, b) = (ba^{-1}, a)$.

LEMMA 4.1. *Replacing V by an open subset, we may assume the two projections $p_i\colon Z \to V$ $(i = 1, 2)$ are surjective.*

Proof. Note that the two projections $p_i\colon V \times V \to V$, $(i = 1, 2)$ are flat since $V \to \operatorname{spec}(k)$ is flat. So the p_i are open by [EGA] IV, §2.4.6. Hence the $p_i(Z)$ are open. Let $V' = p_1(Z) \cap p_2(Z)$. We claim V' has the property stated in the lemma. Let $C = V - V'$ and let $A = (C \times V) \cup (V \times C)$. The subset $X_\Phi{}'$ of $V' \times V'$ corresponding to X_Φ is the complement in X_Φ of $S = (X_\Phi \cap A) \cup \Phi^{-1}(Y_\Phi \cap A)$. We claim that if the fiber of $p_1 : X_\Phi \to V$ at $v \in V$ is contained in S, then $v \in C$. Thus $p_1 : X_\Phi{}' \to V'$ is surjective.

Let us prove the claim. Assume $(v \times V) \cap X_\Phi \subset S$, but $v \notin C$. We have

$$(v \times V) \cap X_\Phi \subset S \subset A \cup \Phi^{-1}(A) \subset (C \times V) \cup (V \times C) \cup \Phi^{-1}(C \times V) \cup \Phi^{-1}(V \times C).$$

Since V is irreducible, we must have

$$(v \times V) \cap X_\Phi \subset C \times V, \quad V \times C, \quad \Phi^{-1}(C \times V), \text{ or } \Phi^{-1}(V \times C).$$

Since $v \notin C$, we have

$$(v \times V) \cap X_\Phi \not\subset C \times V, \quad \Phi^{-1}(C \times V).$$

So

$$(v \times V) \cap X_\Phi \subset V \times C \text{ or } \Phi^{-1}(V \times C).$$

Assume $(v \times V) \cap X_\Phi \subset V \times C$. Note that since $v \notin C$, we have $v \in V'$. Hence $(v \times V) \cap X_\Phi$ is not empty. So we have

$$\begin{aligned}\dim V = \dim((v \times V) \cap X_\Phi) &= \dim(((v \times V) \cap X_\Phi) \cap (V \times C)) \\ &\le \dim(v \times C) < \dim V,\end{aligned}$$

that is, $\dim V < \dim V$. This is impossible.

Assume $(v \times V) \cap X_\Phi \subset \Phi^{-1}(V \times C)$. Then $\Phi((v \times V) \cap X_\Phi) \subset V \times C$. Since Φ is birational, we have

$$\begin{aligned}\dim V = \dim \Phi((v \times V) \cap X_\Phi) &= \dim(\Phi((v \times V) \cap X_\Phi) \cap (V \times C)) \\ &\le \dim(v \times C) < \dim V,\end{aligned}$$

which is again impossible. So we must have $v \in C$.

Next we show that if the fiber of $p_2 : X_\Phi \to V$ at $v \in V$ is contained in S, then $v \in C$, and hence $p_2 : X_\Phi{}' \to V'$ is surjective.

Assume $(V \times v) \cap X_\Phi \subset S$ but $v \notin C$. As before we have

$$(V \times v) \cap X_\Phi \subset C \times V, \quad V \times C, \quad \Phi^{-1}(C \times V) \text{ or } \Phi^{-1}(V \times C).$$

Since $v \notin C$, we have $(V \times v) \cap X_\Phi \not\subset V \times C$. By counting dimensions, one can show $(V \times v) \cap X_\Phi \not\subset C \times V$. Since $\Phi^{-1}(C \times V) \subset C \times V$, we have $(V \times v) \cap X_\Phi \not\subset \Phi^{-1}(C \times V)$. So we can only have $(V \times v) \cap X_\Phi \subset \Phi^{-1}(V \times C)$. Then we have a rational map

$$V \xrightarrow{\iota_1} (V \times v) \cap X_\Phi \xrightarrow{\Phi} V \times C \xrightarrow{p_2} C,$$

where $\iota_1(x) = (x, v)$. This map $p_2 \Phi \iota_1 : V \to C$ is nothing but $x \mapsto xv$ and it is birational. (Its birational inverse is $p_1 \Psi^{-1} \iota_2$, where $\iota_2(x) = (v, x)$.) So V is birational to C. This is impossible since $\dim V \neq \dim C$. So we must have $v \in C$. This finishes the proof of the surjectivity of $p_2 \colon X_\Phi{}' \to V'$.

Similarly $p_i \colon X'_\Phi, Y_\Phi{}', X'_\Psi, Y'_\Psi \to V'$ are surjective. Since the fibers of $p_i \colon V \times V \to V$ are irreducible, the projection $p_i \colon Z' = X'_\Phi \cap Y'_\Phi \cap X'_\Psi \cap Y'_\Psi \to V'$ is also surjective.

Having replaced V as in Lemma 4.1, we may assume V satisfies the following properties:

PROPERTY 4.2. *There exists an open set $Z \subset V \times V$ such that Φ, Φ^{-1}, Ψ, and Ψ^{-1} are defined on Z, the restrictions $\Phi|_Z$ and $\Psi|_Z$ are open immersions, and the projections $p_i \colon Z \to V$ are surjective. Hence for every $v \in V$, the maps Φ, Φ^{-1}, Ψ and Ψ^{-1} are defined at (v, x) and at (x, v), provided x is generic, i.e. lies in an open set.*

LEMMA 4.3. *Assume 4.2 holds. Denote the closure of the graph of m in $V \times V \times V$ by Γ. Then the projections $p_{ij} \colon \Gamma \to V \times V$ $(1 \leq i < j \leq 3)$ are open immersions.*

Proof. By [EGA] III, §4.4.9, it suffices to show that the maps p_{ij} are set-theoretically injective. Let x be a point of V. The two rational maps $\Gamma \to V$ defined by

$$(a, b, c) \mapsto (xa)b \quad \text{and} \quad (a, b, c) \mapsto xc$$

are equal by the associative law. Let (a, b, c), $(a, b, c') \in \Gamma$. Choose x so that $(xa)b$ is defined and (x, c), $(x, c') \in Z$. Then $xc = (xa)b = xc'$. Hence $\Phi(x, c) = \Phi(x, c')$. Since Φ is an open immersion on Z, we have $(x, c) = (x, c')$. Hence $c = c'$. This shows that $p_{12} \colon \Gamma \to V \times V$ is injective. Similarly one can show the other projections are injective.

We will now expand V to the group we want by glueing translates of V. Let s be a point of V and let V_s be a copy of V thought of as the translate $V_s = \{vs \mid v \in V\}$. The subset $W_s = (V \times s \times V) \cap \Gamma$ is closed in $V \times s \times V \cong V \times V$, and the two projections $W_s \to V$ are open immersions because they are the base extensions of the open immersions $p_{ij}\colon \Gamma \to V \times V$ by the base changes $V \times s \to V \times V$ and $s \times V \to V \times V$, respectively. Therefore W_s defines glueing data and yields a separated scheme $V' = V \cup_{W_s} V_s$.

LEMMA 4.4. *V is an open dense subset of V' and V' satisfies 4.2.*

Proof. Since xs is defined for generic $x \in V$, the set $V \cap V_s$ is not empty. So V' is irreducible and V is dense in V'. We have

$$V' \times V' = (V \times V) \cup (V \times V_s) \cup (V_s \times V) \cup (V_s \times V_s).$$

For every point $v \in V$, denote by v_s the point v considered as a point in V_s. Note that if $(v, s) \in Z$, then $vs \in V$ and $v_s \in V_s$ are glued together in V'. Define $R_s\colon V \to V_s$ by $v \mapsto v_s$. Let

$$W_1 = \{(a, b) \in V \times V \mid (a, b),\quad (s, a) \text{ and } (b, sa^{-1}) \text{ are all in } Z\}.$$

This is a non-empty open subset of Z. Take $Z_1 = (\mathrm{id} \times R_s)(W_1) \subset V \times V_s$. We define Φ, Ψ, Φ^{-1} and Ψ^{-1} on Z_1 by

$$\begin{aligned}
\Phi(a, b_s) &= (a, (ab)_s) \in V \times V_s,\\
\Psi(a, b_s) &= (b_s, (ab)_s) \in V_s \times V_s,\\
\Phi^{-1}(a, b_s) &= (a, (a^{-1}b)_s) \in V \times V_s,\\
\Psi^{-1}(a, b_s) &= (b(sa^{-1}), a) \in V \times V
\end{aligned}$$

for any $(a, b_s) \in Z_1$. Let

$$\begin{aligned}
W_2 = \{(a, b) \in V \times V&\\
&\mid (a, b),\ (s, b),\ (a, sb),\ (s, a^{-1}b) \text{ and } (bs^{-1}, a) \text{ are all in } Z\}.
\end{aligned}$$

This is a non-empty open subset of Z. Take $Z_2 = (R_s \times \mathrm{id})(W_2) \subset V_s \times V$. We define Φ, Ψ, Φ^{-1}, and Ψ^{-1} on Z_2 by

$$\begin{aligned}
\Phi(a_s, b) &= (a_s, a(sb)) \in V_s \times V,\\
\Psi(a_s, b) &= (b, a(sb)) \in V \times V,\\
\Phi^{-1}(a_s, b) &= (a_s, s^{-1}(a^{-1}b)) \in V_s \times V,\\
\Psi^{-1}(a_s, b) &= ((bs^{-1})a^{-1}, a_s) \in V \times V_s
\end{aligned}$$

for any $(a_s, b) \in Z_2$.

Let $Z' = Z \cup Z_1 \cup Z_2$. It is an open subset of $V' \times V'$, and Φ, Ψ, Φ^{-1}, Ψ^{-1} are defined on it. One can show that $\Phi|_{Z'}$ and $\Psi|_{Z'}$ are open immersions. Given $v \in V'$, we need to show there exists $x \in V'$ such that (x, v) and (v, x) are in Z'. This is true if $v \in V$ by the property of Z. If $v \in V_s$, then $v = a_s$ for some $a \in V$. We leave it to the reader to show that $(x, a_s) \in Z_1$ and $(a_s, x) \in Z_2$ for generic x in V. This completes the proof of the lemma.

The above lemma allows us to replace V by V', hence to expand V whenever there exists a point s in V such that vs is not defined for all $v \in V$, and we can expand V' if there exists a point $s' \in V'$ such that $v's'$ is not defined for all $v' \in V'$. Denote the result of finitely many such expansions also by V', and let $U \subset V \times V \times V'$ be the closure of Γ. By Lemma 4.3 applied to V', the projection $p_{12}\colon U \to V \times V$ is an open immersion. Its image is the set of points (a, b) such that $m\colon V \times V \to V'$ is defined at (a, b). If $V \times s \not\subset p_{12}(U)$ for some point s in V, then replacing V' by $V' \cup V_s'$ increases both V' and $p_{12}(U)$. Using noetherian induction on open subschemes of $V \times V$, we may assume that after finitely many expansions, $V \times s \subset p_{12}(U)$ for all points $s \in V$. Then we have $p_{12}(U) = V \times V$.

PROPOSITION 4.5. *Let V, V', and U be as above. If $p_{12}(U) = V \times V$, then the operation $m\colon V' \times V' \to V'$ is everywhere defined on V' and makes V' an algebraic group.*

Proof. Take (a', b') in $V' \times V'$. Choose a point x so that $a'x$ and $x^{-1}b'$ are both defined and lie in V. Then we can define $m(a', b') = (a'x)(x^{-1}b')$. Similarly one can define $a'^{-1}b'$ and $b'a'^{-1}$. In this way we extend m, Φ, Ψ, Φ^{-1} and Ψ^{-1} to $V' \times V'$. The verification of the group axioms is routine and is omitted.

5. FUNDAMENTAL PROPERTIES OF GENERALIZED JACOBIANS

Keep the notations in §3. We have proved that there is a birational group structure on $(X - S)^{(\pi)}$. The algebraic group associated to this birational group is called the *generalized jacobian* of $X_{\mathfrak{m}}$ and is denoted by $J_{\mathfrak{m}}$. It is a commutative algebraic group.

Let D_0 be a divisor on X prime to S of degree 0. By Lemma 3.3, the set

$$V_{D_0} = \{D \in (X - S)^{(\pi)} \mid l_{\mathfrak{m}}(D + D_0) = 1, \quad l(D + D_0 - \mathfrak{m}) = 0\}$$

is a non-empty open subset of $(X - S)^{(\pi)}$. We have the following