

6.4 Other generalizations

Objektyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **44 (1998)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **24.09.2024**

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern. Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden. Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

frequencies of binary codings: the frequencies of the factors of given length of a coding of an irrational rotation with respect to a partition in two intervals take ultimately at most 5 values.

6.3 THE $3d$ DISTANCE THEOREM

Let us consider another generalization of the three distance theorem, known as the *3d distance theorem*. This result, conjectured by Graham (see [17] and [34]), was first proved by Chung and Graham in [18] and secondly by Liang who gave a very nice proof in [37]. Geelen and Simpson remark in [29] that their proof uses ideas from Liang's proof.

THE $3d$ DISTANCE THEOREM. *Assume we are given $0 < \alpha < 1$ irrational, $\gamma_1, \dots, \gamma_d$ real numbers and n_1, \dots, n_d positive integers. The points $\{n\alpha + \gamma_i\}$, for $0 \leq n < n_i$ and $1 \leq i \leq d$, partition the unit circle into at most $n_1 + \dots + n_d$ intervals, having at most $3d$ different lengths.*

We will give a combinatorial proof of this result in Section 8 and express the corresponding result for frequencies of codings of rotations, i.e., that the frequencies of the factors of given length of a coding of a rotation by the unit circle under a partition in d intervals take ultimately at most $3d$ values.

6.4 OTHER GENERALIZATIONS

Slater has studied in [50] the following generalization of the three gap theorem, which should be compared with Theorem 13: there is a bounded number of gaps between the successive values of the integers n such that $\{n(\eta_1, \dots, \eta_d)\} \in C$, where C is a closed convex region on the d -dimensional torus and where $1, \eta_1, \dots, \eta_d$ are rationally independent. However, Fraenkel and Holzman prove Theorem 13 even in the case where α_1, α_2 and 1 are rationally independent.

Chevallier studies in [16] a d -generalization of the three distance theorem to \mathbf{T}^d , where intervals are replaced by Voronoï cells: the number of Voronoï cells (up to isometries) is shown to be connected to the number of sides of a Voronoï cell. The notion of continued fraction expansion is generalized by properties of best approximation.

Finally, note the unsolved problems quoted in [29] concerning further generalizations of the three distance theorem. For instance, an upper bound for the number of distinct lengths in the partition of the unit circle by the points

$k_1\alpha_1 + k_2\alpha_2 + \cdots + k_d\alpha_d$, for $k_i \leq n_i - 1$ and $1 \leq i \leq d$ is conjectured to be of the form $c_d + \prod_{i=1}^{d-1} n_i$, where c_d is a constant independent of n_1, \dots, n_d .

7. FREQUENCIES OF FACTORS FOR BINARY CODINGS OF ROTATIONS

We will prove in this section the following result, which corresponds to the case $\min\{n_1, n_2\} = 2$ in Theorem 14. The idea of using a reflection of the unit circle can also be found in the original proof in [29].

THEOREM 18. *Let α be an irrational number in $]0, 1[$, $\beta \neq 0$ a real number and n a non-zero integer. The set of points $\{0\}, \{\beta\}, \{\alpha\}, \{\beta + \alpha\}, \dots, \{n\alpha\}, \{\beta + n\alpha\}$ divides the circle into a finite number of intervals, whose lengths take at most five values.*

7.1 A COMBINATORIAL PROOF

We will prove Theorem 18 by introducing a coding of the rotation by angle α with respect to the intervals of the unit circle bounded by the points $\{0\}, \{\beta\}, \{\alpha\}, \{\beta + \alpha\}, \dots, \{n\alpha\}, \{\beta + n\alpha\}$.

Let α be an irrational number, β a non-zero real number and n an integer. Let I_1, \dots, I_p denote the intervals of the unit circle bounded by the points $\{0\}, \{\beta\}, \{\alpha\}, \{\beta + \alpha\}, \dots, \{n\alpha\}, \{\beta + n\alpha\}$. Let $u = (u_n)_{n \in \mathbb{N}}$ be the sequence defined on the alphabet $\Sigma = \{a_1, \dots, a_p\}$ as the coding of the orbit of 0 under the rotation R of angle α under the partition $\{I_1, \dots, I_p\}$:

$$u_n = a_k \iff \{n\alpha\} \in I_k.$$

The frequency of the letter a_k in the sequence u is equal to the length of the interval I_k , by uniform distribution of the sequence $(\{n\alpha\})_{n \in \mathbb{N}}$. We must now prove that the frequencies of the letters of u take at most five values. Let us consider the graph Γ_1 of words of u of length 1. There is one edge from a_k to $a_{k'}$ if $I_{k'}$ is the image of I_k by the rotation R or if $I_{k'}$ contains $\{-\alpha\}$ or $\{-\alpha + \beta\}$. Therefore the graph Γ_1 contains p vertices (one for each letter) and $p + 2$ edges: indeed, every vertex has only one leaving edge, except the ones associated with the intervals containing $\{-\alpha\}$ or $\{-\alpha + \beta\}$, which have two leaving edges (if both of these points belong to the same interval I_k , then a_k has three leaving edges and all the other intervals have only one edge). In other words, we have $p(1) = p$ and $p(2) = p + 2$. As in the proof of Theorem 6, this implies that there are at most 6 branches in Γ_1 : indeed, each