3. Gabber's réduction to regular embeddings

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free resolution of $R/\mathfrak{q}$. Thus $\mathrm{Tor}_i^R(R/\mathfrak{q}, R/\mathfrak{p})$ is the homology $H_i(K_\bullet \otimes R/\mathfrak{p})$. Applying the above theorem with $M = R/\mathfrak{p}$, we deduce that

$$\chi(R/\mathfrak{q}, R/\mathfrak{p}) = e_k(\mathfrak{q}, R/\mathfrak{p}).$$

Since the Samuel multiplicity $e_k(\mathfrak{q}, R/\mathfrak{p})$ is always non-negative and is positive if and only if the dimension of $R/\mathfrak{p}$ is equal to k, this proves the conjectures in this case.

Serre's proof of the multiplicity conjectures in the equicharacteristic case proceeded by reducing to the case of a regular sequence by reduction to the diagonal. If R is a power series ring $k[[X_1, \dots, X_d]]$ and M and N are R-modules with $M \otimes_R N$ of finite length, he introduced a new set of variables $Y_1, \dots, Y_d$ and considered N as a module over $k[[Y_1, \dots, Y_d]]$. He then defined a "complete" tensor product $M \mathbin{\widehat{\otimes}_k} N$ over k as a module over the ring $k[[X_1, \dots, X_d, Y_1, \dots, Y_d]]$ and showed that

$$\mathrm{Tor}_i^R(M, N) \cong \mathrm{Tor}_i^{k[[X_i, Y_j]]}(M \mathbin{\widehat{\otimes}_k} N, k[[X_i, Y_j]]/(X_1 - Y_1, \dots, X_d - Y_d)).$$

Since $X_1 - Y_1, \dots, X_d - Y_d$ form a regular sequence, this proves the result for power series rings, and the conjectures for general equicharacteristic rings can be reduced to this case by completion and the Cohen structure theorems.

3. Gabber's reduction to regular embeddings

In this section we describe Gabber's use of de Jong's theorem on the existence of "regular alterations" to reduce the intersection conjectures to questions on regular embeddings in projective space over R.

As above, let R be a regular local ring and let $\mathfrak{p}$ and $\mathfrak{q}$ be prime ideals of R such that $R/\mathfrak{p} \otimes R/\mathfrak{q}$ has finite length. Let d be the dimension of R, let r be the dimension of $R/\mathfrak{p}$ and let t be the dimension of $R/\mathfrak{q}$.

The following theorem of de Jong [2] makes the reduction to a question on regular embeddings possible:

THEOREM 2. *Let A be a local integral domain which is a localization of a ring of finite type over a discrete valuation ring. Then there exists a projective map $\phi\colon X \to \mathrm{Spec}(A)$ such that*

- *X is an integral regular scheme.*
- *If K is the quotient field of A, then the extension $k(X)$ of K is finite (we say that X is* generically finite *over* $\mathrm{Spec}(A)$).

For the proof of this theorem we refer to [2]. We show below that this reduces the questions on intersections over a regular local ring to corresponding questions on intersections on projective schemes where one of the schemes is regular. We note that the fact that ϕ is projective means that X is a closed subscheme of $\mathrm{Proj}(A[X_0,\dots,X_n])$ for some n. In our application, we apply the theorem to $A = R/\mathfrak{q}$. Suppose first that $\mathrm{Spec}(R/\mathfrak{q})$ is already regular, which means that $R/\mathfrak{q}$ is a regular local ring. In that case, $\mathfrak{q}$ is generated by part of a regular system of parameters, so is in particular generated by a regular sequence. Hence the conjectures follow immediately from the results of the previous section on Koszul complexes.

We note that there is an extra assumption, that the ring be a localization of a ring of finite type over a discrete valuation ring (and there are also assumptions on the discrete valuation ring). However the general case can be reduced to this case (see Berthelot [1] or Hochster [5]), and we assume that our rings have this property.

Let $X = \mathrm{Spec}(R)$, $Z = \mathrm{Spec}(R/\mathfrak{q})$ and $Y = \mathrm{Spec}(R/\mathfrak{p})$. We denote a regular scheme which is projective and generically finite over Z (whose existence follows from de Jong's theorem) by Z'. Then there exists an n such that Z' is a closed subscheme of $\mathrm{Proj}(R/\mathfrak{q}[X_0,\dots,X_n])$ and hence also of $\mathrm{Proj}(R[X_0,\dots,X_n])$. We let P denote $\mathrm{Proj}(R[X_0,\dots,X_n])$ and let ϕ denote both the map from P to X and the induced map from Z' to Z. Let I denote the graded ideal of $R[X_0,\dots,X_n]$ which defines Z'. Let $Y' = \phi^{-1}(Y) = \mathrm{Proj}(R/\mathfrak{p}[X_0,\dots,X_n])$.

The generalization from rings to projective schemes involves a corresponding generalization from modules to sheaves. The sheaves we consider will be coherent (see for example Hartshorne [4] for the general theory of sheaves on projective schemes). We recall that a coherent sheaf $\mathcal{M}$ on W can be defined either by specifying its modules of sections over the open sets in an affine open cover or, alternatively, by taking the sheaf defined by a finitely generated graded A-module M. We will generally use the second definition, as it is usually more convenient in computing examples.

We let $\mathcal{O}_P$, $\mathcal{O}_{Y'}$, and $\mathcal{O}_{Z'}$ denote the structure sheaves of P, Y', and Z' respectively; they are defined by the graded rings $R[X_0,\dots,X_n]$, $R/\mathfrak{p}[X_0,\dots,X_n]$, and $R[X_0,\dots,X_n]/I$. We will also sometimes denote R by $\mathcal{O}_X$ and similarly for $\mathcal{O}_Y$ and $\mathcal{O}_Z$.

If $\mathcal{M}$ and $\mathcal{N}$ are sheaves on a projective scheme W defined by graded modules M and N, we define the sheaves $\mathcal{T}or_i^{\mathcal{O}_W}(\mathcal{M},\mathcal{N})$ by taking a resolution of $\mathcal{M}$ (or $\mathcal{N}$) by locally free sheaves $\mathcal{F}_i$, and letting $\mathcal{T}or_i^{\mathcal{O}_W}(\mathcal{M},\mathcal{N})$ be the i^{th} homology of $\mathcal{F}_\bullet \otimes \mathcal{N}$. Usually we define $\mathcal{F}_\bullet$ by defining a complex of graded modules F_i which define locally free sheaves and which give a resolution of M (or N). In the case where $W = P$, a bounded resolution can be constructed using direct sums of copies of $\mathcal{O}_P(n)$, so this process is quite easy to carry out. We also define the complex $\mathcal{T}or^{\mathcal{O}_W}(\mathcal{M},\mathcal{N})$ to be the complex $\mathcal{F}_\bullet \otimes \mathcal{N}$. This complex is of course not well-defined as a complex, but it is well-defined up to quasi-isomorphism.

The last ingredient in the generalization to projective space is the pushdown of complexes from P to X by the map ϕ, which we denote ϕ_*. In general this functor is the derived functor of the global section functor on sheaves, but in the case of projective space over R it is not difficult to give a direct definition using Cech cohomology. Let $A = R[X_0, \dots, X_n]$, and let $P = \mathrm{Proj}(A)$ as above. Let $C^\bullet$ be the complex

$$0 \to \prod A_{X_i} \to \prod A_{X_iX_j} \to \cdots \to A_{X_0X_1\cdots X_n} \to 0$$

where for any element $Y \in A$, A_Y denotes the localization of A obtained by inverting Y. If $\mathcal{M}_\bullet$ is a bounded complex of coherent sheaves over P represented by a complex of graded modules $M_\bullet$, we then define $\phi_*(\mathcal{M}_\bullet)$ to be the graded part of degree zero of the complex $C^\bullet \otimes_A M_\bullet$. Then $\phi_*(\mathcal{M}_\bullet)$ is a bounded complex of R-modules with finitely generated homology and is well-defined up to quasi-isomorphism.

Now suppose that $\mathcal{M}$ and $\mathcal{N}$ are coherent sheaves on P such that $\mathcal{M} \otimes_{\mathcal{O}_P} \mathcal{N}$ has support which lies over the closed point of R, which we denote s. Then $\mathcal{T}or_i^{\mathcal{O}_P}(\mathcal{M},\mathcal{N})$ has support lying over s for all i, so that the homology of $\phi_*(\mathcal{T}or^{\mathcal{O}_P}(\mathcal{M},\mathcal{N}))$ is supported at the maximal ideal and thus has finite length. Hence we can define

$$\chi(\mathcal{M},\mathcal{N}) = \sum(-1)^i \,\mathrm{length}\left(H^i(\phi_*(\mathcal{T}or^{\mathcal{O}_P}(\mathcal{M},\mathcal{N})))\right).$$

The first part of the reduction is to show that it suffices to show that the Euler characteristic $\chi(\mathcal{O}_{Z'},\mathcal{O}_{Y'})$ is non-negative and is zero if $\dim(R/\mathfrak{p})+\dim(R/\mathfrak{q}) < \dim(R)$. The point is that the assumptions on ϕ imply that this new Euler characteristic is closely related to the Euler characteristic $\chi(R/\mathfrak{p},R/\mathfrak{q})$ defined earlier. Let $G_\bullet$ be a finite free resolution of $R/\mathfrak{p}$ over R. Let $\mathcal{F}_\bullet$ be a finite locally free resolution of $\mathcal{O}_{Z'}$ as above, and let $\mathcal{F}_\bullet$ be defined by a complex $F_\bullet$ of graded modules. We then have the following "projection formula":

$$\phi_*(\mathcal{F}_\bullet) \otimes G_\bullet \cong \phi_*(\mathcal{F}_\bullet \otimes \phi^*(G_\bullet)).$$

To prove this formula, we use the definition of ϕ_* in terms of the complex $C^\bullet$ defined above. The associativity of the tensor product implies that we have isomorphisms of complexes:

$$(C^\bullet \otimes_A F_\bullet) \otimes_R G_\bullet \cong C^\bullet \otimes_A (F_\bullet \otimes_R G_\bullet) \cong C^\bullet \otimes_A (F_\bullet \otimes_A (A \otimes_R G_\bullet)).$$

Since $A \otimes_R G_\bullet$ defines a locally free resolution of $\mathcal{O}_{Y'}$, it is clear that the complex $C^\bullet \otimes_A (F_\bullet \otimes_A (A \otimes_R G_\bullet))$ represents $\phi_*(\mathcal{F}_\bullet \otimes \phi^*(G_\bullet))$. Since $(C^\bullet \otimes_A F_\bullet) \otimes_R G_\bullet$ represents $\phi_*(\mathcal{F}) \otimes G_\bullet$, this proves the above isomorphism.

To complete the proof of the fact that it suffices to prove non-negativity and vanishing for $\chi(\mathcal{O}_{Z'}, \mathcal{O}_{Y'})$, we use induction on the dimension of $R/\mathfrak{q}$ together with the assumption that the map induced by ϕ from Z' to Z is generically finite. Let $\mathcal{F}_\bullet$ be a locally free resolution of $\mathcal{O}_{Z'}$ on P. If we localize at $\mathfrak{q}$, the generic finiteness of ϕ implies that the resulting map from $\mathrm{Proj}((A/I)_\mathfrak{q})$ to $\mathrm{Spec}((R/\mathfrak{q})_\mathfrak{q})$ is defined by a finite field extension of a given degree which we denote n. Thus $\phi_*(\mathcal{F}_\bullet)$ localized at $\mathfrak{q}$ is isomorphic to $((R/\mathfrak{q})_\mathfrak{q})^n$, so the complex $\phi_*(\mathcal{F}_\bullet)$ is isomorphic to the module $(R/\mathfrak{q})^n$ up to a complex with homology of dimension strictly less than the dimension of $R/\mathfrak{q}$.

By the projection formula, we have that

$$\phi_*(\mathcal{F}_\bullet) \otimes G_\bullet \cong \phi_*(\mathcal{F}_\bullet \otimes \phi^*(G_\bullet)),$$

where $G_\bullet$ is a free resolution of $R/\mathfrak{p}$. Since $\mathcal{F}_\bullet$ is a locally free resolution of $\mathcal{O}_{Z'}$ and $\phi^*(G_\bullet)$ is a locally free resolution of $\mathcal{O}_{Y'}$, the complex $\phi_*(\mathcal{F}_\bullet \otimes \phi^*(G_\bullet))$ is quasi-isomorphic to $\phi_*(\mathcal{T}or^{\mathcal{O}_P}(\mathcal{O}_{Z'}, \mathcal{O}_{Y'}))$. Hence, taking Euler characteristics and using the above isomorphism, we have $\chi(\phi_*(\mathcal{F}_\bullet), R/\mathfrak{p}) = \chi(\mathcal{O}_{Z'}, \mathcal{O}_{Y'})$. Applying the induction hypothesis, we have that $\chi(M, R/\mathfrak{p})$ is zero whenever the dimension of M is less than the dimension of $R/\mathfrak{q}$. Thus, since $\phi_*(\mathcal{O}_{Z'})$ is isomorphic to $(R/\mathfrak{q})^n$ up to something of dimension strictly less than the dimension of $R/\mathfrak{q}$, we have that

$$\chi(\mathcal{O}_{Z'}, \mathcal{O}_{Y'}) = \chi(\phi_*(\mathcal{F}_\bullet), R/\mathfrak{p}) = \chi((R/\mathfrak{q})^n, R/\mathfrak{p}) = n(\chi(R/\mathfrak{q}, R/\mathfrak{p})).$$

Thus the vanishing, non-negativity, and positivity of $\chi(\mathcal{O}_{Z'}, \mathcal{O}_{Y'})$ are equivalent to the corresponding properties of $\chi(R/\mathfrak{q}, R/\mathfrak{p})$.

Thus we have reduced the multiplicity conjectures to corresponding conjectures on Euler characteristics defined by subschemes Y' and Z' of projective space over R, where Z' is regular and Y' is the pullback of a subscheme of $\mathrm{Spec}(R)$. In particular, the ideal I defining Z' is locally generated by a regular sequence, and this fact makes it possible to use the Serre spectral sequence to reduce to the case of associated graded rings.

Let $gr_I(A) = A/I \oplus I/I^2 \oplus \ldots$ be the associated graded ring of I. Let B denote $A/\mathfrak{p}A$, and let $gr_I(B)$ denote the associated graded ring of the image of I in B. We note that both $gr_I(A)$ and $gr_I(B)$ are bigraded rings, with one grading induced by the grading on A and the other corresponding to powers of I. The ring $gr_I(B)$ is also a bigraded module over $gr_I(A)$. We make the convention that the i,j component of $gr_I(A)$ is the component of I^j/I^{j+1} of degree i. We let E denote the scheme $\mathrm{Proj}(gr_I(A))$, where $gr_I(A)$ is considered to be a graded module by the grading in the first component (the grading induced from that on A). Then E can be defined locally as follows: if U is an affine open set in Z' and $\mathcal{O}_U$ is the ring such that $U = \mathrm{Spec}(\mathcal{O}_U)$, then the fiber of E over U is defined to be $\mathrm{Spec}(C)$, where C is the associated graded ring of $\mathcal{O}_U$ by the restriction of I to U. Since I is locally generated by a regular sequence, C is locally a polynomial ring over $\mathcal{O}_U$. We note that $\mathcal{O}_{Z'}$ is a quotient of both A and $gr_I(A)$. Let $\mathcal{M}$ denote the sheaf on E defined by the graded module $gr_I(B)$.

We next show that the Serre spectral sequence implies that we have an equality:

$$\chi_E(\mathcal{O}_{Z'}, \mathcal{M})) = \chi_P(\mathcal{O}_{Z'}, \mathcal{O}_{Y'}).$$

Let

$$0 \to \mathcal{F}_k \to \cdots \to \mathcal{F}_1 \to \mathcal{F}_0 \to 0$$

be a locally free resolution of $\mathcal{O}_{Z'}$ over $\mathcal{O}_P$. We apply the argument of section 2 to the filtration of $\mathcal{F}_\bullet$ induced by the powers of I. Since I is locally generated by a regular sequence, the same argument goes through. However, there are two points which are different from the case of the Koszul complex. First of all, $\mathcal{F}_\bullet$ will in general not be a minimal complex locally, so that it is locally a direct sum of a Koszul complex and a trivial (split exact) complex. However, in the local computation, the split exact part is eliminated in the step from E^0 to E^1, so from that point the argument goes through as before. The second point is that in taking the homology at E^1, the homology is no longer of finite length, but only supported at the maximal ideal of R. However, it is still zero except for finitely many i and n and we can conclude that the Euler characteristic is the same using the additivity of ϕ_* and the Euler characteristic on $\mathrm{Spec}(R)$. Thus the argument goes through, and we have the above equality.

There is one more reduction, which reduces to the fibers over $\mathrm{Spec}(R/\mathfrak{m}) = s$. Let $\mathcal{M}$ be the sheaf on E with associated graded ring $gr_I(B)$ considered as a module over $gr_I(A)$. Then, since for the original ideals $\mathfrak{p}$ and $\mathfrak{q}$ we had that $R/\mathfrak{p} \otimes R/\mathfrak{q}$ had finite length, $\mathcal{M}$ is annihilated by a power of the maximal ideal $\mathfrak{m}$ of R. Hence it has a finite filtration with quotients $\mathcal{M}_i$ which are annihilated by $\mathfrak{m}$. It then suffices to show that $\chi_E(\mathcal{O}_{Z'}, \mathcal{M}_i)$ is non-negative for each i. We can compute this Euler characteristic by taking a locally free resolution for $\mathcal{O}_{Z'}$ and tensoring with $\mathcal{M}_i$, and, since $\mathcal{M}_i$ is annihilated by $\mathfrak{m}$, we can tensor first with $R/\mathfrak{m}$. Let s denote the closed point of Spec(R) as above, let $E_s = \operatorname{Proj}(gr_I(A) \otimes_R k)$, where $k = R/\mathfrak{m}$, and let $Z'_s = \operatorname{Proj}(A/I \otimes_R k)$. The above argument shows that for each i we have $\chi_E(\mathcal{O}_{Z'}, \mathcal{M}_i) = \chi_{E_s}(\mathcal{O}_{Z'_s}, \mathcal{M}_i)$. Hence

$$\chi_E(\mathcal{O}_{Z'}, \mathcal{M}) = \sum_i \chi_E(\mathcal{O}_{Z'}, \mathcal{M}_i) = \sum_i \chi_{E_s}(\mathcal{O}_{Z'_s}, \mathcal{M}_i).$$

We recall that the dimension of $\mathcal{M}$ is equal to $\dim(R/\mathfrak{p}) + n$ (where $P = \operatorname{Proj}(R[X_0, \ldots, X_n])$). Thus to prove the vanishing and non-negativity conjectures it suffices to show that whenever $\mathcal{M}$ is a coherent sheaf on E_s and $\dim(\mathcal{M}) + \dim(Z') \leq \dim(R) + n$ we have $\chi_{E_s}(\mathcal{O}_{Z'_s}, \mathcal{M}) \geq 0$, and that we have equality when $\dim(\mathcal{M}) + \dim(Z') < \dim(R) + n$.

To prove the positivity conjecture it would of course suffice to show that if $\dim(\mathcal{M}) + \dim(Z') = \dim(R) + n$, the Euler characteristic is positive. However, this is not true in general (we give an example below). However, assuming the non-negativity conjecture for a moment, we show that there is a simple criterion for positivity.

PROPOSITION 1. *Let notation be as above, and let $\mathcal{M}_0$ be the sheaf defined by $gr_I(B) \otimes_R k$ considered as a module over $gr_I(A) \otimes_R k$. Assume that* $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) = \dim(R)$. *Then the positivity conjecture holds for the ideals $\mathfrak{p}$ and $\mathfrak{q}$ if and only if $\chi_{E_s}(\mathcal{O}_{Z'_s}, \mathcal{M}_0) > 0$.*

Proof. Since $\mathcal{M}_0$ is a quotient of the sheaf $\mathcal{M}$ defined by $gr_I(B)$ and Euler characteristics are non-negative, if $\chi_{E_s}(\mathcal{O}_{Z'_s}, \mathcal{M}_0) > 0$, then $\chi_E(\mathcal{O}_{Z'}, \mathcal{M}) > 0$. Conversely, suppose that $\chi_{E_s}(\mathcal{O}_{Z'_s}, \mathcal{M}_0) = 0$. Since $gr_I(B)$ is annihilated by a power of $\mathfrak{m}$, it has a filtration with quotients which are homomorphic images of direct sums of copies of $gr_I(B) \otimes_R k$. Again using non-negativity, we can deduce that if $\mathcal{M}_i$ is the sheaf defined by any of these quotients, then $\mathcal{M}_i$ is a quotient of a direct sum of copies of $\mathcal{M}_0$, so we have $\chi_{E_s}(\mathcal{O}_{Z'_s}, \mathcal{M}_i) = 0$. Thus the additivity of the Euler characteristic implies that $\chi_E(\mathcal{O}_{Z'}, \mathcal{M}) = 0$.