# 5. Spectral estimates for adjacency operators on Cayley graphs

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **44 (1998)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **26.04.2024**

$$\frac{1}{\epsilon}\sum_i Int(f_i) > \sum_i \#(f_i) = \#(\Delta) + \frac{1}{2}\sum_i Int(f_i),$$

whence $\frac{2-\epsilon}{2\epsilon}\sum_i Int(f_i) > \#(\Delta)$ or $\frac{2-\epsilon}{\epsilon}I(\Delta) > \#(\Delta)$. Since $\epsilon = 2\theta/(1+\theta)$, we obtain $I(\Delta) > \theta\#(\Delta)$, which contradicts the $\theta$-condition.

In fact, if the reduced diagram $\Delta$ is not simple, it is a union of simple diagrams linked by bridges. So each of its parts, which is a simple diagram, defines another reduced diagram (relative to another word), so the inequality holds for every part of $\Delta$ which is a simple diagram. We conclude by saying that increasing the number of external edges does not affect the inequality. □

*Proof of 4.3.* By Lemma 4.2, it is sufficient to prove that the $(\epsilon, x_0)$-balanced and $\theta$-conditions imply that every non trivial reduced word in $\mathbf{F}_X$ which vanishes in $\Gamma$ contains at least one $x_0^{\pm 1}$.

Let us choose such a word $\omega$ and $\Delta$ a reduced diagram of $\omega$. By Lemma 4.4, there exists a cell $f$ with border equal to one $r \in R^*$, such that

$$Int(f) \leq \frac{2\theta}{1+\theta}\#(f) = \frac{2\theta}{1+\theta}|r| < \epsilon|r| \leq n_{x_0}(r),$$

because $\theta < \epsilon/(2-\epsilon)$. As there are more occurences of $x_0$ or $x_0^{-1}$ than the number of internal edges, it means that some occurrences of $x_0$ or $x_0^{-1}$ will be external edges, i.e. will be in the border of $\Delta$ which is $\omega$. □

We are now able to prove the main theorem.

*Proof of theorem 1.1.* By Proposition 4.3, for a finite presentation $\langle X \mid R\rangle$, we know that being $(\epsilon, x_0)$-balanced and satisfying a $\theta$-condition is sufficient to ensure that $X - \{x_0\}$ freely generates a free subgroup in $\Gamma$. But by Corollary 3.2 and [13, Theorem 2], these two conditions are generic and so is the conjunction of these two conditions. □

## 5. Spectral estimates for adjacency operators on Cayley graphs

The existence of a free subgroup generated by $X - \{x_0\}$ gives an upper bound for the spectral value of the adjacency operator on the Cayley graph of $\Gamma = \langle X \mid R\rangle$ associated with the symmetric generating system $S = X \cup X^{-1}$.

We briefly recall some definitions and notations. The Cayley graph $G(\Gamma, X)$ of $\Gamma$ associated with $S$ has its set of vertices in bijection with $\Gamma$ and two vertices $g_1$ and $g_2$ are linked by an edge if and only $g_1^{-1}g_2 \in S$. A graph is completely determined by its adjacency operator and, in the case of Cayley graphs, the adjacency operator $h_S$ can be expressed in terms of the right regular representation $\rho$ acting on $l^2(\Gamma)$ as

$$h_S = \frac{1}{\#S} \sum_{s \in S} \rho(s) .$$

The spectral properties of $h_S$ capture some information about the pair $(\Gamma, S = X \cup X^{-1})$. For example, Kesten proved

THEOREM (Kesten [10], [9]). *Let $\Gamma$ be a finitely generated group, let $X$ be a finite generating system and set $S = X \cup X^{-1}$.*

*a) The following are equivalent:*

*i) $\|h_S\| = 1$;*

*ii) $\Gamma$ is amenable.*

*b) Assume that $\#X \geq 2$; then $\frac{\sqrt{2(\#X)-1}}{\#X} \leq \|h_S\|$. Equality holds if and only if $\Gamma$ is isomorphic to the free group $\mathbf{F}_X$ generated by $X$.*

This enables us to give an easy proposition which was pointed out to us by Pierre de la Harpe.

PROPOSITION 5.1. *Let $\Gamma = \langle X \,|\, R \rangle$ be a finite presentation of a group $\Gamma$ with $\#X \geq 2$. If $X \cap X^{-1} = \varnothing$ and if there exists $x_0 \in X$ such that $X - \{x_0\}$ generates a free subgroup in $\Gamma$ then*

$$\frac{\sqrt{2(\#X)-1}}{\#X} \leq \|h_S\| \leq \frac{\sqrt{2(\#X)-3}+1}{\#X} .$$

*Proof.* The first inequality is just Kesten's. To prove the second one, set $X' = X - \{x_0\}$, $S' = X' \cup (X')^{-1}$. Then we can write

$$(\#S)h_S = \rho(x_0) + \rho(x_0)^{-1} + \sum_{s \in S'} \rho(s) .$$

As $X'$ freely generates a free group, by Kesten's result, we obtain that

$$\Big\|\sum_{s \in S'} \rho(s)\Big\| = 2\sqrt{2(\#X') - 1} = 2\sqrt{2(\#X) - 3} .$$

So

$$\|(\#S)h_S\| \leq 2 + \Big\|\sum_{s \in S'} \rho(s)\Big\| = 2 + 2\sqrt{2(\#X) - 3} .$$

And as $X \cap X^{-1} = \varnothing$, $\#S = 2\#X$ and we get

$$\|h_S\| \leq \frac{1 + \sqrt{2(\#X) - 3}}{\#X}\,.$$

□

The last proposition and Theorem 1.1 permit us to give generic upper bounds for $\|h_S\|$. Note that this upper bound is non trivial only for $\#X \geq 3$.

COROLLARY 5.2. *For a presentation* $\Gamma = \langle X \,|\, R\rangle$ *with* $\#X = k \geq 3$ *and* $\#R = m$ *fixed, the inequalities* $\frac{\sqrt{2(\#X)-1}}{\#X} \leq \|h_S\| \leq \frac{\sqrt{2(\#X)-3}+1}{\#X}$ *are generically true.*

*Proof.* Let $x, y \in X$. Since $k \geq 3$, there exists $x_0 \in X$ distinct from $x$ and $y$. By Theorem 1.1, the subgroup generated by $X - \{x_0\}$ is generically free; in particular $xy \neq e$ in $\Gamma$. This shows that, generically, $X \cap X^{-1} = \varnothing$. The corollary follows then by combining Theorem 1.1 with Proposition 5.1. □

It was proved by Grigorchuk [7, Theorem 7.1] that for any fixed $\epsilon > 0$, any group satisfying the small cancellation hypothesis $C'(1/6)$ and such that the length of every relation is sufficiently large satisfies

$$\frac{\sqrt{2(\#X) - 1}}{\#X} < \|h_S\| < \frac{\sqrt{2(\#X) - 1}}{\#X} + \epsilon\,.$$

This corresponds to the intuitive idea that when the relations are long and do not cancel too much, the Cayley graph looks like a tree in some ball of large radius.

Champetier (in [4]) generalised this theorem, by replacing the small cancellation $C'(1/6)$, by a weaker condition defined by:

DEFINITION (Champetier). A finite presentation $\langle x_1, \ldots, x_k \mid r_1, \ldots, r_m\rangle$ satisfies the $A(C)$ condition for $C > 0$, if for every word $\omega$ in $\mathbf{F}_X$ representing the identity in $\langle X \,|\, R\rangle$, there exists a diagram $\Delta$ representing $\omega$ such that, if there are $l_i$ 2-cells contained in $\Delta$ having the relation $r_i$ as border, then

$$\sum_{i=1}^{m} l_i |r_i| \leq C|\omega|\,.$$

With that definition, the precise statement of Champetier's theorem is:

THEOREM (Champetier). *Let $C$ be a positive constant. For every $\epsilon > 0$, there exists an integer $n_0$ such that for every presentation $\Gamma = \langle X \mid R \rangle$, with $\#R = m$, satisfying $A(C)$ and $n_0 \leq \inf\{|r| \mid r \in R\}$, the following inequalities hold:*

$$\frac{\sqrt{2(\#X) - 1}}{\#X} \leq \|h_S\| \leq \left(1 + \frac{\epsilon}{2}\right) \frac{\sqrt{2(\#X) - 1}}{\#X} .$$

Assume the presentation satisfies a $\theta$-condition: then $I(\Delta) < \theta\#(\Delta) = \theta(I(\Delta) + |\omega|)$, for any reduced diagram associated with $\omega$. As

$$\sum_{i=1}^{m} l_i |r_i| = \sum_{2\text{-cell } f \subset \Delta} \#(f) \leq 2I(\Delta) + E(\Delta) ,$$

it is easy to see that the $\theta$-condition implies $A(\frac{2\theta}{1-\theta} + 1)$. So Champetier's theorem and the genericity of the $\theta$-condition imply:

COROLLARY 5.3. *For every $\epsilon > 0$, every fixed $\#X = k$ and every fixed $\#R = m$, $\|h_S\|$ is generically close to $\frac{\sqrt{2(\#X)-1}}{\#X}$.*

## REFERENCES


[1] ARZHANTSEVA, G. and A. OL'SHANSKII. Generality of the class of groups in which subgroups with a lesser number of generators are free (Russian). *Mat. Zametki 59 no. 4* (1996), 489–496.

[2] BENDER, E. A. Central and local limit theorems applied to asymptotic enumeration. *J. Comb. Theory, Ser. A 15* (1973), 91–111.

[3] CHAMPETIER, C. *Introduction à la petite simplification.* Proceedings of the congress 'Cayley graphs', École Normale Supérieure de Lyon, France, 13–15 décembre 1993.

[4] —— Cocroissance des groupes à petite simplification. *Bull. London Math. Soc. 25* (1993), 438–444.

[5] —— Propriétés statistiques des groupes de présentation finie. *Adv. Maths. 116* (1995), 197–262.

[6] CHERIX, P.-A. Generic result for the existence of free semi-group. In: *Séminaire de théorie spectrale et géométrie, no. 13, Université de Grenoble I, Institut Fourier* (1994–1995), 123–133.

[7] GRIGORCHUK, R. Symmetrical random walks on discrete groups. Multicomponent random systems (1978), 285–325.

[8] GROMOV, M. Hyperbolic groups. In: *Essays in Group Theory,* S.M. Gersten Ed. *M.S.R.I. Publ. 8* (1987), 75–263.