

## 4. SOME SUFFICIENT CONDITIONS FOR THE EXISTENCE OF FREE SUBGROUPS

Objektyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **44 (1998)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **25.04.2024**

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

### **Haftungsausschluss**

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

COROLLARY 3.2. *For  $\#X = k$ ,  $\#R = n$ ,  $x_0 \in X$  and  $0 < \epsilon < 1/k$  fixed, being  $(\epsilon, x_0)$ -balanced is generic for  $\Gamma = \langle X | R \rangle$ .*

*Proof of corollary.* We choose  $n$  relations at random; by Lemma 3.1, every  $r \in R$  is generically  $(\epsilon, x_0)$ -balanced, but the conjunction of finitely many generic properties is also generic.  $\square$

#### 4. SOME SUFFICIENT CONDITIONS FOR THE EXISTENCE OF FREE SUBGROUPS

We first begin by a very easy proposition.

PROPOSITION 4.1. *Let  $\Gamma = \langle X | R \rangle$  be a finite presentation, which has a Dehn algorithm and such that for some  $y \in X$  every subword  $u$  of every  $r \in R^*$  with  $|u| > |r|/2$  contains either  $y$  or  $y^{-1}$ , then  $X - \{y\}$  generates a free subgroup in  $\Gamma$ .*

The proof of this proposition will follow from Lemma 4.2 below.

LEMMA 4.2. *For  $\langle X | R \rangle$  a finite presentation of a group  $\Gamma$  and  $y \in X$ , the following are equivalent:*

- $X - \{y\}$  freely generates a free subgroup of  $\Gamma$ ;
- every non trivial element  $\omega \in \mathbf{F}_X$ , which represents the identity in  $\Gamma$ , contains either  $y$  or  $y^{-1}$ .

*Proof.* 1)  $\Rightarrow$  2): By contraposition, suppose that there exists a non trivial reduced element  $\omega \in \mathbf{F}_{X-\{y\}}$  such that  $\bar{\omega} = e$  (where  $\bar{\omega}$  is the canonical projection of  $\omega$  in  $\Gamma$ ), then  $X - \{y\}$  does not freely generate a free subgroup in  $\Gamma$ .

2)  $\Rightarrow$  1): Let  $\omega_1, \omega_2 \in \mathbf{F}_{X-\{y\}}$  be two reduced elements such that  $\bar{\omega}_1 = \bar{\omega}_2 \in \Gamma$ . Then  $\overline{\omega_1 \omega_2^{-1}} = e \in \Gamma$ . So  $\omega_1 \omega_2^{-1}$  is an element of  $\mathbf{F}_{X-\{y\}}$  which represents the identity in  $\Gamma$ . By hypothesis, this implies  $\omega_1 = \omega_2$  in  $\mathbf{F}_X$ . Hence  $X - \{y\}$  freely generates a free subgroup in  $\Gamma$ .  $\square$

*Proof of Proposition 4.1.* By Lemma 4.2, it is sufficient to show that every non trivial reduced word on  $\mathbf{F}_X$  which represents the identity in  $\Gamma$  contains either  $y$  or  $y^{-1}$ . By assumption,  $\Gamma = \langle X | R \rangle$  satisfies a Dehn algorithm, so such a word contains at least one half of a relator  $r$  in  $R$  which contains at least one occurrence of  $y$  or  $y^{-1}$ .  $\square$

The interest of this proposition appears when we replace “having a Dehn’s algorithm” by “satisfying the small cancellation condition  $C'(1/6)$ ”, because  $C'(1/6)$  and the fact that every subword  $u$  of any relation  $r$  with  $|u| > |r|/2$  contains at least one  $y$  or  $y^{-1}$  are easy to check on a given presentation.

Unfortunately, as explained before, it is not known if the small cancellation hypothesis is generic, so we need other sufficient conditions to ensure that  $X - \{y\}$  generates a free subgroup in  $\Gamma$ .

**PROPOSITION 4.3.** *Let  $\Gamma = \langle X \mid R \rangle$  be a finite presentation with  $k$  generators and  $l$  relations, which is  $(\epsilon, x_0)$ -balanced for some  $0 < \epsilon < 1/k$  and some  $x_0 \in X$ , and which satisfies a  $\theta$ -condition such that  $\theta < \epsilon/(2 - \epsilon)$ . Then  $X - \{x_0\}$  freely generates a free group in  $\Gamma$ .*

To prove the proposition we need the following lemma and the following notations. For a cell  $f_i$  of the diagram, we denote by  $Int(f_i)$  (resp. by  $Ext(f_i)$ ) the number of edges of  $f_i$  which are internal to the diagram (resp. which are on the border of the diagram). We denote also by  $\#(f_i)$  the total number of edges of the cell  $f_i$ .

**LEMMA 4.4.** *Let  $\Gamma = \langle X \mid R \rangle$  be a finite presentation of a group  $\Gamma$  which satisfies a  $\theta$ -condition for some  $0 < \theta < 1$ , then for every reduced diagram, there exists a 2-cell  $f$  of  $\Delta$  satisfying*

$$Int(f) \leq \frac{2\theta}{1 + \theta} \#(f).$$

*Proof.* First we prove it for simple diagrams. Let  $\epsilon = 2\theta/(1 + \theta)$ . Because the diagram is simple we have the following equalities:

- I)  $\sum_i Ext(f_i) = E(\Delta) = |\partial\Delta|,$
- II)  $\sum_i Int(f_i) = 2I(\Delta),$  because every internal edge belongs to two different cells.

So we get:

$$\#(\Delta) = \frac{1}{2} \sum_i Int(f_i) + \sum_i Ext(f_i) = \sum_i \#(f_i) - \frac{1}{2} \sum_i Int(f_i).$$

To obtain a contradiction, we suppose that every cell  $f_i$  of one diagram  $\Delta$  is such that  $(1/\epsilon)Int(f_i) > \#(f_i)$ . Then we have

$$\frac{1}{\epsilon} \sum_i \text{Int}(f_i) > \sum_i \#(f_i) = \#(\Delta) + \frac{1}{2} \sum_i \text{Int}(f_i),$$

whence  $\frac{2-\epsilon}{2\epsilon} \sum_i \text{Int}(f_i) > \#(\Delta)$  or  $\frac{2-\epsilon}{\epsilon} I(\Delta) > \#(\Delta)$ . Since  $\epsilon = 2\theta/(1+\theta)$ , we obtain  $I(\Delta) > \theta\#(\Delta)$ , which contradicts the  $\theta$ -condition.

In fact, if the reduced diagram  $\Delta$  is not simple, it is a union of simple diagrams linked by bridges. So each of its parts, which is a simple diagram, defines another reduced diagram (relative to another word), so the inequality holds for every part of  $\Delta$  which is a simple diagram. We conclude by saying that increasing the number of external edges does not affect the inequality.  $\square$

*Proof of 4.3.* By Lemma 4.2, it is sufficient to prove that the  $(\epsilon, x_0)$ -balanced and  $\theta$ -conditions imply that every non trivial reduced word in  $\mathbf{F}_X$  which vanishes in  $\Gamma$  contains at least one  $x_0^{\pm 1}$ .

Let us choose such a word  $\omega$  and  $\Delta$  a reduced diagram of  $\omega$ . By Lemma 4.4, there exists a cell  $f$  with border equal to one  $r \in R^*$ , such that

$$\text{Int}(f) \leq \frac{2\theta}{1+\theta} \#(f) = \frac{2\theta}{1+\theta} |r| < \epsilon |r| \leq n_{x_0}(r),$$

because  $\theta < \epsilon/(2-\epsilon)$ . As there are more occurrences of  $x_0$  or  $x_0^{-1}$  than the number of internal edges, it means that some occurrences of  $x_0$  or  $x_0^{-1}$  will be external edges, i.e. will be in the border of  $\Delta$  which is  $\omega$ .  $\square$

We are now able to prove the main theorem.

*Proof of theorem 1.1.* By Proposition 4.3, for a finite presentation  $\langle X | R \rangle$ , we know that being  $(\epsilon, x_0)$ -balanced and satisfying a  $\theta$ -condition is sufficient to ensure that  $X - \{x_0\}$  freely generates a free subgroup in  $\Gamma$ . But by Corollary 3.2 and [13, Theorem 2], these two conditions are generic and so is the conjunction of these two conditions.  $\square$

## 5. SPECTRAL ESTIMATES FOR ADJACENCY OPERATORS ON CAYLEY GRAPHS

The existence of a free subgroup generated by  $X - \{x_0\}$  gives an upper bound for the spectral value of the adjacency operator on the Cayley graph of  $\Gamma = \langle X | R \rangle$  associated with the symmetric generating system  $S = X \cup X^{-1}$ .

We briefly recall some definitions and notations. The Cayley graph  $G(\Gamma, X)$  of  $\Gamma$  associated with  $S$  has its set of vertices in bijection with  $\Gamma$  and two