# 4. SOME SUFFICIENT CONDITIONS FOR THE EXISTENCE OF FREE SUBGROUPS


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COROLLARY 3.2. *For $\#X = k$, $\#R = n$, $x_0 \in X$ and $0 < \epsilon < 1/k$ fixed, being $(\epsilon, x_0)$-balanced is generic for $\Gamma = \langle X | R\rangle$.*

*Proof of corollary.* We choose $n$ relations at random; by Lemma 3.1, every $r \in R$ is generically $(\epsilon, x_0)$-balanced, but the conjunction of finitely many generic properties is also generic. $\square$

## 4. SOME SUFFICIENT CONDITIONS FOR THE EXISTENCE OF FREE SUBGROUPS

We first begin by a very easy proposition.

PROPOSITION 4.1. *Let $\Gamma = \langle X \mid R\rangle$ be a finite presentation, which has a Dehn algorithm and such that for some $y \in X$ every subword $u$ of every $r \in R^*$ with $|u| > |r|/2$ contains either $y$ or $y^{-1}$, then $X - \{y\}$ generates a free subgroup in $\Gamma$.*

The proof of this proposition will follow from Lemma 4.2 below.

LEMMA 4.2. *For $\langle X | R\rangle$ a finite presentation of a group $\Gamma$ and $y \in X$, the following are equivalent:*

- *$X - \{y\}$ freely generates a free subgroup of $\Gamma$;*
- *every non trivial element $\omega \in \mathbf{F}_X$, which represents the identity in $\Gamma$, contains either $y$ or $y^{-1}$.*

*Proof.* 1) $\Rightarrow$ 2) : By contraposition, suppose that there exists a non trivial reduced element $\omega \in \mathbf{F}_{X-\{y\}}$ such that $\overline{\omega} = e$ (where $\overline{\omega}$ is the canonical projection of $\omega$ in $\Gamma$), then $X - \{y\}$ does not freely generate a free subgroup in $\Gamma$.

2) $\Rightarrow$ 1) : Let $\omega_1, \omega_2 \in \mathbf{F}_{X-\{y\}}$ be two reduced elements such that $\overline{\omega}_1 = \overline{\omega}_2 \in \Gamma$. Then $\overline{\omega_1\omega_2^{-1}} = e \in \Gamma$. So $\omega_1\omega_2^{-1}$ is an element of $\mathbf{F}_{X-\{y\}}$ which represents the identity in $\Gamma$. By hypothesis, this implies $\omega_1 = \omega_2$ in $\mathbf{F}_X$. Hence $X - \{y\}$ freely generates a free subgroup in $\Gamma$. $\square$

*Proof of Proposition 4.1.* By Lemma 4.2, it is sufficient to show that every non trivial reduced word on $\mathbf{F}_X$ which represents the identity in $\Gamma$ contains either $y$ or $y^{-1}$. By assumption, $\Gamma = \langle X | R\rangle$ satisfies a Dehn algorithm, so such a word contains at least one half of a relator $r$ in $R$ which contains at least one occurrence of $y$ or $y^{-1}$. $\square$

The interest of this proposition appears when we replace "having a Dehn's algorithm" by "satisfying the small cancellation condition $C'(1/6)$", because $C'(1/6)$ and the fact that every subword $u$ of any relation $r$ with $|u| > |r|/2$ contains at least one $y$ or $y^{-1}$ are easy to check on a given presentation.

Unfortunately, as explained before, it is not known if the small cancellation hypothesis is generic, so we need other sufficient conditions to ensure that $X - \{y\}$ generates a free subgroup in $\Gamma$.

PROPOSITION 4.3. *Let $\Gamma = \langle X \mid R \rangle$ be a finite presentation with $k$ generators and $l$ relations, which is $(\epsilon, x_0)$-balanced for some $0 < \epsilon < 1/k$ and some $x_0 \in X$, and which satisfies a $\theta$-condition such that $\theta < \epsilon/(2-\epsilon)$. Then $X - \{x_0\}$ freely generates a free group in $\Gamma$.*

To prove the proposition we need the following lemma and the following notations. For a cell $f_i$ of the diagram, we denote by $Int(f_i)$ (resp. by $Ext(f_i)$) the number of edges of $f_i$ which are internal to the diagram (resp. which are on the border of the diagram). We denote also by $\#(f_i)$ the total number of edges of the cell $f_i$.

LEMMA 4.4. *Let $\Gamma = \langle X | R \rangle$ be a finite presentation of a group $\Gamma$ which satisfies a $\theta$-condition for some $0 < \theta < 1$, then for every reduced diagram, there exists a 2-cell $f$ of $\Delta$ satisfying*

$$Int(f) \leq \frac{2\theta}{1+\theta} \#(f) .$$

*Proof.* First we prove it for simple diagrams. Let $\epsilon = 2\theta/(1+\theta)$. Because the diagram is simple we have the following equalities:

I) $\sum_i Ext(f_i) = E(\Delta) = |\partial \Delta|$,

II) $\sum_i Int(f_i) = 2I(\Delta)$, because every internal edge belongs to two different cells.

So we get:

$$\#(\Delta) = \frac{1}{2} \sum_i Int(f_i) + \sum_i Ext(f_i) = \sum_i \#(f_i) - \frac{1}{2} \sum_i Int(f_i) .$$

To obtain a contradiction, we suppose that every cell $f_i$ of one diagram $\Delta$ is such that $(1/\epsilon) Int(f_i) > \#(f_i)$. Then we have

$$\frac{1}{\epsilon}\sum_i Int(f_i) > \sum_i \#(f_i) = \#(\Delta) + \frac{1}{2}\sum_i Int(f_i),$$

whence $\frac{2-\epsilon}{2\epsilon}\sum_i Int(f_i) > \#(\Delta)$ or $\frac{2-\epsilon}{\epsilon}I(\Delta) > \#(\Delta)$. Since $\epsilon = 2\theta/(1+\theta)$, we obtain $I(\Delta) > \theta\#(\Delta)$, which contradicts the $\theta$-condition.

In fact, if the reduced diagram $\Delta$ is not simple, it is a union of simple diagrams linked by bridges. So each of its parts, which is a simple diagram, defines another reduced diagram (relative to another word), so the inequality holds for every part of $\Delta$ which is a simple diagram. We conclude by saying that increasing the number of external edges does not affect the inequality. □

*Proof of 4.3.* By Lemma 4.2, it is sufficient to prove that the $(\epsilon, x_0)$-balanced and $\theta$-conditions imply that every non trivial reduced word in $\mathbf{F}_X$ which vanishes in $\Gamma$ contains at least one $x_0^{\pm 1}$.

Let us choose such a word $\omega$ and $\Delta$ a reduced diagram of $\omega$. By Lemma 4.4, there exists a cell $f$ with border equal to one $r \in R^*$, such that

$$Int(f) \leq \frac{2\theta}{1+\theta}\#(f) = \frac{2\theta}{1+\theta}|r| < \epsilon|r| \leq n_{x_0}(r),$$

because $\theta < \epsilon/(2-\epsilon)$. As there are more occurences of $x_0$ or $x_0^{-1}$ than the number of internal edges, it means that some occurrences of $x_0$ or $x_0^{-1}$ will be external edges, i.e. will be in the border of $\Delta$ which is $\omega$. □

We are now able to prove the main theorem.

*Proof of theorem 1.1.* By Proposition 4.3, for a finite presentation $\langle X \mid R\rangle$, we know that being $(\epsilon, x_0)$-balanced and satisfying a $\theta$-condition is sufficient to ensure that $X - \{x_0\}$ freely generates a free subgroup in $\Gamma$. But by Corollary 3.2 and [13, Theorem 2], these two conditions are generic and so is the conjunction of these two conditions. □

## 5. Spectral estimates for adjacency operators on Cayley graphs

The existence of a free subgroup generated by $X - \{x_0\}$ gives an upper bound for the spectral value of the adjacency operator on the Cayley graph of $\Gamma = \langle X \mid R\rangle$ associated with the symmetric generating system $S = X \cup X^{-1}$.

We briefly recall some definitions and notations. The Cayley graph $G(\Gamma, X)$ of $\Gamma$ associated with $S$ has its set of vertices in bijection with $\Gamma$ and two