2. Invariant and symmetric functions

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by the *power sum forms* in the results; to our knowledge this phenomenon has not been observed before. The combinatorial identities involving harmonic numbers that we encounter are also interesting. Sections 2-6 contain results in hermitian complex geometry and may be read without prior knowledge of Arakelov theory. § 8 applies our calculations to obtain a presentation of the Arakelov Chow ring of the arithmetic Grassmannian.

This should be regarded as a companion paper to [T]; both papers will be part of the author's 1997 University of Chicago thesis. I wish to thank my advisor William Fulton for many useful conversations and exchanges of ideas.

2. Invariant and symmetric functions

The symmetric group S_n acts on the polynomial ring $\mathbf{Z}[x_1, x_2, \ldots, x_n]$ by permuting the variables, and the ring of invariants $\Lambda(n) = \mathbf{Z}[x_1, x_2, \ldots, x_n]^{S_n}$ is the ring of symmetric polynomials. For $B = \mathbf{Q}$ or $\mathbf{C}$, let $\Lambda(n, B) = \Lambda(n) \otimes_{\mathbf{Z}} B$.

Let $e_k(x_1, \ldots, x_n)$ be the k-th elementary symmetric polynomial in the variables $x_1, \ldots, x_n$ and $p_k(x_1, \ldots, x_n) = \sum_i x_i^k$ the k-th power sum. The fundamental theorem on symmetric functions states that $\Lambda(n) = \mathbf{Z}[e_1, \ldots, e_n]$ and that $e_1, \ldots, e_n$ are algebraically independent. For λ a partition, i.e. a decreasing sequence $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_m$ of nonnegative integers, define $p_\lambda := \prod_{i=1}^{m} p_{\lambda_i}$. It is well known that the p_λ's form an additive $\mathbf{Q}$-basis for the ring of symmetric polynomials (cf. [M], § I.2). The two bases are related by Newton's identity:

$$p_k - e_1 p_{k-1} + e_2 p_{k-2} - \cdots + (-1)^k k e_k = 0\,. \tag{2}$$

Another important set of symmetric functions related to the cohomology ring of grassmannians are the Schur polynomials. For a partition λ as above, the Schur polynomial s_λ is defined by

$$s_\lambda(x_1, \ldots, x_n) = \frac{1}{\Delta} \cdot \det(x_i^{\lambda_j + n - j})_{1 \leqslant i,j \leqslant n}\,,$$

where $\Delta = \prod_{1 \leqslant i < j \leqslant n} (x_i - x_j)$ is the Vandermonde determinant. The s_λ for all λ of length $m \leqslant n$ form a $\mathbf{Z}$-basis of $\Lambda(n)$ (cf. [M], § I.3).

Let $\mathbf{C}[T_{ij}]$ $(1 \leqslant i,j \leqslant n)$ be the coordinate ring of the space $M_n(\mathbf{C})$ of $n \times n$ matrices. $GL_n(\mathbf{C})$ acts on matrices by conjugation, and we let

$I(n) = \mathbf{C}[T_{ij}]^{GL_n(\mathbf{C})}$ denote the corresponding graded ring of invariants. There is an isomorphism $\tau\colon I(n) \to \Lambda(n, \mathbf{C})$ given by evaluating an invariant polynomial ϕ on the diagonal matrix $\mathrm{diag}(x_1, \ldots, x_n)$. We will often identify ϕ with the the symmetric polynomial $\tau(\phi)$. We will need to consider invariant polynomials with rational coefficients; let $I(n, \mathbf{Q}) \simeq \mathbf{Q}[x_1, x_2, \ldots, x_n]^{S_n}$ be the corresponding ring.

Given $\phi \in I(n)_k$, let ϕ' be a k-multilinear form on $M_n(\mathbf{C})$ such that

$$\phi'(gA_1g^{-1}, \ldots, gA_kg^{-1}) = \phi'(A_1, \ldots, A_k)$$

for $g \in GL(n, \mathbf{C})$ and $\phi(A) = \phi'(A, A, \ldots, A)$. Such forms are most easily constructed for the power sums p_k by setting

$$p'_k(A_1, A_2, \ldots, A_k) = \mathrm{Tr}(A_1A_2 \cdots A_k).$$

For p_λ we can take $p'_\lambda = \prod p'_{\lambda_i}$. Since the p_λ's are a basis of $\Lambda(n, \mathbf{Q})$, it follows that one can use the above constructions to find multilinear forms ϕ' for any $\phi \in I(n)_k$.

An explicit formula for ϕ' is given by polarizing ϕ:

$$\phi'(A_1 \ldots, A_k) = \frac{(-1)^k}{k!} \sum_{j=1}^{k} \sum_{i_1 < \ldots < i_j} (-1)^j \phi(A_{i_1} + \ldots + A_{i_j}).$$

Although above formula for ϕ' is symmetric in $A_1 \ldots, A_k$, this property is not needed for the applications that follow.

3. Hermitian differential geometry

Let X be a complex manifold, E a rank n holomorphic vector bundle over X. Denote by $A^k(X, E)$ the C^∞ sections of $\Lambda^k T^*X \otimes E$, where T^*X denotes the cotangent bundle of X. In particular $A^k(X)$ is the space of smooth complex k-forms on X. Let $A^{p,q}(X)$ the space of smooth complex forms of type (p, q) on X and $A(X) := \bigoplus_p A^{p,p}(X)$. The decomposition $A^1(X, E) = A^{1,0}(X, E) \bigoplus A^{0,1}(X, E)$ induces a decomposition $D = D^{1,0} + D^{0,1}$ of each connection D on E. Let $d = \partial + \overline{\partial}$ and $d^c = (\partial - \overline{\partial})/(4\pi i)$.

Assume now that E is equipped with a hermitian metric h. The pair (E, h) is called a *hermitian vector bundle*. The metric h induces a canonical connection $D = D(h)$ such that $D^{0,1} = \overline{\partial}_E$ and D is *unitary*, i.e.

$$d\,h(s, t) = h(Ds, t) + h(s, Dt), \qquad \text{for all } s, t \in A^0(X, E).$$