AMENABILITY AND GROWTH OF ONE-RELATOR GROUPS

L'Enseignement Mathématique, t. 43 (1997), p. 337–354

AMENABILITY AND GROWTH OF ONE-RELATOR GROUPS

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ABSTRACT. An algorithm showing whether a group given by a one-relator presentation is amenable or not is constructed. Sufficient conditions for a one-relator group of exponential growth to have uniformly exponential growth are also given.

0. INTRODUCTION

A one-relator group is a group G which admits a presentation

$$(*) \qquad G = \langle a_1, a_2, \ldots, a_m \ : \ R(a_1, a_2, \ldots, a_m) = 1 \rangle$$

with one defining relation.

The paper by G. Baumslag [B 1] is a comprehensive survey of results about one-relator groups. In particular this paper stresses the role of algorithmic problems in the theory of one-relator groups.

Recently the interest in functional-analytical and asymptotical properties of one-relator groups has increased. For instance, the entropy of one-relator groups was discussed in [GrLP], random walks and Markov operators on one-relator groups where investigated in [CV], [BCCH], [BC], and the K-functor of reduced C^*-algebras of one-relator groups was computed in [BBV]. Also the growth functions of the groups $\Gamma_n = \langle t, a : tat^{-1} = a^n \rangle$, $n \neq 0, \pm 1$, and of some other one-relator groups were calculated in [CEG] and [EJ].

Recall that a discrete group G is amenable if there exists a finitely additive measure $\mu \colon \mathcal{P}(G) = \{0,1\}^G \longrightarrow [0,1]$ which is G-(left)-invariant ($\mu(gE) = \mu(E)$ for all $g \in G$ and $E \subset G$) and such that, in addition, $\mu(G) = 1$. For our purpose it will be enough to know that a group containing a free subgroup of rank two is not amenable, and that, on the contrary, any solvable group is amenable ([G]).

As easily follows from the paper of Karrass and Solitar [KS], all amenable one-relator groups are in the following list:

$$(**)\quad \begin{cases} 1.\ \langle a : a^n = 1\rangle \cong \mathbf{Z}_n, \text{ cyclic groups of finite order } n = 1,2,\ldots; \\ 2.\ \langle a,b : b = 1\rangle \cong \mathbf{Z}, \text{ the infinite cyclic group;} \\ 3.\ \langle a,b : bab^{-1} = a^n\rangle,\ n \neq 0. \\ \quad \text{This class splits into two subclasses:} \\ \quad 3_a.\ n = +1 :\ \langle a,b : bab^{-1} = a\rangle \cong \mathbf{Z}^2; \\ \qquad\ \ n = -1 :\ \langle a,b : bab^{-1} = a^{-1}\rangle : \\ \qquad\qquad \text{this group contains a subgroup } \cong \mathbf{Z}^2 \text{ of index two,} \\ \qquad\qquad \text{but it is not } \cong \mathbf{Z}^2; \\ \quad 3_b.\ n \neq 0, \pm 1 :\ \langle a,b : bab^{-1} = a^n\rangle : \\ \qquad\qquad \text{these groups are 2 step-solvable and of exponential} \\ \qquad\qquad \text{growth (pairwise non-isomorphic).} \end{cases}$$

Also Tits' alternative does hold for one-relator groups: any one-relator group either contains a free subgroup of rank two or is solvable (and from the above list).

But in the Karrass-Solitar paper no algorithm is given answering the question whether, given a one-relator presentation, the corresponding group is solvable or not. In Section 1 we present a simple algorithm and, as a consequence, we re-obtain the above list of all amenable one-relator groups.

In the second part of the paper we investigate the uniformly exponential growth for one-relator groups of exponential growth.

Recall that if G is a group with a finite generating system A,

$$|g|_A = \min\{n : g = a_1 a_2 \cdots a_n,\ a_i \in A\}$$

is the *length* of an element $g \in G$ with respect to A and $\gamma_A^G(n) = |\{g \in G : |g|_A \leq n\}|$ is the *growth function* of G with respect to the generating system A. The limit

$$\lambda_A(G) = \lim_{n \longrightarrow \infty} \sqrt[n]{\gamma_A^G(n)}$$

exists and $\lambda_A(G) \geq 1$. The group G is said to have *exponential growth* (respectively *sub-exponential growth*) if $\lambda_A(G) > 1$ (resp. $\lambda_A(G) = 1$) for some (and therefore for any other) finite system of generators A.

Denoting now by

$$\lambda_*(G) = \inf_A \lambda_A(G)$$

the *minimal growth rate* of G, where the infimum is taken over all finite generating systems, the group G has *uniform exponential growth* if $\lambda_*(G) > 1$. This last concept is due to Avez [A] where the number

$$h(G) = \log(\lambda_*(G))$$

is called the *entropy* of the group G and it is discussed in [GrLP], [SW] and in the survey paper [GH].

The simplest example of a group with uniformly exponential growth is the free group $\mathbf{F}_m$ of finite rank $m \geq 2$ for which the minimal growth rate is $\lambda_*(\mathbf{F}_m) = 2m - 1$, see for instance [GH].

It is not known whether a group of exponential growth has necessarily uniformly exponential growth or not. We formulate the following:

0.1. CONJECTURE. *All one-relator groups of exponential growth have uniformly exponential growth.*

Conjecture 0.1 is true for one-relator groups of rank $m \geq 3$ and for one-relator groups with torsion, therefore we focus our attention on two-generated one-relator groups and give sufficient conditions for such groups to have uniformly exponential growth. We present a new method for estimating the minimal growth rate of a finitely generated group using growth functions of the corresponding graded Lie algebra and apply it to one-relator groups.

1. AN ALGORITHM FOR CHECKING AMENABILITY

Let G be a one-relator group with presentation $(*)$; the number m of the generators of G in the presentation is called the rank of the presentation. Untill Section 4 we shall assume that R is cyclically reduced and non trivial.

The next observation is well known. We shall include the proof stressing the algorithmic aspect of the statement.

1.1. LEMMA. *Let $G = \langle a, b, \ldots : R(a, b, \ldots)\rangle$ be a one-relator group with at least two generators. Then G has a presentation $\langle t, \ldots : R'(t, \ldots)\rangle$ with $\sigma_t(R') = 0$, where $\sigma_t(R')$ denotes the sum of the exponents of t in the word R'. This second presentation can in fact be produced, starting from the original one, in an algorithmical way.*

Proof. Let a and b be two generators involved in R; if $\sigma_a(R) = 0$ or $\sigma_b(R) = 0$ we are already done. If not, suppose that $0 < |\sigma_a(R)| \leq |\sigma_b(R)|$; by exchanging a with a^{-1} and/or b with b^{-1} if necessary, we can suppose that $0 < \sigma_a(R) \leq \sigma_b(R)$. Set $a' = ab$ and $b' = b$; then, if $R'(a', b')$ is the expression of R in terms of the new generators a' and b', one has $\sigma_{a'}(R') = \sigma_a(R)$ and $|\sigma_{b'}(R')| < \sigma_b(R)$. Applying this procedure inductively for at most $|\sigma_a(R)| + |\sigma_b(R)|$ times one gets the claimed presentation. □

Note that the rank of the second presentation in the previous lemma coincides with the rank of the initial one.

1.2. THEOREM. *The following is an algorithm which establishes if a given one-relator group G with presentation $(*)$ is amenable or not:*

Step 1: If $m \geq 3$ then G is not amenable. If $m = 1$ then G is amenable; if $m = 2$ go to next step.

Step 2: Check if R is a power of one of the generators. If this is the case and the power is proper then G is not amenable, if R coincides, up to inversion, with one of the generators then G is amenable. Otherwise go to next step.

Step 3: Using the algorithm from the above lemma, change the presentation of G so that the sum of the exponents of one of the generators in the relator is zero. Then G is amenable iff, up to a relabeling and inversion of the generators, and up to a cyclic permutation of the relator, the presentation obtained is of the form $\langle t, s : tst^{-1}s^{-n} = 1\rangle$, with $n \in \mathbf{Z} \setminus \{0\}$.

Proof. Recall that the Freiheitssatz of Wilhelm Magnus ([MKS: Thm. 4.10] and [LS: IV Thm. 5.1]) states that, if $R = R(a_1, a_2, \ldots, a_m)$ is a cyclically reduced word in $a_1, a_2, \ldots, a_m$ and involves a_m, then the subgroup of $G = \langle a_1, a_2, \ldots, a_m : R(a_1, a_2, \ldots, a_m) = 1\rangle$ generated by $a_1, a_2, \ldots, a_{m-1}$ is freely generated by them.

(1) If $m \geq 3$ then, by Magnus' Theorem, G contains the free group on two generators and thus it is not amenable. If $m = 1$ then $G = \langle a : a^n = 1\rangle$ is cyclic and therefore amenable.

(2) Let $m = 2$. If R is a proper power of one of the generators, say $R = a^n$ with $|n| \geq 2$, then G is isomorphic to the free product $\mathbf{Z} * \mathbf{Z}_{|n|}$ of the infinite cyclic group and the cyclic group of order $|n| \geq 2$ and it is not amenable because its commutator subgroup is a free group of infinite rank. If R coincides, up to inversion, with one of the generators then G is infinite cyclic and therefore amenable.

(3) Suppose now that $\langle a, b : R(a,b) = 1 \rangle$ is a presentation of G with $\sigma_a(R) = 0$. If we denote by $b_i = a^i b a^{-i}$, $i \in \mathbf{Z}$, then the relator R can be expressed as a word in the b_i's just replacing each b^k in $R(a,b)$ by b_j^k, where j is the sum of the exponents of a in the subword of R preceding the given occurrence of b^k. We shall denote this word by $R'(b_m, b_{m+1}, \ldots, b_M)$, where m and M are the minimum and, respectively, the maximum subscript occurring in the expression of R'. Note that since $R(a,b)$ is cyclically reduced, then R' is cyclically reduced as well and $m < M$.

It is known [LS: IV, proof of Thm. 5.1] that any one-relator group with ≥ 2 generators is an HNN-extension $(H; A, B, \phi)$ of another one-relator group H. In our situation

$$\begin{aligned} H &= \langle b_m, b_{m+1}, \ldots, b_M; R'(b_m, b_{m+1}, \ldots, b_M) \rangle \\ A &= \text{ subgroup of } H \text{ generated by } b_m, b_{m+1}, \ldots, b_{M-1} \\ B &= \text{ subgroup of } H \text{ generated by } b_{m+1}, b_{m+2}, \ldots, b_M \\ \phi &: A \ni b_i \longmapsto b_{i+1} \in B, \quad i = m, m+1, \ldots, M-1\,. \end{aligned}$$

Therefore G also admits the following presentation

$$G = \left\langle a, b_m, \ldots, b_M : R'(b_m, \ldots, b_M) = 1,\ a b_i a^{-1} = b_{i+1},\ i = m, \ldots, M-1 \right\rangle.$$

The subgroups A and B are free of rank $M - m$ and if $M - m \geq 2$ then G is not amenable.

Suppose now that $M - m = 1$, so that $A = \langle b_m \rangle \cong B = \langle b_M \rangle \cong \mathbf{Z}$. It is known ([H: Prop. 3.3]) that an HNN-extension $(H; A, B, \phi)$, such that A and B are both proper subgroups of the base group H, contains the free group $\mathbf{F}_2$. Thus, if $A \neq H \neq B$, then G is non amenable.

Suppose that $A = H$ (the case $B = H$ is similar). Then $H = \langle b_m \rangle \cong \mathbf{Z}$ and $b_M = b_m^k$ for a suitable $k \in \mathbf{Z} \setminus \{0\}$. Replacing a by t and b_m by s in the above presentation for G, one gets the presentation

$$G = \left\langle t, s : tst^{-1} = s^k \right\rangle$$

of type 3_b. from the list $(**)$ and so G is amenable. □

1.3. COROLLARY. *For amenable one-relator groups the isomorphism problem is solvable.*

Proof. Suppose two one-relator groups which are amenable are given. Then, in the algorithmical way described above, one gets two presentations from the list $(**)$ and the procedure of recognition becomes obvious since any two groups from the list with different presentations are in fact non-isomorphic. □

2. TWO-GENERATED ONE-RELATOR GROUPS

Let G be a one-relator group with $m \geq 3$ generators, or with torsion. It is known that G has a subgroup of finite index G_0 which surjects homomorphically onto the free group $\mathbf{F}_2$ of rank 2 (see [BP 1,2]). As $\lambda_*(\mathbf{F}_2) = 3$ one has $\lambda_*(G_0) \geq 3$, and it follows from Prop. 3.3 in [SW] that $\lambda_*(G) > 1$.

In the sequel of this section we study the growth of two-generated one-relator groups.

As we remarked in the proof of Theorem 1.2 a group

$$G = \langle a, b : R(a,b) = 1 \rangle ,$$

with the relator R having zero a-exponent sum, is an HNN-extension $H^* = (H; A, B, \phi)$, where $H = \langle b_m, \dots, b_M; R'(b_m, \dots, b_M) = 1 \rangle$ is another one-relator group and the associated subgroups A and B are the free subgroups of H freely generated by $\{b_m, \dots, b_{M-1}\}$ and, respectively, $\{b_{m+1}, \dots, b_M\}$. We can distinguish 3 cases:

I) A and B are proper subgroups of H: $A \neq H \neq B$;

II) only one associated subgroup is proper: $A = H \neq B$ or $A \neq H = B$;

III) the associated subgroups coincide with the base group: $A = H = B$.

Accordingly we say that the group G is of type (I), (II) or (III).

2.1. LEMMA. *Let $G = \langle a, b : R(a,b) = 1 \rangle$ where R has zero exponent sum on a. Then*

(i) G is of type (I) if and only if each of the symbols b_m and b_M occurs in R' at least two times.

(ii) G is of type (II) if and only if either b_m or b_M occurs in R' exactly once i.e., up to inversion and cyclic permutation of R'

$$R' = \begin{cases} b_M U, & \textit{or} \\ b_m V, \end{cases}$$

where $U = U(b_m, b_{m+1}, \dots, b_{M-1})$ (respectively $V = V(b_{m+1}, \dots, b_M)$) involves b_m (respectively b_M).

(iii) G is of type (III) if and only if both b_m and b_M occur in R' exactly once, i.e., up to a cyclic permutation and inversion of R',

$$R = W_1 b_m W_2 b_M^{\pm 1}$$

where $W_i = W_i(b_{m+1}, \dots, b_{M-1})$, $i = 1, 2$.

Proof. (ii) Recall that an element x of a free group F is *primitive* if it can be included in a basis for F. If the basis is $\{x, y, z, \ldots\}$, then x is called a *primitive element associated with* $\{y, z, \ldots\}$.

Now $A = H$ (respectively $B = H$) if and only if R' is a primitive element associated with $\{b_m, b_{m+1}, \ldots, b_{M-1}\}$ (respectively $\{b_{m+1}, \ldots, b_M\}$) in the free group $F(b_m, b_{m+1}, \ldots, b_M)$, and this holds if and only if $R' = U_1 b_M^{\pm 1} U_2$ (respectively $R' = V_1 b_m^{\pm 1} V_2$) where $U_i = U_i(b_m, b_{m+1}, \ldots, b_{M-1})$ (respectively $V_i = V_i(b_{m+1}, \ldots, b_M)$), $i = 1, 2$. Using cyclic permutations and/or inverse operations if needed we get the conclusion.

(iii) The same arguments used for (ii) can be applied.

(i) Follows immediately by exclusion from (ii) and (iii). □

In the remaining part of this section it will be shown that all groups of exponential growth of types (II) and (III) have uniformly exponential growth (Proposition 2.7). In particular we will show that all amenable one-relator groups of exponential growth have uniformly exponential growth (Proposition 2.6).

The proof of the following statement is trivial.

2.2. LEMMA. *Let G be a finitely generated group, $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_m\}$ two systems of generators. Let*

$$L = \max\left\{|b_i|_A, |a_j|_B : 1 \leq i \leq m,\ 1 \leq j \leq n\right\}.$$

Then

$$\lambda_A(G)^{\frac{1}{L}} \leq \lambda_B(G) \leq \lambda_A(G)^L$$

$$\lambda_B(G)^{\frac{1}{L}} \leq \lambda_A(G) \leq \lambda_B(G)^L.$$

2.3. LEMMA. *Let G be a finitely generated group such that there exists a short exact sequence*

$$1 \longrightarrow F \longrightarrow G \longrightarrow \mathbf{Z} \longrightarrow 1$$

where F is a non abelian free group. Then G has uniformly exponential growth and $\lambda_*(G) \geq \sqrt[6]{3}$.

Proof. Let A be a finite set of generators of G. Set

$$C = \left\{c \in F : \text{ there exist } a_1, a_2 \in A \cup A^{-1} \text{ such that } c = [a_1, a_2]\right\},$$

$$B = \left\{b \in F : \text{ there exist } a \in A \cup A^{-1} \cup \{1\} \text{ and } c \in C \text{ s.t. } b = aca^{-1}\right\},$$

and let F_C (respectively F_B) denote the subgroup of F generated by C (resp. B). Then F_C and F_B are free (as subgroups of a free group), $F_C \leq F_B$ and F_C is non trivial. We are going to show that F_B is not abelian.

Assume that F_C is abelian. Then there exists a simple element (i.e. not a proper power) $t \in F$ generating a cyclic subgroup $T < F$ such that $F_C < T$. If $aca^{-1} \in T$ for all $a \in A \cup A^{-1}$ and $c \in C$, then $aTa^{-1} < T$ for all $a \in A$. Thus T is normal in G, and therefore in F. But this is impossible because non abelian free groups do not have normal cyclic subgroups. It follows that there exist $a \in A \cup A^{-1}$ and $c \in C$ such that aca^{-1} does not commute with c. This shows that F_B is not abelian and

$$\lambda_{A \cup B}(G) \geq \lambda_B(F_B) \geq 3 .$$

As $\lambda_{A\cup B}(G) \leq \lambda_A^6(G)$ by Lemma 2.2, this ends the proof. $\square$

2.4. LEMMA. *Let G be a finitely generated group and suppose we have an exact sequence*

$$1 \longrightarrow F \longrightarrow G \longrightarrow \mathbf{Z} \longrightarrow 1$$

where F is the union of an ascending chain of free groups of rank ≥ 2. Then G has uniformly exponential growth.

Proof. Suppose that $F^{(1)} \leq F^{(2)} \leq F^{(3)} \leq \cdots \leq F^{(n)} \leq F^{(n+1)} \leq \ldots$ and $F = \bigcup_{n=1}^{\infty} F^{(n)}$. Then, with the same notations as in previous lemma, $B \subset F^{(n)}$ for n sufficiently large, and the arguments as above can be applied. $\square$

The following statement is a reformulation of [M: Thm. 2].

2.5. LEMMA. *Consider an exact sequence of groups*

$$1 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 1$$

where A is abelian and B is finitely generated. Suppose there exist $a \in A$ and $b \in B$ such that the group generated by $\{b^i a b^{-i} : i \in \mathbf{Z}\}$ is not finitely generated. Then b and ba generate a free semigroup.

2.6. PROPOSITION. *The group $\Gamma_n = \langle t, s : tst^{-1} = s^n \rangle$, $|n| \geq 2$, has uniformly exponential growth.*

Proof. Consider, for $n \in \mathbf{Z}, n \neq 0$, the abelian group

$$\mathbf{Z}\left[\frac{1}{n}\right] = \left\{ \frac{k}{n^s} : k, s \in \mathbf{Z} \right\}.$$

The group Γ_n is isomorphic to the semidirect product $\mathbf{Z}\left[\frac{1}{n}\right] \times_\phi \mathbf{Z}$, where $\phi : x \longmapsto nx$.

Let A be a finite system of generators for Γ_n and suppose that there exist $\alpha, \beta \in A$ such that $\alpha \in \Gamma_n \setminus \mathbf{Z}\left[\frac{1}{n}\right]$ and $\beta \in \mathbf{Z}\left[\frac{1}{n}\right]$. Then it is easy to check that the subgroup

$$\left\langle \alpha^j \beta \alpha^{-j};\, j \in \mathbf{Z} \right\rangle \leq \mathbf{Z}\left[\frac{1}{n}\right]$$

is not finitely generated. Thus, according to the previous lemma, the set $\{\beta, \beta\alpha\}$ generates a free semigroup and setting $A' = A \cup \{\beta\alpha\}$ one gets

$$\lambda_A(\Gamma_n) \geq \sqrt{\lambda_{A'}(\Gamma_n)} \geq \sqrt{2}\,.$$

Suppose now that the generating system $A = \{a_1, \ldots, a_m\}$ is contained in $\Gamma_n \setminus \mathbf{Z}\left[\frac{1}{n}\right]$. For all $i, j = 1, \ldots, m$ one has $[a_i, a_j] \in \mathbf{Z}\left[\frac{1}{n}\right]$ and since Γ_n is not abelian there exist i_0, j_0 such that $\alpha = [a_{i_0}, a_{j_0}] \neq 0$. Setting $A'' = A \cup \{\alpha, \alpha a_1\}$ one obtains, as before,

$$\lambda_A(\Gamma_n) \geq (\lambda_{A''}(\Gamma_n))^{\frac{1}{5}} \geq 2^{\frac{1}{5}}\,. \qquad \square$$

2.7. PROPOSITION. *If G is a two-generated one-relator group of exponential growth of type (II) or (III) then it has uniformly exponential growth.*

Proof. Consider G as an HNN-extension $H^* = (H; A, B, \phi)$ and denote by N the kernel of the homomorphism $H^* \longrightarrow \mathbf{Z}$, $h \longmapsto \sigma_t(h)$ (here $\sigma_t(h)$ denotes the sum of exponents in h of the stable letter t). Then one has the short exact sequence

$$1 \longrightarrow N \longrightarrow G \longrightarrow \mathbf{Z} \longrightarrow 1\,.$$

If G is of type (II) and H is free non abelian, then $N = \bigcup_{i=1}^{\infty} t^n H t^{-n}$ is an increasing union of free groups of the same rank $m \geq 2$ and Lemma 2.4 can be applied. If $H \cong \mathbf{Z}$ then $G \cong \Gamma_n$, where $|n|$ equals the index of the proper associated subgroup in H and sign(n) = sign($\phi(1)$), and the statement follows from the previous proposition.

If G is of type (III) then $N = H$ is free of finite rank ≥ 2 and Lemma 2.3 can be applied. $\square$

3. Uniformly exponential growth and growth of graded algebras

In this section we describe a method of estimating growth functions of a group in terms of its graded Lie, and associative algebras defined via dimension subgroups. We begin by recalling some concepts and notations.

As in [Gri] considerations were given with respect to a Galois field $\mathbf{GF}_p$, here we modify the arguments for a field of characteristic 0, namely $\mathbf{Q}$.

Let G be a group; denote by $\mathbf{Q}[G]$ the group algebra of G over $\mathbf{Q}$, and by $\Delta \subset \mathbf{Q}[G]$ the augmentation ideal, that is the ideal generated by the elements of the form $g-1$, with $g \in G$. Recall that the *lower central series* of G is the sequence of subgroups $\{\gamma_n(G)\}_{n=1}^{\infty}$ of G defined by $\gamma_1(G)=G$ and, for $n \geq 2$, $\gamma_n(G) = [G, \gamma_{n-1}(G)]$.

The subgroup

$$G_n = \{g \in G : g-1 \in \Delta^n\}$$

is called the *n-th dimension subgroup* of G over $\mathbf{Q}$ and it has the following characterisation due to Jennings [J] (see also [P: IV, Thm. 1.5] or [Pm: 11, Thm. 1.10])

$$G_n = \sqrt{\gamma_n(G)} := \left\{g \in G : \exists k \in \mathbf{N}, g^k \in \gamma_n(G)\right\}.$$

For any group G one defines as usual an associative graded algebra $\mathcal{A}(G)$ and two graded Lie algebras $L(G)$ and $\mathcal{L}(G)$ by

$$\mathcal{A}(G) = \bigoplus_{n=1}^{\infty} \Delta^n/\Delta^{n+1}$$

$$L(G) = \bigoplus_{n=1}^{\infty} \left[(G_n/G_{n+1}) \otimes_{\mathbf{Z}} \mathbf{Q}\right]$$

$$\mathcal{L}(G) = \bigoplus_{n=1}^{\infty} \left[(\gamma_n(G)/\gamma_{n+1}(G)) \otimes_{\mathbf{Z}} \mathbf{Q}\right]$$

(see for instance [P], [Pm]). Quillen's Theorem [Q] states that $\mathcal{A}(G)$ is the universal enveloping algebra of $L(G)$.

Assume now that G is finitely generated and set

$$a_n(G) = \dim(\Delta^n/\Delta^{n+1})$$

$$b_n(G) = \operatorname{rank}(G_n/G_{n+1})$$

$$c_n(G) = \operatorname{rank}(\gamma_n(G)/\gamma_{n+1}(G))$$

where, by rank, we mean the torsion free rank of the corresponding abelian group. Then the following relations hold

$$\sum_{n=0}^{\infty} a_n(G)z^n = \prod_{n=1}^{\infty}(1-z^n)^{-b_n(G)} = \prod_{n=1}^{\infty}(1-z^n)^{-c_n(G)} .$$

The first equality follows easily from Quillen's Theorem [Pm: Thm. 4.10, Chapter 3] and the second one follows from the equality $b_n(G) = c_n(G)$ as proved in [Be].

In [Be] it is also proved that

$$\limsup_{n\longrightarrow\infty} \sqrt[n]{a_n} = \limsup_{n\longrightarrow\infty} \sqrt[n]{c_n} .$$

3.1. LEMMA. *For any finite system of generators A of a group G the following inequality holds:*

$$a_n(G) \leq \gamma_A^G(n), \quad n \geq 1 .$$

Proof. For $x, y \in G$ we have

$$xy - 1 = (x-1) + (y-1) + (x-1)(y-1)$$
$$x^{-1} - 1 = -(x-1) - (x-1)(x^{-1}-1)$$

so that

$$xy - 1 \equiv (x-1) + (y-1) \mod \Delta^2$$
$$x^{-1} - 1 \equiv -(x-1) \mod \Delta^2 .$$

The ideal Δ^n is spanned, over $\mathbf{Q}$, by the elements of the form

$$y_1(x_1-1)y_2(x_2-1)\cdots y_n(x_n-1)y_{n+1} ,$$

where $x_i \in G$ and $y_j \in \mathbf{Q}[G]$, $1 \leq i \leq n$, $1 \leq j \leq n+1$. Since

$$y = \sum_{g\in G} k_g g \equiv \sum_{g\in G} k_g \mod \Delta,\ k_g \in \mathbf{Q}$$

a basis for the quotient space Δ^n/Δ^{n+1} can be chosen among the images modulo Δ^{n+1} of the elements of the form

$$(a_{i_1}-1)(a_{i_2}-1)\cdots(a_{i_n}-1) ,$$

where $a_{i_j} \in A$. But $(a_{i_1}-1)(a_{i_2}-1)\cdots(a_{i_n}-1) = \sum_{g\in G} k'_g g$, where the summation extends over elements g of length at most n with respect to the system of generators A. $\square$

3.2. COROLLARY. *Let G be a finitely generated group and suppose that the ranks of $\gamma_n(G)/\gamma_{n+1}(G)$ grow exponentially. Then G has uniformly exponential growth and the estimate*

$$\lambda_*(G) \geq \limsup_{n \longrightarrow \infty} \sqrt[n]{\operatorname{rank}(\gamma_n(G)/\gamma_{n+1}(G))}$$

holds.

Recall that a group G is *parafree of para-rank* m if it is residually nilpotent and the factors of consecutive groups in its lower central series equal the corresponding ones of a free group of rank m. There are parafree groups which are not isomorphic to free groups [B 2,3].

3.3. PROPOSITION. *A finitely generated parafree group G of para-rank $m \geq 2$ has uniformly exponential growth and*

$$\lambda_*(G) \geq m\,.$$

Proof. It is known (see for instance [MKS: Thms. 5.11 (Witt's Formulae) and 5.12]) that for a free group $\mathbf{F}_m$ the rank of $(\gamma_n(\mathbf{F}_m)/\gamma_{n+1}(\mathbf{F}_m))$ equals the n-th coefficient of the Maclaurin power series of the function $U(z) = 1/(1-mz)$ and the previous corollary can be applied. $\square$

Given a parafree group G of para-rank $m \geq 2$ it would be interesting to compare $\lambda_*(G)$ with $\lambda_*(\mathbf{F}_m) = 2m - 1$.

3.4. PROBLEM. *Is it true that, for a finitely generated para-free group G of para-rank $m \geq 2$ which is not free, one has $\lambda_*(G) > 2m - 1$?*

In order to formulate the next statement we recall the following

3.5. DEFINITION. An element $R \in F$ is said to be *primitive with respect to the lower central series* if, for all $n \geq 2$, it is not an n-th power modulo $\gamma_{\omega(R)+1}(F)$ where $\omega(R)$ is the weight of R. (The latter is defined by $R \in \gamma_{\omega(R)}(F)$ but $R \notin \gamma_{\omega(R)+1}(F)$.)

3.6. THEOREM ([L 1,2]). *Let R be an element of the free group F of finite rank m which is primitive with respect to the lower central series. Denote by $k = \omega(R)$ its weight and by $\langle R \rangle$ the normal closure of R in F. Let $G = F/\langle R \rangle$ and let $\mathcal{L}(F)$ and $\mathcal{L}(G)$ be the corresponding Lie algebras. Let then r be the image of R in $\mathcal{L}_k(F)$, the k-th component of $\mathcal{L}(F)$ and denote by I the ideal of $\mathcal{L}(F)$ generated by r.*

Then I is the kernel of the canonical homomorphism of $\mathcal{L}(F)$ onto $\mathcal{L}(G)$, i.e.

$$\mathcal{L}(G) = \mathcal{L}(F)/I.$$

Moreover for all $n \geq 1$ the abelian group $\mathcal{L}_n(G)$ is a torsion free group whose rank is the n-th coefficient of the Maclaurin power series of the function

$$U(z) = \frac{1}{1 - mz + z^k} \cdot$$

4. MORE ON UNIFORMLY EXPONENTIAL GROWTH OF ONE-RELATOR GROUPS

Any two-generated one-relator group G can be presented in the form $G = \langle a, b : a^k w(a, b) = 1 \rangle$ where $k \in \mathbf{Z}$ and $w(a, b)$ belongs to the commutator subgroup $[F, F]$ of the free group $F = F(a, b)$ freely generated by a and b (this follows from Lemma 1.1). Since a and b constitute a basis in $F/\gamma_2(F)$ and $[a, b]$ generates $\gamma_2(F)/\gamma_3(F)$, one can also present G in the form

$$G = \langle a, b : a^k [a, b]^l w(a, b) = 1 \rangle$$

where $k, l \in \mathbf{Z}$ and $w(a, b) \in \gamma_3(F)$.

In this section we shall see that, under suitable assumptions on k, l and $w(a, b)$, the corresponding group has uniformly exponential growth.

As an application of Labute's Theorem we get the following:

4.1. PROPOSITION. *Let $G = \langle a, b : R(a, b) = 1 \rangle$ be such that R is primitive with respect to $\{\gamma_n(F)\}_{n=1}^{\infty}$ and $R \in \gamma_3(F)$. Then G has uniformly exponential growth.*

Proof. If $\omega(R) \geq 3$, Theorem 3.6 shows that the corresponding function $U(z)$ has a pole z_0 with $0 < z_0 < 1$. It follows that the coefficients $c_n(G)$ grow exponentially. By Corollary 3.2, $\lambda_*(G) > 1$. □

For Proposition 4.3 we need the following notations. Let ξ be a positive rational number such that $\xi \neq 1$ and denote by Q_ξ the smallest subgroup of the additive group of the rationals, which contains 1 and is invariant under multiplication by ξ and ξ^{-1}. In other words if $\xi = \frac{p}{q}$ with $p, q \in \mathbf{Z}$ and $\gcd(p, q) = 1$ then $Q_\xi \equiv \mathbf{Z}[\frac{1}{p}, \frac{1}{q}]$. Consider now the automorphism α of Q_ξ defined by $\alpha(x) = \xi x$, $x \in Q_\xi$. Let $\mathbf{Z}$ act on Q_ξ by powers of α. Denote by $G_\xi = Q_\xi \rtimes_\alpha \mathbf{Z}$ the corresponding semidirect product. The group G_ξ is a two-generated group with system of generators $\{\bar{a}, \bar{b}\}$, where $\bar{a} = 1 \in Q_\xi$ and the element $\bar{b}$ implements the automorphism $\alpha : \bar{b}^{-1} x \bar{b} = \alpha(x)$, $x \in Q_\xi$.

Let now d be a natural number ≥ 2 and set $B_d = \prod_{\mathbf{Z}} \mathbf{Z}_d$. The group $\mathbf{Z}$ acts on B_d by shifts. The corresponding semidirect product $\Gamma(d)$, also denoted by $\mathbf{Z}_d \wr \mathbf{Z}$, is called the *wreath product* of $\mathbf{Z}$ and $\mathbf{Z}_d$. We shall consider $\Gamma(d)$ as generated by $\bar{a} = (\dots, 0, 0, 1, 0, 0, \dots)$ where 1 denotes a generator of $\mathbf{Z}_d$ (in the expression of $\bar{a}$ it appears at the 0-th coordinate place), and by $\bar{b}$, the element which implements the shift.

We have short exact sequences

$$0 \longrightarrow Q_\xi \longrightarrow G_\xi \longrightarrow \mathbf{Z} \longrightarrow 0$$

$$0 \longrightarrow B_d \longrightarrow \Gamma(d) \longrightarrow \mathbf{Z} \longrightarrow 0$$

so that G_ξ and $\Gamma(d)$ are two-step solvable. Slightly modifying the proof of Proposition 2.6 one gets

4.2. LEMMA. *The groups G_ξ and $\Gamma(d)$ have uniformly exponential growth.*

Our last class of two-generated one-relator groups of uniformly exponential growth is determined in the following statement.

4.3. PROPOSITION. *Let $G = \langle a, b; a^k[a, b]^l w(a, b) = 1 \rangle$ with $k, l \in \mathbf{Z}$ and $w(a, b) \in F^{(2)}$ where $F^{(2)} = [[F, F], [F, F]]$ denotes the second commutator subgroup of the free group $F = F(a, b)$ on a and b. Suppose that $(k, l) \notin \{\pm(2, 1), \pm(1, 1), \pm(1, 0), \pm(0, 1)\}$. Then G has uniformly exponential growth.*

Proof. Set $G_{k,l} = \langle a, b; a^k[a, b]^l w(a, b) \rangle$. Set also

$$\overline{G}_{k,l} = \langle a, b; a^k[a, b]^l w(a, b), F^{(2)} \rangle = \langle a, b; a^k[a, b]^l, F^{(2)} \rangle$$

which is a 2-step solvable quotient group of $G_{k,l}$. We shall show that $\overline{G}_{k,l}$ can be mapped homomorphically onto either G_ξ or $\Gamma(d)$ for a suitable positive rational number $\xi \neq 1$ or natural number $d \geq 2$.

Suppose first that $k \neq l, 2l$ and $lk \neq 0$. These assumptions guarantee that $\xi := \left|\frac{l-k}{l}\right| \neq 0, 1$. Then the map $a \longmapsto (\bar{a})^{sgn(\frac{l-k}{l})}, b \longmapsto \bar{b}$ from F onto G_ξ factorizes through $\overline{G}_{k,l}$. Indeed if we suppose, for instance, that $\frac{l-k}{l} > 0$, then the image of $a^k[a,b]^l$ is the number $k + l(-1+\xi) \in \mathbf{Q}_\xi$ which is zero. Thus $\overline{G}_{k,l}$ maps onto G_ξ.

Suppose now that $\gcd(k,l) = d$ or $(k,l) \in \{\pm(d,0), \pm(0,d)\}$ for some $d \geq 2$. Then, the same arguments as before show that $\overline{G}_{k,l}$ can be mapped onto $\Gamma(d)$ via the map $a \longmapsto \bar{a}, b \longmapsto \bar{b}$.

Finally observe that $\overline{G}_{0,0}$ is the free two-generated two-step solvable group $F/F^{(2)}$ and thus maps homomorphically onto $\Gamma(d)$ for any $d \geq 2$.

The proof follows from Lemma 4.2. □

Remark that the two-generated one-relator groups that are not covered by our statements have their relator that can be reduced to one of the form bw, $[a,b]w$ or $ba^{-1}baw$, where $w = w(a,b) \in F^{(2)}$.

Let us finish the paper by the following observation.

In [GrLP] it was conjectured that if G is a group with m generators and p relations, then

$$\lambda_*(G) \geq 2(m-p) - 1.$$

For one-relator groups there is one case when Gromov's conjecture holds true.

4.4. PROPOSITION. *Let $G = \langle a_1, a_2, \ldots, a_m : R(a_1, a_2, \ldots, a_m) = 1\rangle$, with $m \geq 2$, be a one-relator group such that the relator R does not belong to the commutator subgroup F' of the free group F of rank m freely generated by $a_1, a_2, \ldots, a_m$. Then $\lambda_*(G) \geq 2m - 3$.*

Proof. We may assume that G is torsion-free. Indeed if $U, V \in F$ are such $U = V^k$ for some $k \in \mathbf{Z}$, then $U \in F'$ iff $V \in F'$. If the relator R is a proper power, say $R = W^k$, then G maps onto $G_1 = \langle a_1, a_2, \ldots, a_m : W(a_1, a_2, \ldots, a_m) = 1\rangle$, which is torsion-free, and $\lambda_*(G) \geq \lambda_*(G_1)$.

Under our assumptions on R, $H_1(G, \mathbf{Q}) \cong \mathbf{Z}^{m-1}$ and the second rational homology group $H_2(G, \mathbf{Q})$ vanishes.

In [S] it is proven that if $H_2(G, \mathbf{k}) = 0$, where $\mathbf{k}$ is a field, then any subset $\{x_j\} \in G$, whose image in $H_1(G, \mathbf{k})$ is linearly independent, freely generates a free group.

Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite system of generators for G. Then $\overline{X} = \{\bar{x}_1, \ldots, \bar{x}_n\}$, where $\bar{x}_i$ denotes the image of x_i in $H_1(G, \mathbf{Q})$, generates

$H_1(G, \mathbf{Q})$. We can find an independent subsystem $\{\bar{x}_{i_1}, \dots \bar{x}_{i_{m-1}}\}$ in $H_1(G, \mathbf{Q})$ such that its pre-image $\{x_{i_1}, \dots, x_{i_m}\}$ freely generates a free group. Therefore $\lambda_X(G) \geq 2(m-1) - 1 = 2m - 3$. $\square$

It seems to us that for a one-relator group G of rank $m \geq 3$ the inequality $\lambda_*(G) \geq 2m - 3$ cannot be deduced directly from Magnus' Theorem as it is claimed in [GrLP].

ACKNOWLEDGMENTS. This paper was written during our stay at the Section de Mathématiques de l'Université de Genève. We thank Professor Pierre de la Harpe for his invitation and kind hospitality. We also thank him for his encouragement, stimulating conversations and suggestions.

We are grateful for various supports: the CNR for the first author, the RFFI grant 96-01-00713 for the second author, and the "Fonds National Suisse de la Recherche Scientifique" for both of us.

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(Reçu le 21 septembre 1996; version révisée reçue le 16 juin 1997)

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