I. Quaternary Cubic Forms

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accessible. On the other hand, the results for $b = 4$ are scattered in the literature of over 100 years. Hence, we have written an extensive summary of the theory of complex quaternary cubic forms. Being interested in (Cubic forms over $\mathbf{Z})/\mathrm{GL}_b(\mathbf{Z})$, it is more reasonable to consider the action of $\widetilde{\mathrm{SL}}_b(\mathbf{C}) := \{m \in \mathrm{GL}_b(\mathbf{C}) \mid \det(m) = \pm 1\}$. To simplify things we will consider the action of $\mathrm{SL}_b(\mathbf{C})$ instead. This is the content of Part I.

In the second part, we treat the following weakened form of our original problem:

Which quaternary cubic forms can occur as cup forms of simply connected projective threefolds?

For the case $b \leq 3$, we refer the reader to [OV]. In this part, we have collected a number of examples. We also show that there is a simply connected projective threefold with $b_2 = 3$ whose cup form defines a plane cubic with a node, a problem which remained unsolved in [OV]. We conclude our notes by a brief summary of the author's results concerning the non-realizability of certain *real* cubic polynomials as cup forms of projective threefolds.

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I. QUATERNARY CUBIC FORMS

In this section, we will be concerned with the space $S^3(\mathbf{C}^{4^\vee})$ of quaternary cubic forms on which $\mathrm{SL}_4(\mathbf{C})$ acts by substitution of variables. In particular, we will treat the following problems:

1) Find "good" representatives for the orbits in $S^3(\mathbf{C}^{4^\vee})$;

2) Describe the categorical quotient $S^3(\mathbf{C}^{4^\vee})/\!/\,\mathrm{SL}_4(\mathbf{C})$.

(The categorical quotient is an affine algebraic variety whose set of points is in natural bijection with the closed orbits in $S^3(\mathbf{C}^{4^\vee})$. A good introduction to this kind of constructions can be found in [Ne].)

1. Normal Forms for Quaternary Cubic Forms

1.1. *Normal Forms for Quaternary Cubic Forms Defining Non-Singular Cubic Surfaces.* Here, the result is as follows:

THEOREM 1. *Every homogeneous polynomial of degree 3 in four variables defining a non-singular cubic surface can be brought into one of the following canonical forms ($r_i, r, s, t \in \mathbf{C}^*$):*

$$(*) \qquad r_1x_1^3 + r_2x_2^3 + r_3x_3^3 + r_4x_4^3 + r_5(-x_1 - x_2 - x_3 - x_4)^3,$$

$$\text{where } \sum_{i=1}^{5} \pm 1/\sqrt{r_i} \neq 0 \qquad \text{(Sylvester's pentahedral form)}$$

$$(*_1) \qquad r(x_1^3 + x_2^3 + x_3^3 + x_4^3) \qquad \text{(diagonal form)}$$

$$(*_2) \qquad rx_1^3 + x_2^3 + x_3^3 + x_4^3 - 3sx_2x_3x_4,$$

$$\text{where } (s^3 - 1)(s^3 + 8) \neq 0 \qquad \text{(non-equianharmonic form)}$$

$$(*_3) \qquad x_2^3 + x_3^3 + x_4^3 - 3x_1^2(r_2x_2 + r_3x_3 + r_4x_4)$$

$$(*_4) \qquad x_2^3 + x_3^3 + x_4^3 - 3x_1^2(r_1x_1 + r_2x_2 + r_3x_3 + r_4x_4)$$

$$(*_5) \qquad 2rx_1^3 + x_2^3 + x_3^3 - 3x_1(sx_1x_2 + x_1x_3 + tx_4^2),$$

$$\text{where } st(r \pm s^{\frac{3}{2}} \pm 1) \neq 0.$$

For a proof of this theorem, we refer the reader to Segre's book [Se]. We will also call a form being equivalent to a form of type $(*)$ a *Sylvestrian pentahedral form.* Such a form determines a configuration of five planes which is called the *Sylvester pentahedron.* Forms being equivalent to diagonal or non-equianharmonic forms will be called *degenerate Sylvestrian pentahedral forms.*

REMARK 1. Given a cubic form f defining a non-singular cubic surface, one is led to ask to which of the above forms f is equivalent. This problem is related to the geometry of the Hessian surface $H_f = 0$ in the following way:

If the Hessian surface is reducible, there are two possibilities: Either it consists of four different planes or of a cone over a smooth plane cubic and a plane. In the first case, f is equivalent to a diagonal form, and in the second case, f is equivalent to a non-equianharmonic form.

If the Hessian surface is irreducible, we have to look at its singularities. If there are precisely ten A_1-singularities, f is equivalent to a Sylvestrian pentahedral form, and the Sylvester pentahedron is determined by the configuration of the singular points. If there are seven singular points, one A_1-singularity and six A_k-singularities with $k \geq 2$, then f is equivalent to a form $(*_3)$ or $(*_4)$ depending on whether the intersection of the Hessian surface with the tangent cone to the A_1-singularity consists of a double line and an irreducible conic or of a double line and a reducible conic. If there are four singular points on the Hessian surface, then f can be brought into a form of type $(*_5)$. In any case, much information on the canonical form can be read off the configuration of the singular points of $H_f = 0$. We refer the reader to [Se] and [Sch1] for the details.

The following results on canonical forms of quaternary cubic forms can be easily derived from the treatment of Bruce and Wall [BW] of the classification of singular cubic surfaces.

1.2. *Normal Forms for Quaternary Cubic Forms Defining Cubic Surfaces with Isolated Singularities.* Here, the normal form of f depends on the configuration of the singularities on the surface $f = 0$, and we obtain:

THEOREM 2. *The table overleaf lists the normal forms for quaternary cubic forms defining cubic surfaces with isolated singularities. The configuration of singularities on the respective surface is noted in the first column. Here, A_1 etc. refer to the classification of singularities (see e.g. [AGV], 242ff). Thus, $2A_1A_2$ means that there are two A_1-singularities and one A_2-singularity on the respective surface. It is assumed throughout that $l \in \mathbf{C}^*$.*

REMARK 2. The two different forms with a D_4-singularity are again distinguished by the geometry of the Hessian surface: The Hessian surface consists in the first case of a double plane and an irreducible quadric cone and in the second case of a double plane and two simple planes.

A_1	$lx_4(x_2^2 - x_1x_3) +$ $+ x_2\big(x_1 - (1+\rho_1)x_2 + \rho_1x_3\big)\big(x_1 - (\rho_2+\rho_3)x_2 + \rho_2\rho_3x_3\big),$ $\rho_i \in \mathbf{C} \setminus \{0,1\}$ *pairwise different*
$2A_1$	$lx_4(x_2^2 - x_1x_3) + x_2\big(x_1 - (1+\rho_1)x_2 + \rho_1x_3\big)(x_1 - \rho_2x_2),$ $\rho_i \in \mathbf{C} \setminus \{0,1\}$ *not equal*
$3A_1$	$lx_4(x_2^2 - x_1x_3) + x_2^2\big(x_1 - (1+\rho)x_2 + \rho x_3\big),\ \ \rho \in \mathbf{C} \setminus \{0,1\}$
$4A_1$	$lx_4(x_2^2 - x_1x_3) + x_2^2(x_1 - 2x_2 + x_3)$
A_1A_2	$lx_4(x_2^2 - x_1x_3) + x_1x_2\big(x_1 - (1+\rho)x_2 + \rho x_3\big),$ $\rho \in \mathbf{C} \setminus \{0,1\}$
$2A_1A_2$	$lx_4(x_2^2 - x_1x_3) + x_2^2(x_1 - x_2)$
A_12A_2	$lx_4(x_2^2 - x_1x_3) + x_2^3$
A_1A_3	$lx_4(x_2^2 - x_1x_3) + x_1^2x_2 - x_1x_2^2$
$2A_1A_3$	$lx_4(x_2^2 - x_1x_3) + x_1x_2^2$
A_1A_4	$lx_4(x_2^2 - x_1x_3) + x_1^2x_2$
A_1A_5	$lx_4(x_2^2 - x_1x_3) + x_1^3$
A_2	$lx_4x_1x_2 - x_3(x_1^2 + x_2^2 + x_3^2 + \rho_1x_1x_3 + \rho_2x_2x_3),$ $\rho_1, \rho_2 \in \mathbf{C} \setminus \{-2,+2\}$
$2A_2$	$lx_4x_1x_2 - x_3(x_1^2 + x_3^2 + \rho x_1x_3),\ \ \rho \in \mathbf{C} \setminus \{-2,+2\}$
$3A_2$	$lx_4x_1x_2 - x_3^3$
A_3	$lx_4x_1x_2 + x_1(x_1^2 - x_3^2) + \rho x_2(x_2^2 - x_3^2),\ \ \rho \in \mathbf{C}^*$
A_4	$lx_4x_1x_2 + x_1^2x_3 + x_2(x_2^2 - x_3^2)$
A_5	$lx_4x_1x_2 + x_1^3 + x_2(x_2^2 - x_3^2)$
D_4^I	$lx_4x_1^2 + x_2^3 + x_3^3 + x_1x_2x_3$
D_4^{II}	$x_4x_1^2 + x_2^3 + x_3^3$
D_5	$x_4x_1^2 + x_1x_3^2 + x_2^2x_3$
E_6	$x_4x_1^2 + x_1x_3^2 + x_2^3$
$\widetilde{E}_6$	$x_1^3 + x_2^3 + x_3^3 - 3lx_1x_2x_3,\quad l^3 \neq 1$

1.3. *Normal Forms for Quaternary Cubic Forms Defining Irreducible Cubic Surfaces with Non-Isolated Singularities.*

PROPOSITION 1. *The canonical forms for quaternary cubic forms defining irreducible cubic surfaces with non-isolated singularities are the following:*

Canonical form	*The surface $f = 0$*
$x_1^2x_3 + x_2^2x_4$	*Whitney's ruled surface*
$x_1^2x_3 + x_1x_2x_4 + x_2^3$	*Cayley's ruled surface*
$x_1x_3^2 + x_1x_2^2 + x_2^3$	*Cone over a nodal cubic*
$x_1^2x_3 + x_2^3$	*Cone over Neil's parabola*

REMARK 3. Cayley's ruled surface is actually a degeneration of Whitney's surface. Explicit constructions of those surfaces can be found in [Ha1], 330f, for Whitney's surface and in [Ha2], 80, for Cayley's surface.

1.4. *Normal Forms for Quaternary Cubic Forms Defining Reducible Cubic Surfaces.* Here, one obtains the following obvious result:

PROPOSITION 2. *A quaternary cubic form defining a reducible cubic surface can be brought into one of the following canonical forms:*

Canonical form	*The surface $f = 0$*
$(x_1 + x_2)(x_1x_2 + x_3x_4)$	*Non-sing. quadric w. transversal plane*
$x_1(x_1x_2 + x_3x_4)$	*Non-sing. quadric w. tangent plane*
$x_1(x_2^2 + x_3x_4)$	*Quadric cone w. transversal plane*
$x_2(x_2^2 + x_3x_4)$	*Cone over plane conic w. transversal line*
$x_3(x_2^2 + x_3x_4)$	*Cone over plane conic w. tangent*
$x_1x_2x_3$	*Three different planes*
$x_1x_2(x_1 + x_2)$	*Three different planes in a pencil*
$x_1^2x_2$	*Double plane and simple plane*
x_1^3	*Triple plane*

2. The Invariant Theory of Quaternary Cubic Forms

2.1. *Stable, Semistable and Nullforms.* The stable and semistable quaternary cubic forms and the quaternary cubic nullforms were determined by Hilbert [Hi] (for the definition of semistable and stable see [Ne], *nullform* means non-semistable form):

THEOREM 3. i) *A quaternary cubic form f is stable (resp. semistable) if and only if the surface $\{f=0\}$ has at most singularities of type A_1 (resp. A_2).*

ii) *A quaternary cubic form f is a nullform if and only if the surface $\{f=0\}$ has isolated singularities of type A_k ($k \geq 3$), D_4, D_5, E_6, or $\widetilde{E}_6$, or if it has non-isolated singularities.*

2.2. *Degenerations of Orbits of Semistable Forms.* First, one observes that the semistable forms with closed orbit are precisely the forms whose associated cubic surfaces have three A_2-singularities. Applying Luna's slice theorem, one then computes the following table of degenerations where we characterize a form by the configuration of singularities on the corresponding cubic surface:

$$\begin{tikzcd}
A_2 \arrow[dr] & A_2A_1 \arrow[dr] & 2A_1A_2 \arrow[d] \\
 & 2A_2 \arrow[dr] & 2A_2A_1 \arrow[d] \\
 & & 3A_2
\end{tikzcd}$$

The details can be found in [Sch1], 58ff.

2.3. *The Ring of Invariants.* Proofs of the following results can be found in the paper [Be]. We want to describe the ring $A := \mathbf{C}[S^3(\mathbf{C}^{4^\vee})]^{\mathrm{SL}_4(\mathbf{C})}$. This is the coordinate ring of the categorical quotient $S^3(\mathbf{C}^{4^\vee})/\!/\,\mathrm{SL}_4(\mathbf{C})$. It is the ring of polynomial expressions in the coefficients of cubic polynomials which are constant on all $\mathrm{SL}_4(\mathbf{C})$-orbits. In order to describe the ring A, we first introduce the following vector space

$$S := \left\{ r_1x_1^3 + r_2x_2^3 + r_3x_3^3 + r_4x_4^3 + r_5x_5^3 \mid \sum x_i = 0 \right\}.$$

On S, there is a natural action of the alternating group $\mathfrak{A}_5$, and $A \subset \mathbf{C}[S]^{\mathfrak{A}_5}$. This inclusion is constructed as follows: The group of automorphisms H of the Sylvester pentrahedron naturally acts on S, and it can be shown that the natural morphism $S/\!/H \longrightarrow S^3(\mathbf{C}^{4^\vee})/\!/\,\mathrm{SL}_4(\mathbf{C})$ is birational. This induces the inclusion $A \subset \mathbf{C}[S]^H$. Now, H is a finite group of order 480 obviously containing $\mathfrak{A}_5$. Denote by σ_i, $i = 1, 2, 3, 4, 5$, and v the i-th symmetric function and the Vandermonde determinant in the r_i. Then $\mathbf{C}[S]^{\mathfrak{A}_5} = \mathbf{C}[\sigma_1, \dots, \sigma_5, v]$.

THEOREM 4. *The ring of invariants A is the subring of $\mathbf{C}[S]^{\mathfrak{A}_5}$ generated by the following invariant polynomials*

$$I_8 := \sigma_4^2 - 4\sigma_3\sigma_5, \quad I_{16} := \sigma_5^3\sigma_1, \quad I_{24} := \sigma_5^4\sigma_4,$$

$$I_{32} := \sigma_5^6\sigma_2, \quad I_{40} := \sigma_5^8, \quad I_{100} := \sigma_5^{18}v,$$

which satisfy a relation

$$I_{100}^2 = P(I_8, I_{16}, I_{24}, I_{32}, I_{40}).$$

2.4. *The Discriminant.* Using techniques from the paper [BC], one obtains the following

PROPOSITION 3. *The discriminant of quaternary cubic forms is given by the formula*

$$\Delta = (I_8^2 - 64 I_{16})^2 - 2^{11}(I_8 I_{24} + 8 I_{32}).$$

2.5. *Moduli Spaces of Cubic Surfaces.* Define $\overline{\mathcal{M}}$ to be the hypersurface $\{I_{100}^2 - P(I_8, I_{16}, I_{24}, I_{32}, I_{40}) = 0\}$ in the weighted projective space $\mathbf{P}(8, 16, 24, 32, 40) = \mathbf{P}(1, 2, 3, 4, 5)$. Then $\mathcal{M} := \overline{\mathcal{M}} \setminus \{\Delta = 0\}$ is a moduli space for non-singular cubic surfaces. On the other hand, every non-singular cubic surface can be obtained as the blow up of $\mathbf{P}_2$ in six points in general position. The sextuples of points in general position form an open subset $\mathcal{U} \subset S^6\mathbf{P}_2$ of the sixth symmetric power of $\mathbf{P}_2$. Furthermore, there is an action of $\mathrm{PGL}_3(\mathbf{C})$ on $\mathcal{U}$, and the geometric quotient $\mathcal{N} := \mathcal{U}/\!/\,\mathrm{PGL}_3(\mathbf{C})$ does exist [Is]. By [Is], §6, $\mathcal{N}$ is a coarse moduli space for pairs (X, L) consisting of a cubic surface X and a globally generated line bundle L which defines a blow down $X \longrightarrow \mathbf{P}_2$. Forgetting the line bundle L provides us with a morphism $\mathcal{N} \longrightarrow \mathcal{M}$, so that there is a surjection $f\colon \mathcal{U} \longrightarrow \mathcal{M}$. Hence, we can view the invariants of quaternary cubic forms as regular functions on $\mathcal{U}$. This relates the geometry of the cubic surface to the set of six points. One obtains, e.g., PROPOSITION 4. *The set of sextuples in $\mathcal{U}$ whose associated cubic surface is given by an equation which is not a (nondegenerate) Sylvestrian pentahedral form is the Zariski-closed subset* $\{f^*I_{40} = 0\}$.

Of course, a better understanding of the geometric meaning of the other invariants should allow to extend this result.

II. CUBIC FORMS OF PROJECTIVE THREEFOLDS

1. PRELIMINARIES

For the convenience of the reader, we have collected the crucial theorems which we will use in the construction of our examples.

1.1. *The Lefschetz Theorem on Hyperplane Sections.* We summarize Bertini's Theorem and Lefschetz' Theorem in:

THEOREM 5. *Let Y be a projective manifold, L a very ample line bundle on Y, and $X := Z(s)$ the zero-set of a general section $s \in H^0(X, L)$. Then X is a manifold (connected if* $\dim Y \geq 2$*), and the inclusion $\iota\colon X \hookrightarrow Y$ induces isomorphisms*

$$\begin{aligned} \iota^*: H^i(Y, \mathbf{Z}) &\longrightarrow H^i(X, \mathbf{Z}), \qquad && i = 1, \ldots, \dim Y - 2; \\ \iota_*: \pi_i(X) &\longrightarrow \pi_i(Y), \qquad && i = 1, \ldots, \dim Y - 2\,. \end{aligned}$$

Proof. [La], Th. 3.6.7 & Th. 8.1.1. $\square$

1.2. *Formulas for Blow Ups.* A very simple way to obtain a new manifold from a given one is the blow up in a point or along a smooth curve. The cup form behaves as follows (we will suppose for simplicity that $H^2(Y, \mathbf{Z})$ is without torsion):

THEOREM 6. i) *Let $\sigma\colon X \longrightarrow Y$ be the blow up of Y in a point. Let $q(x_1, \ldots, x_n)$ be the cubic polynomial which describes the cup form of Y w. r. t. the basis $(\kappa_1, \ldots, \kappa_n)$ of $H^2(Y, \mathbf{Z})$. If $h_0 \in H^2(X, \mathbf{Z})$ is the cohomology class of the exceptional divisor, then $(h_0, \sigma^*\kappa_1, \ldots, \sigma^*\kappa_n)$ is a basis of $H^2(Y, \mathbf{Z})$ w. r. t. which the cup form of X is given by*

$$x_0^3 + q(x_1, \ldots, x_n)\,.$$