# 6. Toric manifold structures on $mP_+^3(\alpha)(a)$ for m = 4,5,6

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$$M = \begin{pmatrix} a_2 & b_2 \\ a_3 & b_3 \\ a_4 & b_4 \end{pmatrix}$$

for the diagonal $\partial_{2,4} := \rho(2) + \rho(3) + \rho(4)$. The bending flows around two diagonals $\partial_{p,q}$ and $\partial_{p',q'}$ commute if and only if the pairs $\{p,q\}$ and $\{p',q'\}$ intersect or are unlinked in $\mathbf{R}/m\mathbf{Z}$.

## 6. TORIC MANIFOLD STRUCTURES ON ${}^m\mathcal{P}^3_+(\alpha)$ FOR $m = 4, 5, 6$

In this section, we study examples of $\mathcal{P}^3_+(\alpha) \subset {}^m\mathcal{P}^3$ such that the $m-3$ diagonal functions $d_2, \ldots, d_{m-2} : \mathcal{P}^3_+(\alpha) \longrightarrow \mathbf{R}$ never vanish. The whole space $\mathcal{P}^3_+(\alpha)$ consists of prodigal polygons and, by §5, the bending flows give an action of a big (i.e. half-dimensional) torus on $\mathcal{P}^3_+(\alpha)$. By Delzant's theorem (see [De], or [Gu, § 1]), we can construct from the moment polytope $\Delta_\alpha$ alone a toric manifold which is equivariantly symplectomorphic to the space $\mathcal{P}^3_+(\alpha)$. This can be achieved also by [DJ,§ 1.5], though only up to equivariant diffeomorphism. The latter also gives the real part, the planar polygon space $\mathcal{P}^2(\alpha)$, as a $2^{m-3}$-sheeted branched cover of $\Delta_\alpha$. We sum up below some results of these constructions without writing all the details.

Without explicit mention of the contrary, $\alpha$ is supposed to be generic. Contrary to the previous sections, we do not require that the perimeter of our polygons is 2. It was necessary to fix the perimeter in order to define the map $\ell$ and the value 2 is the natural choice to deal with the map $\Phi : \mathbf{V}_2(\mathbf{C}^m) \longrightarrow {}^m\widetilde{\mathcal{P}}^k$. But ${}^m\mathcal{F}^k(\alpha)$ makes sense for any $\alpha \in \mathbf{R}^m_{\geq 0}$ and so do the various moduli spaces ${}^m\mathcal{P}^k(\alpha)$, etc. When $\sum \alpha_i = 2$, the polytope $\Delta_\alpha$ is a slice through the Gel'fand-Cetlin moment polytope $\Gamma_m$ of §5: for general $\alpha$ it is a homothetic copy of this section.

(6.1) $m = 4$: The condition which guarantees that $d_2$ never vanishes is $\alpha_1 \neq \alpha_2$ or $\alpha_3 \neq \alpha_4$. The space of quadrilaterals ${}^4\mathcal{P}^3_+(\alpha)$ is then a compact toric manifold of dimension 2, therefore diffeomorphic to $\mathbf{C}P^1$. The moment map $d_2$ has image the interval $\Delta_\alpha := I_1 \cap I_2$ where

$$I_1 := [|\alpha_1 - \alpha_2|, \alpha_1 + \alpha_2] \quad \text{and} \quad I_2 := [|\alpha_4 - \alpha_3|, \alpha_4 + \alpha_3]\,.$$

The space ${}^4\mathcal{P}^2(\alpha)$ is $\mathbf{R}P^1$. The quadrilateral spaces ${}^4\mathcal{P}^2(\alpha)_+$ have long since been classified (see for instance [Ha]). One has

$${}^4\mathcal{P}^2(\alpha)_+ = \begin{cases} S^1 \sqcup S^1 & \text{when } I_1 \subset I_2 \text{ or } I_2 \subset I_1 \\ S^1 & \text{otherwise} \end{cases}.$$

Observe also that $\alpha$ is generic if and only if the boundaries of the intervals $I_1$ and $I_2$ do not meet.

By the Duistermaat-Heckman Theorem [Gu, §2], the symplectic volume of ${}^4\mathcal{P}^3(\alpha)$ is equal to the length of $\Delta_\alpha$. We would then obtain the same length if we had used the other diagonal $|\rho(2)+\rho(3)|$. This produces a statement of elementary Euclidean geometry: the variation intervals of the two diagonals of a quadrilateral with given sides in $\mathbf{R}^3$ are the same length.

(6.2) $m=5$: Conditions for which both $d_2$ and $d_3$ never vanish are for instance $\alpha_1 \neq \alpha_2$ and $\alpha_4 \neq \alpha_5$. The space of pentagons ${}^5\mathcal{P}^3_+(\alpha)$ is then a toric manifold of dimension 4. The moment polytope $\Delta_\alpha \in \mathbf{R}^2$ for $(d_2, d_3)$ is the intersection of the rectangle $I_\alpha$

$$I_\alpha := [|\alpha_1 - \alpha_2|\ ,\ \alpha_1 + \alpha_2] \times [|\alpha_5 - \alpha_4|, \alpha_5 + \alpha_4]$$

with the non-compact rectangular region

$$\Omega_\alpha := \{(x,y) \in (\mathbf{R}_{\geq 0})^2 \mid x+y \geq \alpha_3 \quad \text{and} \quad y \geq x - \alpha_3 \quad \text{and} \quad y \leq x + \alpha_3\}.$$

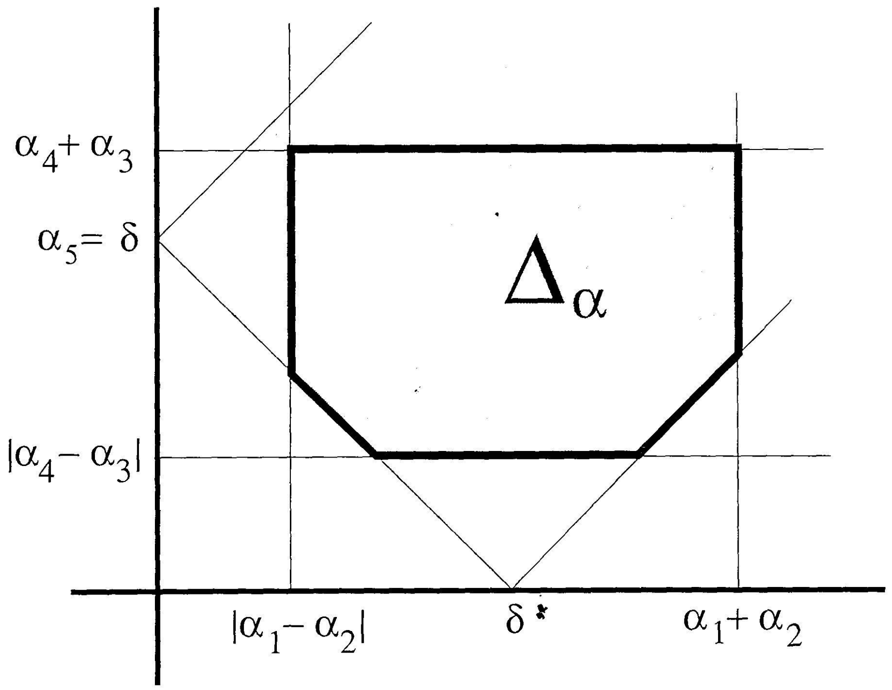


FIGURE 2: The moment polytope $\Delta_\alpha$

(see Figure 2). One sees that $\Delta_\alpha$ has at most 7 sides. The generic $\alpha$ are exactly those for which the boundary of $\Omega_\alpha$ contains no corner of $I_\alpha$ and ${}^5\mathcal{P}^3_+(\alpha)$ is then obtained by symplectic blowings up from $\mathbf{C}P^2$ or $S^2 \times S^2$. The space of planar polygons ${}^5\mathcal{P}^2_+(\alpha)$ is a closed surface obtained by gluing 4 copies of $\Delta_\alpha$ and its Euler characteristic is given by the formula

$$\chi\big({}^5\mathcal{P}^2(\alpha)\big) = 4 - \#\ (\text{sides of } \Delta_\alpha)$$

(see [DJ], Example 1.20) and is orientable if and only if $I_\alpha \subset \omega_\alpha$. One has of course $\chi\big({}^5\mathcal{P}^2_+(\alpha)\big) = 2\chi\big({}^5\mathcal{P}^2(\alpha)\big)$ and ${}^5\mathcal{P}^2_+(\alpha)$ is an orientable surface (${}^m\mathcal{P}^k_+(\alpha)$ is always orientable). The possible cases, depending on the number of sides of $\Delta_\alpha$, are summed up in the following table.

| # of sides | $\mathcal{P}^3_+(\alpha)$ | $\mathcal{P}^2(\alpha)$ | $\mathcal{P}^2_+(\alpha)$ | Ex. of $\alpha$ |
|---|---|---|---|---|
| 3 | $\mathbf{C}P^2$ | $\mathbf{R}P^2$ | $S^2$ | (2,1,5,1,2) |
| 4 | a) $\mathbf{C}P^2 \# \overline{\mathbf{C}P^2}$ | Klein bottle | $T^2$ | (3,2,5,1,2) |
| | or | | | |
| | b) $S^2 \times S^2$ | $T^2$ | $T^2 \sqcup T^2$ | (3,1,3,1,3) |
| 5 | $(S^2 \times S^2) \# \overline{\mathbf{C}P^2}$ | $T^2 \# \mathbf{R}P^2$ | $\Sigma_2$ | (2,1,3,1,2) |
| 6 | $(S^2 \times S^2) \# 2\overline{\mathbf{C}P^2}$ | $T^2 \# 2\mathbf{R}P^2$ | $\Sigma_3$ | (2,1,1,1,2) |
| 7 | $(S^2 \times S^2) \# 3\overline{\mathbf{C}P^2}$ | $T^2 \# 3\mathbf{R}P^2$ | $\Sigma_4$ | (4,3,4,3,4) |

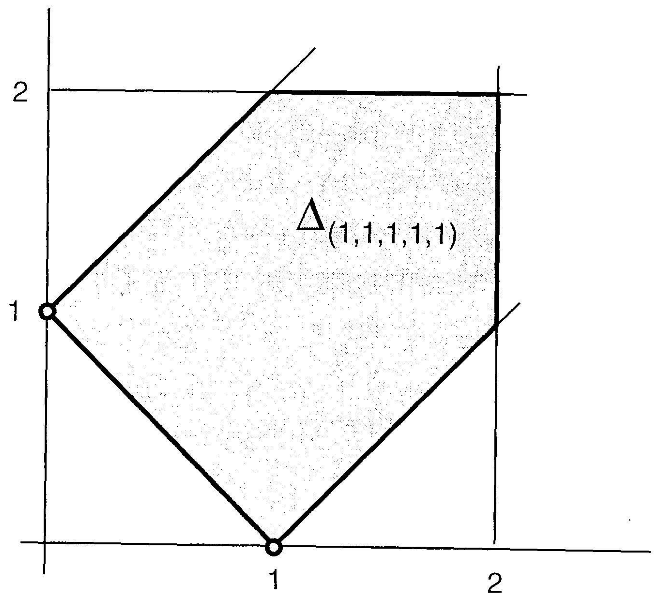


FIGURE 3: $\Delta_{(1,1,1,1,1)}$

(6.3) Some embeddings of the regular pentagon $\alpha = (1,1,1,1,1)$ are not prodigal. However none are lined and thus the moduli space $V_0 := {}^5\mathcal{P}^3(\alpha)$ is diffeomorphic for small $\varepsilon$ to $V_\varepsilon$ where $V_\varepsilon := {}^5\mathcal{P}^3(\alpha_\varepsilon)$ and

$\alpha_\varepsilon := (1+\varepsilon, 1, 1, 1, 1+\varepsilon)$. The moment polytope for $\alpha_\varepsilon$ has then 7 sides and thus $V_0 \simeq V_\varepsilon$ is diffeomorphic to $(S^2 \times S^2) \# 3\overline{\mathbf{C}P^2}$ (if $k=2$, ${}^5\mathcal{P}^2(\alpha)_+ \simeq \Sigma_4$). The "limit moment polytope" $\Delta_{(1,1,1,1,1)}$ is shown in Figure 3.

The pre-image in $V_\varepsilon$ of the segments $\{x = \varepsilon\} \cap \Delta'_\alpha$ and $\{y = \varepsilon\} \cap \Delta'_\alpha$ are 2-spheres of symplectic volume proportional to $\varepsilon$, by the Duistermaat-Heckman Theorem. Passing to the limit $V_0$, these spheres become Lagrangian, and so cannot be complex. This shows that the action of the bending torus is not complex — these polygon spaces are only equivariantly symplectomorphic, not equivariantly isometric, to toric varieties.

(6.4) Any class $r \in {}^5\mathcal{P}^{k=2,3}(\alpha)$ has a unique representative in $\rho \in {}^5\widetilde{\mathcal{P}}^k(\alpha)$ with $\rho(5) = (-\alpha_5, 0, 0)$ and $\gamma(r) := \rho(1) + \rho(2)$ in the half-plane $\mathcal{H} = \{z = 0, y \geq 0\}$. This provides a map $\gamma : {}^5\mathcal{P}^3(\alpha) \longrightarrow \mathcal{H}$ whose image $\widetilde{\Delta}_\alpha$ is the intersection $R_1 \cap R_2 \cap \mathcal{H}$ where $R_1$ and $R_2$ are the rings

$$R_1 := \{v \in \mathbf{R}^2 \mid |\alpha_1 - \alpha_2| \leq |v| \leq \alpha_1 + \alpha_2\},$$
$$R_2 := \{v \in \mathbf{R}^2 \mid |\alpha_4 - \alpha_3| \leq |v| \leq \alpha_4 + \alpha_3\}.$$

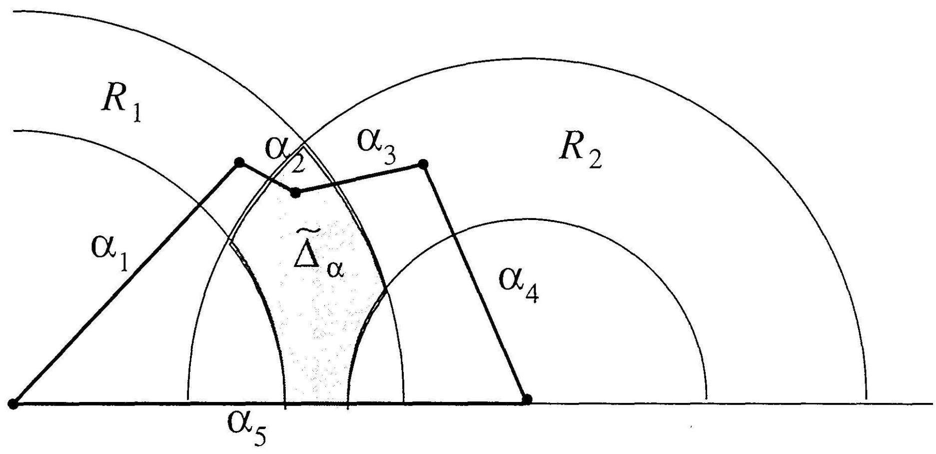


FIGURE 4: $\widetilde{\Delta}_\alpha$

The idea of reconstructing ${}^5\mathcal{P}^2(\alpha)$ by gluing copies of $\widetilde{\Delta}_\alpha$ goes back to the early works of W. Thurston on planar linkages (see [TW, p.100]). The relationship with our theory is the following: the domain $\widetilde{\Delta}_\alpha$ is straightened up into a PL-polytope $\Delta_\alpha$ in $\mathbf{R}^2$ by the map $v \mapsto (|v|, |v - (0, \alpha_5)|)$ and $\Delta_\alpha$ is just the moment polytope for the bending Hamiltonians $\partial_1(\rho) = |\rho(1) + \rho(2)|$ and $\partial_2(\rho) = |\rho(3) + \rho(4)|$.

(6.5) $m = 6$: The conditions $\alpha_1 \neq \alpha_2$ and $\alpha_5 \neq \alpha_6$ imply that $d_2$ and $d_4$ never vanish. However, one cannot guarantee generically $d_3 \neq 0$. But we can replace the $d = (d_1, d_2, d_3)$ by $\delta := (\partial_1, \partial_2, \partial_3)$ where

$$\partial_1 := d_1 = |\rho(1)+\rho(2)| \;,\; \partial_2 := |\rho(3)+\rho(4)| \;,\; \partial_3 := d_3 = |\rho(5)+\rho(6)|$$

and guarantee non-vanishing of the $\delta_i$'s by the generic condition $\alpha_{2i-1} \neq \alpha_{2i}$. Observe that $\partial_i \circ \Phi : \mathbf{V}_2(\mathbf{C}^m) \longrightarrow \mathbf{R}$ $(i = 1, 2, 3)$ are the functions on $\mathbf{V}_2(\mathbf{C}^m)$ given (on $(a, b) \in \mathbf{V}_2(\mathbf{C}^m)$) by the difference of the eigenvalues of the $(2 \times 2)$-matrices $M_i^* M_i$, where

$$M_1 := \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \qquad M_2 := \begin{pmatrix} a_3 & b_3 \\ a_4 & b_4 \end{pmatrix} \qquad M_3 := \begin{pmatrix} a_5 & b_5 \\ a_6 & b_6 \end{pmatrix}.$$

The moment polytope in $\mathbf{R}^3$ is the intersection of the rectangular parallelepiped

$$I_\alpha := [|\alpha_1 - \alpha_2|, \alpha_1 + \alpha_2] \times [|\alpha_4 - \alpha_3|, \alpha_4 + \alpha_3] \times [|\alpha_6 - \alpha_5|, \alpha_6 + \alpha_5]$$

with the region

$$\Omega := \{(x, y, z) \in \mathbf{R}^3 \mid 0 \le z \le x + y \;,\; 0 \le x \le y + z \text{ and } 0 \le y \le x + z\}.$$

The domain $\Omega$ can be described as the convex hull of the three half-lines

$$\{0 \le x = y \text{ and } z = 0\} \;,\; \{0 \le y = z \text{ and } x = 0\} \;,\; \{0 \le z = x \text{ and } y = 0\}$$

or the cone $\mathbf{R}_+ \cdot \Xi_3$ on the hypersimplex $\Xi_3$. The polytope $\Delta_\alpha$ has then at most 9 facets. The length-system $\alpha$ is generic when the boundary of $\Omega$ does not contain corners of $I_\alpha$. As 6 is even, the regular hexagon is not generic: ${}^6\mathcal{P}^1(1, \dots, 1)$ contains 10 elements.

(6.6) The bending flows $\partial$ occuring in (6.4) and 6 admit the following generalization. For $m = 2n - 1$ or $2n$, we define the *even-step* map $e : {}^m\mathcal{F}^k \longrightarrow {}^n\mathcal{F}^k$ by $e(\rho)(i) := \rho(2i-1) + \rho(2i)$ taking $e(\rho)(n) := \rho(m)$ if $m$ is odd. We also call $e$ the induced maps ${}^m\widetilde{\mathcal{P}}^k \xrightarrow{e} {}^n\widetilde{\mathcal{P}}^k$, ${}^m\mathcal{P}^k_+ \xrightarrow{e} {}^n\mathcal{P}^k_+$ and ${}^m\mathcal{P}^k \xrightarrow{e} {}^n\mathcal{P}^k$. We call $\rho \in {}^m\mathcal{F}^k$ *even generic* if $e(\rho)$ is a proper polygon. Above the space of proper polygons, the map $e$ is a smooth locally trivial bundle whose fiber is a product of $(k-1)$-spheres. Define $\partial = (\partial_1, \dots, \partial_n) : {}^m\mathcal{F}^k \longrightarrow \mathbf{R}^n$ by $\partial := \ell \circ e$. The map $\partial$ gives the side lengths of the new polygon $e(\rho)$. It is always continuous and smooth when $e(\rho)$ is a proper polygon. As the map $e$ is a submersion on even-generic polygons, the critical values of $\partial$ are the same as those of $\ell$, the walls of 4.3. As for the map $\ell$, the map $\partial$ can be defined on each ${}^m\mathcal{P}^k(\alpha)$. Call $\alpha \in \mathbf{R}^m$ *even generic* if ${}^m\mathcal{P}^k(\alpha)$ only consists of even-generic polygons. For instance, $\alpha$ is even-generic if $\alpha_{2i-1} \neq \alpha_{2i}$ for all $i$. When $k = 3$, $\partial$ is a moment map for the corresponding bending action of $T^n$ defined on even-generic polygons.

Restrict to ${}^m\mathcal{P}^3(\alpha)_+$ for an even-generic $\alpha$. Define the right-angled polytope

$$I_\alpha := \prod_{i=1}^{n} [|\alpha_{2i} - \alpha_{2i-1}|, \alpha_{2i} + \alpha_{2i-1}]$$

and consider the convex polytope $\Delta_\alpha \subset \mathbf{R}^n$

$$\Delta_\alpha := \begin{cases} I_\alpha \cap (\mathbf{R}_+ \cdot \Xi_n) & \text{when } m = 2n \\ I_\alpha \cap (\mathbf{R}_+ \cdot \Xi_n) \cap \{x_n = |\rho(m)|\} & \text{when } m = 2n-1 \end{cases}.$$

PROPOSITION 6.7. *1) The image of $\partial : {}^m\mathcal{P}^k(\alpha)_+ \longrightarrow \mathbf{R}^n$ is the whole polytope $\Delta_\alpha$.*

*2) If $x \in \Delta_\alpha$ is a regular value of $\partial$, the even-step map $e$ induces, for $m = 3$, a symplectomorphism from the symplectic reduction $T^n \backslash \partial^{-1}(x)$ onto ${}^n\mathcal{P}^k_+(x)$.* □

## 7. REMARKS AND OPEN PROBLEMS

(7.1) Is there an octonionic version of Section 3 ? Alternately, are there $U_1(\mathbf{H})$ bendings in dimension 5 (like the $U_1(\mathbf{C})$ bending flows in dimension 3 and $U_1(R)$ flippings in dimension 2) ?

(7.2) Observe that the inclusion ${}^m\mathcal{P}^k \subset {}^m\mathcal{P}^{k+1}$ becomes a bijection when $k \geq m-1$ (triangles are always planar, etc.). In what ways are these spaces ${}^m\mathcal{P}^{m-1}$ more natural than the unstable ones ?

(7.3) The $m$-polygons whose first diagonal is of a given length forms a sphere bundle over a space of $(m-1)$-polygons. (For $k = 3$ this is just symplectic reduction by the first bending circle.) This gives an inductive way to construct the space of $m$-polygons by gluing together (sphere bundles over) the spaces of $(m-1)$-polygons; it would require identification of these sphere bundles, which in $k = 3$ might be done using the Duistermaat-Heckman theorem (where the circle bundle is determined by its Euler class).

Alternately one might work out the fibers of the whole map $d$ of section 5. Unfortunately in dimensions above 3 these are always singular (at, in particular, the planar polygons).

(7.4) In [KM1] and [Wa] there are presented "wall-crossing arguments" for identifying the spaces ${}^m\mathcal{P}^2(\alpha)$. It would be nice to relate these to a combination of [Du] and the paper [GS2], which presents its own wall-crossing arguments for any symplectic reduction by a torus.