§3. Presentations for F

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§3. Presentations for F

Two presentations for F will be given in this section.

Now two groups F_1 and F_2 will be defined by generators and relations. The generators $A, B, X_0, X_1, X_2, \ldots$ will be referred to as *formal symbols*, as opposed to the functions defined above. Given elements x, y in a group, $[x,y] = xyx^{-1}y^{-1}$.

$$F_1 = \langle A, B : [AB^{-1}, A^{-1}BA], [AB^{-1}, A^{-2}BA^2] \rangle$$
$$F_2 = \langle X_0, X_1, X_2, \ldots : X_k^{-1}X_nX_k = X_{n+1} \quad \text{for} \quad k < n \rangle$$

THEOREM 3.1. *There exists a group isomorphism from F_1 to F_2 which maps A to X_0 and B to X_1.*

Proof. There is a group homomorphism from the free group generated by the formal symbols A and B to F_2 such that A maps to X_0 and B maps to X_1. This homomorphism is surjective because $X_n = X_0^{-(n-1)}X_1X_0^{n-1}$ for $n \geq 2$. To see that the defining relations of F_1 are in the kernel of this homomorphism, note that

$$X_1^{-1}X_2X_1 = X_0^{-1}X_2X_0 \qquad \text{and} \qquad X_1^{-1}X_3X_1 = X_0^{-1}X_3X_0\,,$$

hence

$$[X_0X_1^{-1}, X_2] = 1 \qquad \text{and} \qquad [X_0X_1^{-1}, X_3] = 1\,,$$

hence

$$[X_0X_1^{-1}, X_0^{-1}X_1X_0] = 1 \qquad \text{and} \qquad [X_0X_1^{-1}, X_0^{-2}X_1X_0^2] = 1\,.$$

Thus to complete the proof of Theorem 3.1 it suffices to prove that there exists a group homomorphism from F_2 to F_1 which maps X_0 to A and X_1 to B. To prove this it in turn suffices, after setting $Y_0 = A$ and $Y_n = A^{-(n-1)}BA^{n-1}$ for $n \geq 1$, to prove that

$$Y_k^{-1}Y_nY_k = Y_{n+1} \qquad \text{for } k < n\,. \tag{3.2}$$

A closely related statement is that

$$[A^{-1}B, Y_m] = 1 \qquad \text{for } m \geq 3\,. \tag{3.3}$$

Lines (3.2) and (3.3) will be proved in this paragraph. To see that line (3.3) is true for $m = 3$ note that

$$[AB^{-1}, A^{-1}BA] = 1 \Rightarrow A^{-1}[AB^{-1}, A^{-1}BA]A = 1 \Rightarrow [B^{-1}A, A^{-2}BA^2] = 1$$
$$\Rightarrow [A^{-1}B, A^{-2}BA^2] = 1 \Rightarrow [A^{-1}B, Y_3] = 1.$$

The same argument gives line (3.3) for $m = 4$. The following equations show that line (3.2) is true if line (3.3) is true for $m = n - k + 2$.

$$\begin{aligned} Y_nY_k &= A^{-n+1}BA^{n-1}A^{-k+1}BA^{k-1} = A^{-k+2}A^{-(n-k+1)}BA^{n-k+1}A^{-1}BA^{k-1} \\ &= A^{-k+2}Y_{n-k+2}A^{-1}BA^{k-1} = A^{-k+2}A^{-1}BY_{n-k+2}A^{k-1} \\ &= A^{-k+1}BA^{k-1}A^{-k+1}Y_{n-k+2}A^{k-1} = Y_kY_{n+1} \end{aligned}$$

Thus line (3.2) is true for every positive integer n and $k = n-1$. In particular, $Y_3^{-1}Y_4Y_3 = Y_5$. Because line (3.3) is true for $m = 3$ and $m = 4$, it follows that line (3.3) is true for $m = 5$. An obvious induction argument now gives line (3.3) for every $m \geq 3$. This proves lines (3.2) and (3.3).

The proof of Theorem 3.1 is now complete. □

THEOREM 3.4. *There exist group isomorphisms from F_1 and F_2 to F which map the formal symbols $A, B, X_0, X_1, X_2, \ldots$ to the corresponding functions in F.*

Proof. Example 1.2 shows that the interior of the support of the function AB^{-1} in F is disjoint from the supports of the functions $A^{-1}BA$, $A^{-2}BA^2$ in F, and so the functions A, B in F satisfy the defining relations of F_1. Thus there exists a group homomorphism from F_1 to F which maps the formal symbols A, B to the corresponding functions in F. Corollary 2.6 shows that this group homomorphism is surjective. Theorem 3.1 shows that this surjective group homomorphism induces a surjective group homomorphism from F_2 to F which maps the formal symbols $X_0, X_1, X_2, \ldots$ to the corresponding functions in F. To prove Theorem 3.4 it suffices to prove that this latter group homomorphism is injective.

It will be proved that this latter group homomorphism is injective in this paragraph. The defining relations of F_2 imply that

$$X_k^{-1}X_n = X_{n+1}X_k^{-1}, \qquad X_n^{-1}X_k = X_kX_{n+1}^{-1}, \qquad X_nX_k = X_kX_{n+1} \qquad \text{for } k < n.$$

It follows that every nontrivial element x of F_2 can be expressed as a positive element times a negative element as in Corollary-Definition 2.7. If X_k occurs in both the positive and negative part of x but X_{k+1} occurs in neither, then because $X_kX_{n+1}X_k^{-1} = X_n$ for $n > k$, it is possible to simplify x by deleting one occurrence of X_k from both the positive and negative part of x and replacing every occurrence of X_{n+1} in x by X_n for $n > k$. Thus every nontrivial element of F_2 can be put in normal form as in Corollary-Definition 2.7. It follows from Corollary-Definition 2.7 that every nontrivial element of F_2 maps to a nontrivial element of F, as desired.

This proves Theorem 3.4. □