2. Functions of moderate growth

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2. Functions of moderate growth

In this section, we investigate uniform distribution on divisors and effective ppl upper bounds for the discrepancy in the case of functions f for which the sets $\mathscr{A}(z; f)$ defined in (2) may be tackled by Theorem 4 or techniques of similar strength.

We say that a function $f: \mathbf{R}^+ \to \mathbf{R}^+$ has *moderate growth* if it satisfies

$$f(t) \ll R(t^{o(1)}) \quad (t \to \infty) \tag{30}$$

for some increasing function R satisfying (6) and having the property that

$$\exists b > 0 : R(\sqrt{t}) \ll R(t)^{1-b} \quad (t \geqslant 1)\,. \tag{31}$$

An easy calculation shows that this implies $R(x) \gg \exp\{(\log x)^c\}$ for some positive c.

Our first result establishes a connection between usual uniform distribution modulo 1 and uniform distribution on divisors. It was announced, with a sketched proof (and incidentally a slightly deficient statement), in [13].

THEOREM 9 (Hall & Tenenbaum). *Let* $F: \mathbf{R}^+ \to \mathbf{R}^+$ *be differentiable and satisfy*

(i) $F'(x) = o(1) \quad (x \to \infty)$,

(ii) $\{F(n)\}_{n=1}^{\infty}$ *is uniformly distributed modulo 1.*

Suppose that $\theta: \mathbf{R}^+ \to \mathbf{R}^+$ *has moderate growth and is ultimately of class* C^1. *Furthermore assume that, for large* x,

$$x \mapsto x\theta'(x) \text{ is monotonic, and } \theta(x) \ll x\theta'(x)\log x\,. \tag{32}$$

Then $f := F \circ \theta$ *is* erd.

Proof. We observe that the assumptions on $\theta(x)$ imply that $\theta(x) \to \infty$ and in fact $\theta(x) \gg (\log x)^c$ for some positive c. Moreover, we may modify θ on any fixed, finite interval and hence assume without loss of generality that $\theta'(x)$ exists and is positive for all $x > 0$, and that (32) holds for all $x > \frac{3}{2}$.

Let $z \in (0, 1)$. We shall show that $\mathrm{D}\mathscr{A}(z; f) = z$, which implies the stated result in view of Theorem 1: indeed, the cases $z = 0$ or 1 then follow by a straightforward argument. For fixed $\varepsilon \in (0, \min(z, 1-z))$, we set

$$\mathscr{A}^{\pm}(\varepsilon) := \{d \geqslant 1 : \langle F([\theta(d)])\rangle \leqslant z \pm \varepsilon\}\,.$$

Our first aim is to prove that

(33) $$\delta\mathscr{A}^{\pm}(\varepsilon) = z \pm \varepsilon .$$

We only consider $\mathscr{A}^{+}(\varepsilon)$ since the other case is similar. Let x be large and put $N = [\theta(x)]$. Denoting by ψ the inverse function of θ, we have

(34) $$\sum_{\substack{d \leqslant x \\ d \in \mathscr{A}^{+}(\varepsilon)}} \frac{1}{d} = \sum_{\substack{1 \leqslant n < N \\ \langle F(n)\rangle \leqslant z+\varepsilon}} \sum_{\psi(n) < d \leqslant \psi(n+1)} \frac{1}{d} + O(1 + \log(x/\psi(N))) ,$$

where the error term corresponds to those d with $d \leqslant \psi(1)$ or $\psi(N) \leqslant d \leqslant x$.

The inner sum is

$$a(n) + O(1/\psi(n)), \quad \text{with} \quad a(n) := \log(\psi(n+1)/\psi(n)) .$$

Since θ has moderate growth, we certainly have $\theta(x) \ll x^{o(1)} \ll \sqrt{x}$, whence $\psi(n) \gg n^2$. Therefore the double sum on the right-hand side of (34) is equal to

(35) $$\sum_{\substack{1 \leqslant n < N \\ \langle F(n)\rangle \leqslant z+\varepsilon}} a(n) + O(1) = (z+\varepsilon) \sum_{n<N} a(n) + \sum_{n<N} \chi(n) a(n) + O(1) ,$$

where

$$\chi(n) := \begin{cases} 1 - (z+\varepsilon) & \text{if } \langle F(n)\rangle \leqslant z+\varepsilon, \\ -(z+\varepsilon) & \text{otherwise.} \end{cases}$$

We have $t\theta'(t) \geqslant t_0\theta'(t_0) \gg 1$ for $t \geqslant t_0$, so $\theta(x) - \theta(cx) > 1$ for sufficiently small c and large x. This implies $cx \leqslant \psi(N) \leqslant x$ and hence

$$\log(x/\psi(N)) \ll 1, \qquad \sum_{n<N} a(n) = \log\psi(N) + O(1) = \log x + O(1) .$$

Inserting these estimates into (35), we see that proving (33) reduces to showing the asymptotic formula

(36) $$\sum_{n<N} \chi(n) a(n) = o(\log x) .$$

This will follow by partial summation, noting that the assumption that $\{F(n)\}_{n=1}^{\infty}$ is uniformly distributed modulo 1 immediately implies

(37) $$H(y) := \sum_{n \leqslant y} \chi(n) = o(y) .$$

We first observe that we have for all $y \leqslant N-1$

$$a(y) = \int_y^{y+1} \frac{\psi'(t)}{\psi(t)}\,\mathrm{d}t = \int_y^{y+1} \frac{\mathrm{d}t}{\psi(t)\theta'(\psi(t))} \tag{38}$$
$$\ll \int_y^{y+1} \frac{\log\psi(t)}{t}\,\mathrm{d}t \ll \frac{\log\psi(y+1)}{y} \ll \frac{\log x}{y},$$

where we have used (32) in the third stage. Now the left-hand side of (36) is

$$\int_1^{N-1} a(y)\,\mathrm{d}H(y) = a(N-1)H(N-1) - \int_1^{N-1} a'(y)H(y)\,\mathrm{d}y\,.$$

By (37) and (38), and since

$$a'(y) = 1/\psi(y+1)\theta'(\psi(y+1)) - 1/\psi(y)\theta'(\psi(y))$$

has constant sign for large y by the monotonicity assumption on $x\theta'(x)$, this is

$$\ll \frac{\log x}{N}\,o(N) + o\left(\int_1^{N-1} a'(y)y\,\mathrm{d}y\right) = o(\log x) + o\left(\int_1^{N-1} a(y)\,\mathrm{d}y\right),$$

where we estimated the integral over $a'(y)$ by another partial summation. Now

$$\int_1^{N-1} a(y)\,\mathrm{d}y = \int_1^{N-1}\int_y^{y+1} \frac{\psi'(t)}{\psi(t)}\,\mathrm{d}t\,\mathrm{d}y \leqslant \int_1^{N} \frac{\psi'(t)}{\psi(t)}\,\mathrm{d}t = \log x + O(1)\,.$$

This shows that (36) holds and hence establishes (33).

We may now apply Theorem 4 to the sequences $\mathscr{A}^{\pm}(\varepsilon)$: indeed they are composed of at most $[\theta(x)]+1$ blocks, and this has the required order of magnitude since θ is of moderate growth. Thus we obtain

$$\mathrm{D}\mathscr{A}^{\pm}(\varepsilon) = z \pm \varepsilon\,. \tag{39}$$

From the facts that $\theta(d) \to \infty$ and $F'(x) = o(1)$, we deduce that, for each $\varepsilon > 0$, there exists a $d_0(\varepsilon)$ such that

$$|F([\theta(d)]) - F(\theta(d))| < \varepsilon \quad (d > d_0(\varepsilon))\,.$$

This implies that for all n

$$\tau(n, \mathscr{A}^-(\varepsilon)) - d_0(\varepsilon) \leqslant \tau(n, \mathscr{A}) \leqslant \tau(n, \mathscr{A}^+(\varepsilon)) + d_0(\varepsilon) ,$$

whence, in view of (39),

$$\{z - \varepsilon + o(1)\}\tau(n) \leqslant \tau(n, \mathscr{A}) \leqslant \{z + \varepsilon + o(1)\}\tau(n) \qquad \text{pp} .$$

Since ε is arbitrary, a routine argument yields $\tau(n, \mathscr{A}) = \{z + o(1)\}\tau(n)$ pp, as required. This completes the proof of Theorem 9.

The following corollary was also stated (with a slight oversight in the monotonicity assumption) in [13].

COROLLARY 4 (Hall & Tenenbaum). *Let* $f: \mathbf{R}^+ \to \mathbf{R}^+$ *be differentiable and such that, for some function* $\theta(x)$ *satisfying the conditions of Theorem 9,*

(i) $\theta'(x)/f'(x) + 1/xf'(x)$ *is ultimately monotonic,*

(ii) $|\theta'(x)/f'(x)| + |xf'(x)| = o(\theta(x)) \quad (x \to \infty)$.

Then f *is* erd.

Proof. Set $\theta_1(x) := \theta(x) \log x$. Then θ_1 also satisfies the assumptions of Theorem 9. The only condition which is non-trivial to check is that (30) holds for $f = \theta_1$; however we have by (31), for some function $\eta(x) \to 0$ sufficiently slowly,

$$\theta_1(x) \ll R(x^{\eta(x)}) \log x \ll R(x^{2\eta(x)})^{1-b} \log x \ll R(x^{2\eta(x)}) .$$

We also observe that $\theta_1'(x)/\theta_1(x) \asymp \theta'(x)/\theta(x)$ by (32).

It is clear that θ_1 is ultimately strictly increasing, and hence ultimately one-to-one. Let ψ_1 denote the inverse of θ_1, and put $F(x) = f \circ \psi_1(x)$ for sufficiently large x, so that $f(x) = F(\theta_1(x))$. We want to apply Theorem 9 to F and hence must check that $F'(x) = o(1)$ and that $\{F(n)\}_{n=1}^{\infty}$ is uniformly distributed modulo 1.

By a well-known criterion of Fejér (see e.g. Rauzy [20], corollary II.1.2) we only need to prove, in addition to $F'(x) = o(1)$, that F' is ultimately monotonic and that $xF'(x) \to \infty$. Since, for large x,

$$F'(\theta_1(x)) = f'(x)/\theta_1'(x) \asymp f'(x)\theta(x)/\theta'(x)\theta_1(x) \ll xf'(x)/\theta(x) ,$$
$$\theta_1(x)F'(\theta_1(x)) \asymp f'(x)\theta(x)/\theta'(x) ,$$

our assumption (ii) implies that $F'(x) \to 0$, $xF'(x) \to \infty$. Moreover, replacing $\theta_1'(x)$ by $\theta'(x) + 1/x$ in the first equality above yields, by assumption (i), that F' is monotonic. This completes the proof of Corollary 4.

Our next result provides effective ppl bounds for the discrepancy under slightly stronger assumptions. This is a refinement of theorem 5 of [13] and is obtained by the same technique. The proof has been (re)written jointly with R.R. Hall.

THEOREM 10 (Hall & Tenenbaum). *Let* $f : \mathbf{R}^+ \to \mathbf{R}^+$ *be continuously differentiable, such that* $xf'(x)$ *is ultimately monotonic. Assume that, for some increasing function* R *satisfying (6) and (31), there is a further non-decreasing function* $\varphi : \mathbf{R}^+ \to \mathbf{R}^+$ *such that* $\varphi_1(x) := (\log x)/\varphi(x)$ *is ultimately non-decreasing and*

$$\varphi(x) \gg (\log_2 x)^2, \quad \varphi_1(x) \to \infty, \tag{40}$$

$$1/\varphi(x) \ll x\,|f'(x)| \ll R(e^{\varphi(x)}). \tag{41}$$

Then f *is* erd *and we have for any* $\xi(n) \to \infty$

$$\Delta(n; f) < \tau(n)\xi(n)\left(1 - \frac{\log \varphi_1(n)}{2\log_2 n}\right)^{\Omega(n)/2} \log_2 \varphi_1(n) \quad \text{ppl}. \tag{42}$$

We note that the upper bound (42) is always non trivial under the conditions of the theorem. It yields in fact

$$\Delta(n; f) \ll \tau(n)\varphi_1(n)^{-1/4 + o(1)} \quad \text{ppl}, \tag{43}$$

since the normal order of $\Omega(n)$ is $\log_2 n$. It is clear that the theorem only applies to functions of moderate growth, and one can get a fairly precise idea of the quality of the quantitative result by considering the functions $f(x) = (\log x)^\alpha$ with $\alpha > 0$ and $f(x) = (\log_2 x)^\beta$ with $\beta > 1$. In the first instance we may choose $\varphi(x) = (\log x)^{1-\alpha} + (\log_2 x)^2$, and hence obtain

$$\Delta(n; \log^\alpha) < \tau(n)^{1 - \kappa(\alpha) + o(1)} \quad \text{ppl}, \tag{44}$$

with $\kappa(\alpha) = -\log\left(1 - \frac{1}{2}\min(1, \alpha)\right)/\log 4 > 0$. In the second instance, we select $\varphi(x) = (\log x)/(\log_2 x)^{\beta - 1}$ and get similarly

$$\Delta(n; \log_2^\beta) < \tau(n)(\log_2 n)^{-(\beta - 1)/4 + o(1)} \quad \text{ppl}. \tag{45}$$

From the point of view of constructing Behrend sequences, the uniform distribution approach is usually weaker than the block sequences technique developed in the author's recent paper [24], which rests upon a probabilistic argument. This is to be expected since, in the former case, one derives the conclusion from a very strong hypothesis (namely that $f(d)$ is occasionally small modulo 1 because the corresponding frequency is asymptotically equal to the expectation), whereas, in the latter case, the density of the set of multiples is tackled by an *ad hoc* method. Thus, from (44) one can only infer, via Theorem 3, that

$$\mathscr{A}(\alpha, t) := \{d > 1 : \langle(\log d)^{\alpha}\rangle \leqslant (\log d)^{-t}\} \tag{46}$$

is Behrend for $t < \kappa(\alpha)\log 2 = -\frac{1}{2}\log\left(1 - \frac{1}{2}\min(1, \alpha)\right)$, whereas Theorem 1 of [24] provides, after a straightforward calculation, the larger range

$$t < t_0(\alpha) := (\log 2)\min\{1, \alpha/(1 - \log 2)\}, \tag{47}$$

which is sharp except for the possibility of taking $t = t_0(\alpha)$. However, some upper bounds methods for exponential sums are so powerful that the discrepancy approach enables one to deal with block sequences composed of intervals which are far too short for the induction technique of [24] to be applicable. We shall discuss some examples of this situation in the next two sections.

At this stage, it is worthwhile to note that one can deduce *lower bounds* for the discrepancy from theorem 1 of Hall & Tenenbaum [15], which provides a necessary condition for block sequences to be Behrend. Indeed, if a block sequence $\mathscr{A}$ is defined, for some function $\varepsilon(n)$ which fulfils the assumptions of Theorem 3, by a formula of the type

$$\mathscr{A} = \{d \geqslant 1 : \langle f(d)\rangle \leqslant \varepsilon(d)\}$$

and yet does not satisfy the corresponding necessary condition of [15], we may deduce that

$$\Delta(n; f) \geqslant \tfrac{1}{2}\varepsilon(n)\tau(n)$$

on a set of positive logarithmic density. Actually the necessary condition of [15] and the sufficient condition of [24] are "adjacent" (in a sense precisely described in [24]), and it follows in particular that the sequence $\mathscr{A}(\alpha, t)$ of (46) is not Behrend when $t > t_0(\alpha)$. As a consequence, we obtain that, for all $\alpha < 1 - \log 2$, the lower bound

$$\Delta(n; \log^{\alpha}) > (\log n)^{\log 2 - t_0(\alpha) + o(1)} = (\log n)^{(\log 2)(1 - \alpha/(1 - \log 2)) + o(1)} \tag{48}$$

holds on a set of positive logarithmic density. It is not very difficult to show by a direct argument (using theorem 07 of [14] or exercise III.5.6 of [25]) that (48) in fact holds pp.

The true order of magnitude of $\Delta(n, \log^{\alpha})$ pp$l$ is an interesting open problem, especially in the case $\alpha = 1$. From (44) we have

$$\Delta(n; \log) < \tau(n)^{1/2 + o(1)} \quad \text{pp}l\,,$$

and, as shown in section 5, the exponent $\frac{1}{2}$ can be further reduced to $\log(4/\pi)/\log 2 \approx 0\cdot 34850$ by exploiting the additivity of $\log d$. However, in view of the fact explained above that $\mathscr{A}(1,t)$ is Behrend for all $t < \log 2$, it seems not unreasonable to conjecture that

$$\Delta(n;\log) = \tau(n)^{o(1)} \quad \text{pp}l\,.$$

For the sake of further reference, we make the following formal and more general statement.

CONJECTURE. *Let* $l(\alpha)$ *be the infimum of the set of those real numbers* ξ *such that* $\Delta(n;\log^\alpha) < \tau(n)^\xi \text{pp}l$. *Then for all positive* α *we have*

$$l(\alpha) = 1 - t_0(\alpha)/\log 2 = \max\{0, 1 - \alpha/(1-\log 2)\}\,.$$

It follows from (48) that $l(\alpha) \geqslant \max\{0, 1-\alpha/(1-\log 2)\}$.

Proof of Theorem 10. We use Theorem 7 with $0 < y_0 < 4$, and set out to find an upper bound for

$$S_\nu(k) := \sum_{m=1}^{\infty} \left(\frac{y}{4}\right)^{\Omega(m)} \frac{e(\nu f(km))}{m^{1+\sigma}}$$

where $0 \leqslant y \leqslant y_0$, $\sigma = 1/\log x$ and ν, k are positive integers. Let x_0 be so large that $xf'(x)$ is monotonic, and $\varphi_1(x)$ is decreasing, for $x > x_0$. It will be convenient to introduce a parameter $M = M(k)$ such that

$$M > x_0, \quad \varphi(kM) \leqslant \tfrac{1}{2}\log M \quad (k \geqslant 1)\,. \tag{49}$$

Such an M exists since $\varphi(kM)/\log M \sim 1/\varphi_1(kM) \to 0$ as $M \to \infty$ for each fixed k. We note that for $u \geqslant M(k)$ we have

$$\begin{aligned}\varphi(ku) &= \frac{\log(ku)}{\varphi_1(ku)} \leqslant \frac{\log(ku)}{\varphi_1(kM)} = \frac{\varphi(kM)\log(ku)}{\log(kM)}\\ &\leqslant \tfrac{1}{2}\,\frac{\log M \log(ku)}{\log(kM)} \leqslant \tfrac{1}{2}\log u\,.\end{aligned} \tag{50}$$

For given $k, \nu \geqslant 1$, put $h(u) := e(\nu f(ku))$, $A(u) := \sum_{m \leqslant u} (y/4)^{\Omega(m)}$, so that

$$S_\nu(k) = \int_{1-}^{\infty} \frac{h(u)}{u^{1+\sigma}}\,\mathrm{d}A(u)\,, \tag{51}$$

and by (6), since $0 \leqslant y \leqslant y_0 < 4$,

$$A(u) = \int_0^{1/2} u^{1-t} \mathrm{d}\lambda_{y/4}(t) + O(u/R(u)) .$$

We insert this into (51), make the trivial estimate $|h(u)| \leqslant 1$ for $u \leqslant M$ and integrate by parts on $[M, \infty)$ the contribution of the remainder term. We obtain

$$S_\nu(k) = O((\log M)^{y/4}) + \int_M^\infty \frac{h(u)}{u^{1+\sigma}} \int_0^{1/2} (1-t) u^{-t} \mathrm{d}\lambda_{y/4}(t) \mathrm{d}u$$

$$+ O\left(\frac{1}{R(M)}\right) - \int_M^\infty \frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{h(u)}{u^{1+\sigma}}\right) O\left(\frac{u}{R(u)}\right) \mathrm{d}u .$$

The last term is

$$\text{(52)} \quad \begin{aligned} &\ll \int_M^\infty \frac{u|h'(u)| + |h(u)|}{u^{1+\sigma} R(u)} \mathrm{d}u \ll \nu \int_M^\infty \frac{R(e^{\varphi(ku)}) + 1}{u^{1+\sigma} R(u)} \mathrm{d}u \\ &\ll \nu \int_M^\infty \frac{R(\sqrt{u})}{u^{1+\sigma} R(u)} \mathrm{d}u \ll \nu \int_M^\infty \frac{\mathrm{d}u}{u R(u)^b} \ll \frac{\nu}{R(M)^{b/2}} , \end{aligned}$$

where we have used (50) in the third step, and (31) in the fourth. Next, we consider the main term in $S_\nu(k)$. This is

$$\text{(53)} \quad \int_0^{1/2} (1-t) \int_M^\infty \frac{h(u)}{u^{1+\sigma+t}} \mathrm{d}u \mathrm{d}\lambda_{y/4}(t) \ll \int_0^{1/2} \left| \int_M^\infty \frac{h(u)}{u^{1+\sigma+t}} \mathrm{d}u \right| t^{-y/4} \mathrm{d}t .$$

We substitute $u = Me^{v}$ in the inner integral which becomes

$$\int_0^\infty \frac{e(\nu f(kMe^{v}))}{M^{\sigma+t} e^{(\sigma+t)v}} \mathrm{d}v = \int_0^\infty \frac{H'(v)}{M^{\sigma+t} e^{(\sigma+t)v}} \mathrm{d}v = \int_0^\infty \frac{(\sigma+t) H(v)}{M^{\sigma+t} e^{(\sigma+t)v}} \mathrm{d}v ,$$

with $H(v) := \int_0^v e(\nu f(kMe^{w})) \mathrm{d}w$. The function

$$\frac{\mathrm{d}}{\mathrm{d}w} \{\nu f(kMe^{w})\} = \nu kMe^{w} f'(kMe^{w})$$

is monotonic on the whole half-line $w \geqslant 0$ by the choice of M, and, by (41), it is $\gg \nu/\varphi(kMe^w) \gg \nu/\varphi(kMe^\upsilon)$ for $w \leqslant \upsilon$. By a well-known lemma on exponential integrals (see e.g. Titchmarsh [26], lemma 4.2), we obtain that

$$H(\upsilon) \ll \nu^{-1}\varphi(kMe^\upsilon) ,$$

so the upper bound in (53) is

$$\ll \frac{1}{\nu}\int_0^{1/2}\int_0^\infty \frac{(\sigma+t)\varphi(kMe^\upsilon)}{M^{\sigma+t}e^{(\sigma+t)\upsilon}t^{y/4}}\,\mathrm{d}\upsilon\,\mathrm{d}t .$$

At this stage, we note that $\varphi_1(x) \ll \varphi_1(x')$ for $1 \leqslant x \leqslant x'$. This readily follows from the facts that φ_1 is non-decreasing for $x > x_0$ and that $\varphi(x) \asymp 1$ for $1 \leqslant x \leqslant x_0$, so we omit the details. Therefore, we have for $\xi \geqslant 1$, $\eta \geqslant 1$,

$$\varphi(\xi\eta) = \frac{\log(\xi\eta)}{\varphi_1(\xi\eta)} \ll \frac{\log\xi}{\varphi_1(\xi)} + \frac{\log\eta}{\varphi_1(\eta)} = \varphi(\xi) + \varphi(\eta) . \tag{54}$$

Thus, the last double integral is

$$\ll \int_0^{1/2}\frac{\varphi(kM)}{M^t t^{y/4}}\,\mathrm{d}t + \int_0^{1/2}\int_0^\infty \frac{(\sigma+t)\varphi(e^\upsilon)}{M^t e^{(\sigma+t)\upsilon}t^{y/4}}\,\mathrm{d}\upsilon\,\mathrm{d}t .$$

The first term can be computed explicitly. In the inner υ-integral of the second term, we substitute $\upsilon = w/(\sigma+t)$ and split the range at $w = 1$. We obtain altogether

$$\ll \varphi(kM)(\log M)^{y/4-1} + \int_0^{1/2}\int_0^\infty \frac{\varphi(e^{w/(\sigma+t)})}{M^t e^w t^{y/4}}\,\mathrm{d}w\,\mathrm{d}t$$

$$\ll \varphi(kM)(\log M)^{y/4-1} + \int_0^{1/2}\frac{\varphi(e^{1/(\sigma+t)})}{M^t t^{y/4}}\,\mathrm{d}t + \int_0^{1/2}\int_1^\infty \frac{\varphi(e^{w/(\sigma+t)})}{M^t e^w t^{y/4}}\,\mathrm{d}w\,\mathrm{d}t$$

$$\ll \{\varphi(kM) + \varphi(e^{1/\sigma})\}(\log M)^{y/4-1} + \int_0^{1/2}\int_1^\infty \frac{w\varphi(e^{1/(\sigma+t)})}{M^t e^w t^{y/4}}\,\mathrm{d}w\,\mathrm{d}t$$

$$\ll \{\varphi(kM) + \varphi(e^{1/\sigma})\}(\log M)^{y/4-1} ,$$

where we have used in the penultimate stage the upper bound

$$\varphi(e^{w/(\sigma+t)}) = \frac{w}{(\sigma+t)\varphi_1(e^{w/(\sigma+t)})} \ll \frac{w}{(\sigma+t)\varphi_1(e^{1/(\sigma+t)})} = w\varphi(e^{1/(\sigma+t)}) .$$

Collecting our estimates so far and inserting them into (52) we obtain, since $e^{1/\sigma} = x$,

$$
\begin{aligned}
S_\nu(k) &\ll (\log M)^{y/4} + \nu R(M)^{-b/2} + \nu^{-1}\{\varphi(kM) + \varphi(x)\}(\log M)^{y/4-1} \\
&\ll (\log M)^{y/4} + \nu R(M)^{-b/2} + \nu^{-1}\varphi(x)\,,
\end{aligned} \tag{55}
$$

as $\varphi(kM) \leqslant \frac{1}{2}\log M$ by (49). Let C be an absolute constant which is at least three times as large as the implicit constant in (54). We select

$$M := e^{C\varphi(k)+\varphi(x)}\,,$$

so that, when $x \geqslant x_1(C)$, we have $\varphi(M) \leqslant (1/2C)\log M$ (because M is large) and hence

$$\varphi(kM) \leqslant \tfrac{1}{3}\, C\{\varphi(M) + \varphi(k)\} \leqslant \tfrac{1}{6}\log M + \tfrac{1}{3}\log M = \tfrac{1}{2}\log M\,.$$

Thus (49) is satisfied with this choice of M. Moreover, we also have $\varphi(x)/\nu \ll \log M$, so we finally obtain from (55) that

$$
\begin{aligned}
S_\nu(k) &\ll (\log M)^{y/4} + \nu R(M)^{-b/2} \ll \varphi(k)^{y/4} + \varphi(x)^{y/4} + \nu R(e^{\varphi(x)})^{-b/2} \\
&\ll \varphi(x)^{y/4}\left\{1 + \left(\frac{\log k}{\log x}\right)^{y/4}\right\} + \nu R(e^{\varphi(x)})^{-b/2}\,,
\end{aligned}
$$

where we have used in the last stage the inequality $\varphi_1(x) \leqslant \varphi_1(k)$ for $k \geqslant x$.

We are now in a position to embark on the final part of the proof. Inserting the above estimate for $S_\nu(k)$ into (25), we find that

$$H_\nu(x,y) = \sum_{k=1}^{\infty}\left(\frac{y}{4}\right)^{\Omega(n)}\frac{1}{k^{1+\sigma}}\,|S_\nu(k)|^2 \ll \varphi(x)^{y/2}(\log x)^{y/4} + \frac{\nu^2(\log x)^{y/4}}{R(e^{\varphi(x)})^b}\,.$$

Hence, for $T \geqslant 2$,

$$\sum_{1\leqslant \nu\leqslant T}\frac{1}{\nu}\,H_\nu(x) \ll \varphi(x)^{y/2}(\log x)^{y/4}\log T + \frac{T^2(\log x)^{y/4}}{R(e^{\varphi(x)})^b}\,.$$

We therefore deduce from (24) that

$$
\begin{aligned}
&(\log x)^{-1}\sum_{n\leqslant x}\frac{\Delta(n;f)^2}{n}\left(\frac{y}{4}\right)^{\Omega(n)} \\
&\qquad \ll \frac{(\log x)^{y-1}}{T^2} + \varphi(x)^{y/2}(\log x)^{y/2-1}(\log T)^2 \\
&\qquad\quad + \frac{T^2(\log T)(\log x)^{y/2-1}}{R(e^{\varphi(x)})^b}\,.
\end{aligned}
$$

We choose $T = \varphi_1(x)^{y/4} = (\log x/\varphi(x))^{y/4}$. The upper bound above becomes

$$(56) \qquad \ll \varphi(x)^{y/2}(\log x)^{y/2-1}(\log \varphi_1(x))^2 .$$

Indeed the last term is easily seen to be negligible by the lower bound (40) imposed on $\varphi(x)$, and because $R(x) = \exp\{(\log x)^{4/7}\}$ is an admissible choice for R. Thus we may define $E_2(x,y)$ as being equal to a suitable constant multiple of the right-hand side of (56), and apply Theorem 7 to obtain that

$$(57) \qquad \Delta(n;f) \ll \xi(n)\tau(n)y^{\Omega(n)/2}\sqrt{E_2(n,y)} \quad \text{pp}l ,$$

provided $0 \leqslant y \leqslant y_0 < 4$ and $y = y(n)$ is such that $E_2(n,y)$ is slowly increasing as a function of n. We choose

$$y = 2\Big/\left(1 + \frac{\log \varphi(n)}{\log_2 n}\right) = 1\Big/\left(1 - \frac{\log \varphi_1(n)}{2\log_2 n}\right),$$

which minimises $(\log n)^{-\log y}E_2(n,y)$ up to a power of $\log_2 \varphi_1(n)$. This value of y is always in the range $[1,2]$. Inserting into (56) yields

$$E_2(n,y) \asymp (\log \varphi_1(n))^2 ,$$

which implies that this function is slowly decreasing. The required estimate (42) hence follows from (57). This completes the proof of Theorem 10.

3. Functions of excessive growth: the case $f(d) = d^\alpha$

Here, we address the problem of bounding the discrepancy pp$l$ for functions which increase too fast for the techniques of the previous section to be applicable. More precisely, let us recall the quantity

$$(58) \qquad H_\nu(x,y) := \sum_{k=1}^{\infty} \left(\frac{y}{4}\right)^{\Omega(k)} \frac{1}{k^{1+\sigma}} \left| \sum_{m=1}^{\infty} \left(\frac{y}{4}\right)^{\Omega(m)} \frac{e(\nu f(km))}{m^{1+\sigma}} \right|^2 ,$$

with $\sigma := 1/\log x$, which appears implicitly in the upper bound (26) of Theorem 7 for the discrepancy $\Delta(n;f)$. This was primarily defined for $y < 8$, but we restrict if here to values of $y \leqslant 4$. The functions of moderate growth are essentially those for which the inner m-sum can be estimated by partial summation, using the available results on the mean value of $m \mapsto (y/4)^{\Omega(m)}$. When the rate of growth of f prohibits such a treatment, we may consider $H_\nu(x,y)$ as a 'type II sum', according to the poetic